ANTICHAINS IN THE POWERSET OF $\mathbb{R}$: REALISATION THROUGH INDUCTION

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Abstract

The existence of a 'large' family of subspaces of the real line, no member of which can be embedded into any other, has been known for almost eight decades. We present a proof, via transfinite induction, that has shown itself adaptable to tackling a wide variety of cognate problems concerning the possible ordertypes of families of substructures of a given structure.

1. Introduction

Two subsets of the real line $\mathbb{R}$ are termed *incomparable* if neither can be homeomorphically embedded into the other. When each two members of a collection $\mathscr{F}$ of subsets of $\mathbb{R}$ are incomparable, we refer to $\mathscr{F}$ as an *antichain* in $P(\mathbb{R})$. The existence of 'large' antichains in $P(\mathbb{R})$ was established three quarters of a century ago by the pioneering work of Banach, Kuratowski and Sierpiński (see, for example, [3] and [5]). Their ingenious methodologies did not, however, transfer readily to other cognate problems such as the construction of a family of sets whose ordertype (under homeomorphic embeddability) was predetermined and substantially more complicated.

In the last ten years there has been a resurgence of interest in this area, attributable to the insight of Stephen Watson that transfinite induction will serve as a highly versatile procedure for resolving a wide range of such questions. The purpose of this short paper is to 'rediscover' some of the antichain results of Kuratowski *et al.* using Watson's 'inductive sabotage' technique: for although the conclusions are old, the devices here exhibited for reaching them are continuing to surprise with their power and diversity. (For illustrations of the last point, the interested reader may wish to consult [9], [10] and [7]. The latter two of these references make use of essentially the same technique to establish that the subspaces of the real line (and of similar spaces) allow us to construct models not merely of large antichains but of any partially ordered or quasi-ordered set whatsoever whose cardinality does not exceed that of the continuum. Article [9], on the other hand, carries out a similar exercise based not in

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topological spaces at all but in ordertypes. Also, a much less formal introduction to transfinite induction and the application of the so-called Bernstein sets will be found in a recent article [8] in the IMS Bulletin.)

Virtually all the subsets of $\mathbb{R}$ which feature in solutions to questions of this type are Bernstein sets. A *Bernstein set* in $\mathbb{R}$ is a set of real numbers which *intersects* every uncountable bounded closed subset of $\mathbb{R}$ but *contains* no such subset. Because every uncountable bounded closed set in $\mathbb{R}$ contains [2] a (homeomorphic) copy of the Cantor middle-thirds set C, an equivalent condition is that a Bernstein set must intersect every copy of C but contain none. It is this criterion that is generally used in constructing these sets, whose name, incidentally, derives from the author who first established their existence [1]. Our objective, then, is to show how induction may be used to generate a family of more-than-continuum-many Bernstein sets that will constitute an antichain.

2. Incomparable Bernstein sets

Let $\mathbb{Q}$ denote the set of rational numbers, $|X|$ the cardinality of a given set X and, in particular, $\mathbf{c}$ the cardinality of $\mathbb{R}$. The first step in the argument consists of establishing the following lemma.

Lemma 1. *Suppose that there is a list of four objects* (A, π, In, Out) *where*

- A *is a Bernstein set that contains* $\mathbb{Q}$,
- π *is a continuous injection from* $\mathbb{Q}$ *to* $\mathbb{R}$,
- In *and* Out *are sets of real numbers such that* $In \cap Out = \emptyset, \mathbb{Q} \subseteq In, |In| < \mathbf{c}$, $|Out| < \mathbf{c}$.

Then there exist disjoint sets In^+, Out^+, *formed from* In, Out *respectively by adjoining at most two irrationals to each, such that if* B *is any set satisfying*

$$In^+ \subseteq B \subseteq \mathbb{R} \backslash Out^+$$

then no continuous injective extension of π *can map* A *into* B *or* B *into* A.

Proof. Recall firstly that (i) if π possesses a continuous real-valued extension $\bar{\pi}$ over a set E such that $\mathbb{Q} \subseteq E \subseteq \mathbb{R}$ then, for each $x \in E$, π has a limit at x (namely $\bar{\pi}(x)$), and (ii) if E is a set (intermediate between $\mathbb{Q}$ and $\mathbb{R}$) at each point x of which π does have a limit $\bar{\pi}(x)$, then the map $\bar{\pi}$ thus identified is a continuous extension of π over E. These elementary remarks effectively permit discussion of the extensibility of π ‘one point at a time’.

Precisely one of the following statements must be true:

(i) there is a point $x \in \mathbb{R} \backslash Out$ at which π has no limit;
(ii) for every $x \in \mathbb{R} \backslash Out$, π has a limit $\bar{\pi}(x)$ *and* there exist distinct real numbers x_1, x_2 belonging to $\mathbb{R} \backslash Out$ with $\bar{\pi}(x_1) = \bar{\pi}(x_2)$;
(iii) for each $x \in \mathbb{R} \backslash Out$, π has a limit $\bar{\pi}(x)$ at x *and* $\bar{\pi}$ is injective on $\mathbb{R} \backslash Out$.

In case (i), declare $In^+ = In \cup \{x\}$. Then π has no continuous extension over any superset of In^+. In case (ii), declare $In^+ = In \cup \{x_1, x_2\}$ and we see that a continuous extension of π over a superset of In^+ must fail to be injective. In case (iii), the extension

$\bar{\pi}: \mathbb{R}\backslash Out \to \mathbb{R}$ is a continuous injection. Now the real line contains **c**-many pairwise-disjoint homeomorphic copies of the Cantor set C (for example, use the notion of space-filling curve to access a continuous function f from $[0, 1]$ onto the unit square in the coordinate plane, observe that the pre-images under f of the horizontal cross-sections of the square constitute **c**-many pairwise-disjoint uncountable bounded closed sets in $\mathbb{R}$, and construct within each of these a copy of C). Since Out has fewer than **c**-many elements, one of these Cantor copies K can be chosen that includes no point of Out. Then $\bar{\pi}(K)$ is uncountable and compact, the Bernstein set A cannot contain it, and we may choose $x \in K$ such that $\bar{\pi}(x) \notin A$. Putting $In^+ = In \cup \{x\}$, we have that no continuous extension of π over a superset of In^+ could map into A.

In each of the three cases that could arise here the set In has been augmented in such a way as to obstruct any possible extension of π from copying the enlarged set *into* A. The next (and easier) stage is likewise to block copying maps *from* A.

Precisely one of the following statements must be true:

(iv) In^+ includes the limit of π at every point of A at which this limit exists;

(v) there is $x \in A$ such that π has a limit y at x and $y \notin In^+$.

In case (iv), if there is any point of A at which π does not possess a limit, then no continuous extension of π over A can exist. If not, then the relative cardinalities of A and In^+ show that distinct points x_1 and x_2 of A may be chosen at which the limits of π will coincide and, just as in case (ii), any continuous extension of π over A must fail to be injective. In case (v), define $Out^+ = Out \cup \{y\}$: a continuous extension of π over A will have to take the value y, so its range cannot be disjoint from Out^+. This completes the proof of Lemma 1. ∎

Lemma 1 shows how to sabotage *one* arbitrary mapping π from being the 'rational skeleton' of an undesired embedding between A and a new set that is under construction. If the process can be implemented for *all* of the **c**-many mappings of which π is the exemplar, then the new set will indeed be incomparable with A. If, in parallel, we inspect the **c**-many homeomorphs of the Cantor set C, and both choose and reject a point from each of them as material for the new set, the end product will also be a Bernstein set. Transfinite induction allows us to run these two procedures side by side, yielding two incomparable Bernsteins.

We can, however, achieve a stronger conclusion with really no additional effort. Given a *list* of Bernstein sets, the induction that has just been outlined can scan through them in the same 'time' as it takes to exhaust the mappings π and the Cantors C *provided only* that the length of that list does not exceed **c**. Thus, a family of at most **c**-many incomparable Bernsteins can always be augmented by one more. We shall now examine the detail of this inductive process.

Lemma 2. *Let* $\mathscr{B}$ *be a set of at most* **c**-*many Bernstein set in* $\mathbb{R}$, *each of which contains* $\mathbb{Q}$. *There is a Bernstein set* (*also containing* $\mathbb{Q}$) *that is incomparable with every member of* $\mathscr{B}$.

Proof. Denote by $\mathscr{C}$ the set of all homeomorphs of C within $\mathbb{R}$, and by Π the set of all continuous injections from $\mathbb{Q}$ to $\mathbb{R}$. It is easy to confirm that **c** is the cardinality of $\mathscr{C}$ and of Π and of $\mathscr{B} \times \mathscr{C} \times \Pi$. Express the last of these as a family $\{(B_\alpha, C_\alpha, \pi_\alpha): \alpha$

$< \mathbf{c}\}$ indexed upon $\mathbf{c}$ as an initial ordinal. We show how to construct, for each ordinal $\gamma < \mathbf{c}$, two sets $In(\gamma)$, $Out(\gamma)$ of real numbers such that

(1) $\gamma_1 \leqslant \gamma_2$ implies that $\mathbb{Q} \subseteq In(\gamma_1) \subseteq In(\gamma_2) \subseteq \mathbb{R}\backslash Out(\gamma_2) \subseteq \mathbb{R}\backslash Out(\gamma_1)$,
(2) $|In(\gamma)| \leqslant |\gamma| + \aleph_o$,
(3) $|Out(\gamma)| \leqslant |\gamma| + \aleph_o$,
(4) C_γ intersects both $In(\gamma)$ and $Out(\gamma)$,
(5) if B is any set such that $In(\gamma) \subseteq B \subseteq \mathbb{R}\backslash Out(\gamma)$ then there is no continuous injective extension of π_γ that maps either B_γ into B or B into B_γ

for all relevant $\gamma, \gamma_1, \gamma_2$.

To initialise the process we apply Lemma 1 to the quartet of objects $(B_o, \pi_o, \mathbb{Q}, \varnothing)$ and inspect the sets $\mathbb{Q}^+, \varnothing^+$ which it produces. Since C_o is uncountable we can select distinct elements i_o, j_o of $C_o\backslash(\mathbb{Q}^+ \cup \varnothing^+)$. Put $In(0) = \mathbb{Q}^+ \cup \{i_o\}, Out(0) = \varnothing^+ \cup \{j_o\}$ and it is seen that conditions (1)–(5) are satisfied. Now assume that, for a particular ordinal $\zeta < \mathbf{c}$, we have already identified $In(\gamma)$ and $Out(\gamma)$ for all $\gamma < \zeta$ and that (1)–(5) are in force for them. Put $I_\zeta = \bigcup_{\gamma<\zeta} In(\gamma)$, $O_\zeta = \bigcup_{\gamma<\zeta} Out(\gamma)$ and note that (whether ζ is a successor ordinal or a limit ordinal) these sets are disjoint and have cardinalities not exceeding $|\zeta| + \aleph_o$. Apply Lemma 1 to $(B_\zeta, \pi_\zeta, I_\zeta, O_\zeta)$ to produce the sets I_ζ^+, O_ζ^+ whose union has smaller cardinality than C_ζ, and select distinct elements i_ζ, j_ζ lying outside that union. Then the definitions $In(\zeta) = I_\zeta^+ \cup \{i_\zeta\}, Out(\zeta) = O_\zeta^+ \cup \{j_\zeta\}$ extend the validity of (1)–(5) to encompass the ordinal ζ.

By induction, this process constructs the sets $In(\gamma)$ and $Out(\gamma)$ for $\gamma < \mathbf{c}$ subject to the five conditions. Lastly, put $W = \bigcup_{\gamma<\mathbf{c}} In(\gamma)$. We note that W includes a (different) point from each Cantor set C_α and from its complement: so W has cardinality $\mathbf{c}$ and is a Bernstein set. Suppose there were a continuous injection j from W into some member B' of the family $\mathscr{B}$ or *vice versa.* Make an arbitrary choice of a Cantor set K and notice that $(B', K, j|_{\mathbb{Q}})$ must have been listed as $(B_\delta, C_\delta, \pi_\delta)$ for some $\delta < \mathbf{c}$. But then $In(\delta) \subseteq W \subseteq \mathbb{R}\backslash Out(\delta)$, so π_δ is not, after all, extensible to a continuous injection between $B_\delta = B'$ and W: a contradiction. ■

The existence of large antichains in $P(\mathbb{R})$ is now almost immediate.

Theorem 3 (compare [3, p. 205]). *There is a family of* $\mathbf{c}^+$ *Bernstein sets in* $\mathbb{R}$ *which are pairwise incomparable under homeomorphic embeddability.*

Proof. Construct an 'initial' Bernstein set B_o that contains $\mathbb{Q}$. Supposing that ξ is an ordinal less than the cardinal (= initial ordinal) $\mathbf{c}^+$ and that $\{B_\varepsilon : \varepsilon < \xi)$ is a family of mutually non-embeddable Bernstein sets, we note that $|\xi| < \mathbf{c}^+$: so Lemma 2 produces another Bernstein $B_\xi \supseteq \mathbb{Q}$ that has no embeddability relationships with the others. This gives us the family $\{B_\varepsilon : \varepsilon < \xi + 1\}$ with the same 'incomparability' characteristic. Transfinite induction now generates the desired family of sets. ■

3. Generalisation and comparison

The argument set out in section 2 exploits only a small number of the many properties that the real line enjoys, and will therefore serve to establish a more general conclusion. To be specific, we used separability to obtain a subset $(\mathbb{Q})$ of low cardinality that would support the skeletons of potential embedding maps and regularity to enable analysis of their extensibility one domain-point at a time, and then the rest of the machinery was driven by the relationships between uncountable compact, Cantor and Bernstein sets. Abstracting this in a routine fashion, we find the following result.

Proposition 4. *Let X be a regular* (T_3) *space of infinite cardinality β, possessing a dense subset D of cardinality α such that $\beta^\alpha = \beta$. Suppose that the family of sets $\mathscr{K} = \{K \subseteq X : K$ is compact and $|K| = \beta\}$ includes β-many pairwise-disjoint members. Suppose also that $\mathscr{K}$ contains a subfamily $\mathscr{L}$ of cardinality β such that each member of $\mathscr{K}$ contains a member of $\mathscr{L}$. Then the powerset of X* (*ordered by embeddability*) *contains an antichain of cardinality β^+.*

The most obvious field in which to seek applications of this result is that of the separable metrisable spaces. If X is such a space, and if X contains even one uncountable compact subset K, then K must contain a Cantor set, whose internal structure then guarantees $\mathbf{c}$-many disjoint copies of itself. Thus all the hypotheses of Proposition 4 are satisfied, and we obtain the following corollary.

Corollary 5. *There is an antichain of cardinality $\mathbf{c}^+$ in the powerset of any separable metrisable space that contains an uncountable compactum.*

In particular, the powersets of the Cantor set itself and of the irrationals must contain such an antichain.

Now let us briefly review how Kuratowski *et al.* tackled this group of problems. Their argument begins (see, for example, [4]) purely combinatorially by showing that, given a set X and a family $\mathscr{F}$ of injective partial mappings from X to X, where X and $\mathscr{F}$ have the *same* infinite cardinality α, there is a family of 2^α-many subsets of X, *none* of which is mapped into any other by any map in $\mathscr{F}$. Next, an invocation is made to a theorem of Lavrentieff [6] that a homeomorphism between two subsets in a complete metrisable space can be extended to become a homeomorphism between G_δ-supersets of them. Lastly, given a separable complete metrisable space X of cardinality $\mathbf{c}$, because there are only $\mathbf{c}$-many G_δ subsets of X, the family of continuous injections defined on G_δ subsets also has cardinality $\mathbf{c}$ and so the first part of the argument generates $2^\mathbf{c}$ subsets of X that are *incomparable in respect of embedding via these mappings*; now, by the Lavrentieff result, these sets are incomparable in the full sense of the term.

The essential difficulty which all such arguments have to circumvent is that the real line, in common with most other non-pathological spaces, has significantly more ‘partial autohomeomorphisms’ than it has elements. There are, in fact, $2^\mathbf{c}$ mappings with domain and range contained in $\mathbb{R}$ that act homeomorphically, and so a naive attempt to build a set of real numbers upon which none of these will function is doomed to run out of ‘building material’ long before the task is complete. The Kuratowski approach uses G_δ envelopes to cut down the list of mappings that must be considered to length $\mathbf{c}$, while the discussion in section 2 employs restriction to $\mathbb{Q}$ to achieve the same objective. Kuratowski’s method yields bigger antichains ($2^\mathbf{c}$ is certainly capable of exceeding $\mathbf{c}^+$) but at the cost of an explicit appeal to completeness which we have avoided. Thus, for example, the existence of large antichains in the powerset of the irrationals does not follow directly from the 1920s argument. However, it does follow indirectly, for the irrational line contains a Cantor set which, being complete, is amenable to the older method. We conclude that the appearance of generality in Proposition 4 is of doubtful value in presently envisaged applications, and that the significant advantages of the style of construction set out in this paper reside in the ease with which it may be modified to model ordered sets that are very different from antichains, as was remarked in the Introduction.

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