NON-UNITAL SEMIGROUP CROSSED PRODUCTS

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Abstract

We construct non-unital semigroup crossed products by Ore semigroups. We discuss their behaviour under short exact sequences and tensor products. Then we specialise to some abelian C^*-algebras, for which we give a description of the ideals that respect the semigroup crossed product.

Introduction

In the study of crossed products of C^*-algebras by semigroups of endomorphisms such as the Bost–Connes Hecke C^*-algebra, some analogues of standard results about short exact sequences and tensor products of ordinary crossed products are needed. Since these questions have also arisen in relation to the application of crossed products by semigroups to Toeplitz algebras (cf. [1] for short exact sequences and [15] for tensor decompositions), it seems worthwhile to seek general versions of these theorems. As Adji [1] pointed out in the case of totally ordered abelian semigroups, new technical hypotheses are required for dealing with exact sequences. The infinite tensor decomposition exhibited by Laca and Raeburn [15] relies on the characterisation of faithful representations of the relevant Toeplitz algebra, viewed as a semigroup crossed product. Here we aim to use the universal properties of semigroup crossed products to establish tensor decompositions. It is our goal to show that the hypotheses identified in [1] and the features of the semigroup actions apparent in [15] are essentially the only prerequisites for building a satisfactory theory. Our theorems apply in most situations involving the current collection of examples [15; 4; 5; 17].

The general setting is that of non-unital C^*-algebras, because we need results that apply to ideals. We work with semigroups that can be embedded in a group, the so-called Ore semigroups. This class is both large (it contains all partially ordered cancellative abelian semigroups as well as a number of non-abelian semigroups) and useful in the context of C^*-algebra theory; many examples of such semigroups acting on C^*-algebras have already arisen, e.g. the semigroup $\mathbb{N}$ [7; 25], the totally ordered abelian semigroups [19; 3; 1], normal semigroups [13], and the direct sums $\mathbb{N}^k$ for k in $\mathbb{N} \cup \{\infty\}$ [15; 5; 17]. We are interested in actions by endomorphisms of non-unital C^*-algebras which admit strictly continuous extensions to multiplier algebras; these are called extendible endomorphisms in [1].

Short exact sequences of C^*-algebras are preserved under crossed products by a totally ordered abelian semigroup if the ideal is *extendibly invariant* with respect to the action [1; 2]. This hypothesis basically means that restricting the extendible endomorphisms of the algebra to the ideal preserves extendibility in a way that is consistent with the embedding of the ideal in the algebra. We prove that this requirement on the ideal is the only obstruction to the existence of short exact sequences of crossed products by Ore semigroups (cf. [21, proposition 2.2] or [23, lemma 2.8.2] for ordinary crossed products).

We want to show that the maximal tensor product of two semigroup crossed products is still such an object. It turns out that representations of two semigroup crossed products commute, as they should when working with the maximal tensor product, if their isometric components $*$-commute. This leads us to the notion of quasi-lattice ordered groups and Nica-covariant isometric representations as introduced in [20]. In order to isolate the covariant representations amongst all isometric representations of a semigroup, we work with extendible endomorphisms that are compatible with the quasi-lattice condition. (A similar compatibility requirement appeared in [11] for a semigroup action on unital C^*-algebras.) Examples of semigroup actions covered by our theorem are contained in [15; 4; 5; 17].

In the first section we show how the Ore condition on the semigroup is decisive for constructing non-unital semigroup crossed products and for dealing with short exact sequences in the way pursued by Adji in the case of totally ordered abelian semigroups [1]. In section 2 we discuss tensor decompositions of semigroup crossed products. In section 3 we exhibit a composition series of ideals in a semigroup crossed product arising from tensoring two short exact sequences of C^*-algebras. This situation is pertinent for the crossed products from [15, section 4] for which the action of a direct product of semigroups on a tensor product of C^*-algebras does not split on the individual semigroups. In the last section we specialise to abelian C^*-algebras with endomorphic actions of the sort studied in [15; 4; 5; 17]. We give necessary and sufficient conditions for the ideals to be extendibly invariant. Example 4.6 illustrates the manner in which we will apply results of the present paper to the crossed products from [18].

1. Non-unital semigroup crossed products

We first introduce some definitions and recall some known facts. An Ore semigroup is a cancellative semigroup S that is right-reversible, in the sense that $Ss \cap St \neq \emptyset$ for all $s, t \in S$. A semigroup S is right-reversible if and only if it can be embedded in a group G such that $G = S^{-1}S$ [6]. We shall always assume that the unit element e in G is contained in S.

If B and I are C^*-algebras, a homomorphism $\phi: I \to M(B)$ is called non-degenerate if there is an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ for I such that $\phi(u_\lambda) \to 1_{M(B)}$ strictly in $M(B)$. If ϕ preserves approximate units as a map $I \to B$ we shall follow [8] and call it proper. If $\phi: I \to M(B)$ is non-degenerate, there is a unique strictly continuous extension $\overline{\phi}: M(I) \to M(B)$ by [16, lemma 1.1]. The same holds if $\phi: I \to B$ is proper. A non-proper endomorphism ϕ of I, which is typical in the context of non-unital semigroup dynamical systems, still has a unique extension $M(I) \to M(I)$ if there are an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ for I and a projection p_ϕ in $M(I)$ such that $\phi(u_\lambda) \to p_\phi$ strictly in $M(I)$ [10, corollary 1.1.15; 1]. Such endomorphisms are called *extendible* in [1]. An extendible endomorphism satisfies the condition that $\phi(u_\lambda) \to \overline{\phi}(1_{M(I)})$ strictly in $M(I)$

for any approximate unit $(u_\lambda)_{\lambda\in\Lambda} \subset I$. In particular, if $\pi: I \to \mathbb{B}(H)$ is a non-degenerate representation on a Hilbert space H, then the extension $\overline{\pi}: M(I) \to \mathbb{B}(H) = M(K(H))$ is strictly continuous.

A semigroup dynamical system is a triple (I, S, α) consisting of a C^*-algebra I, a semigroup S and an action $\alpha: S \to \mathrm{End}(I)$. A covariant representation of (I, S, α) is a pair (π, V), with π a non-degenerate representation of I on a Hilbert space H and V a representation of S by isometries on H such that the covariance condition $\pi(\alpha_s(a)) = V_s \pi(a) V_s^*$ is satisfied for all a in A and s in S [14].

We write ι_I, ι_S in the following definition rather than i_I, i_S as in [14] since we will later need i to stand for the canonical embedding of an algebra into the multiplier algebra of its maximal tensor product with some other algebra.

Definition 1.1. A crossed product for a semigroup dynamical system (I, S, α) is a triple (B, ι_I, ι_S) such that B is a C^*-algebra, $\iota_I: I \to B$ is a proper homomorphism and $\iota_S: S \to$ *Isometries*$(M(B))$ is a semigroup homomorphism satisfying:

(i) $\iota_I(\alpha_s(a)) = \iota_S(s)\iota_I(a)\iota_S(s)^*$, for all $a \in I, s \in S$;

(ii) for any covariant representation (π, V) of (I, S, α) on a Hilbert space H, there exists a non-degenerate representation $\pi \times V$ of B on H such that $\pi \times V \circ \iota_I = \pi$ and $\overline{\pi \times V} \circ \iota_S = V$;

(iii) B is generated by $\{\iota_I(a)\iota_S(s) \mid a \in I, s \in S\}$.

We shall use the notation $I \rtimes_\alpha S$ or just $I \rtimes S$ for the algebra B.

Remark 1.2. Let (I, S, α) be a semigroup dynamical system with extendible endomorphisms, in the sense that each α_s is extendible, and let (π, V) be a covariant representation of the system. For an approximate unit $(u_\lambda)_\lambda$ of I and m in $M(I)$ let $\lambda \to \infty$ in the equation $V_s \pi(u_\lambda m) V_s^* = \pi(\alpha_s(u_\lambda m))$. Since $\pi(u_\lambda m) \to \overline{\pi}(m)$ strictly in $M(I)$ and since α_s is extendible, it follows that $(\overline{\pi}, V)$ is a covariant representation of $(M(I), S, \overline{\alpha})$. Hence,

$$V_s V_s^* = \overline{\pi}(\overline{\alpha}_s(1_{M(I)})), \text{ for all } s \in S. \tag{1.1}$$

Lemma 1.3. *Assume that S is an Ore semigroup and* (I, S, α) *is a semigroup dynamical system with extendible endomorphisms. If* (π, V) *is a covariant representation, then* $C^*(\pi(I)\, V(S)) = \overline{span}\{V_s^* \pi(a) V_t \mid a \in I, s, t \in S\}$.

PROOF. The set $\mathscr{C} := \{V_s^* \pi(a) V_t \mid a \in I, s, t \in S\}$ is closed under taking adjoints and is contained in $C^*(\pi(I)\, V(S))$. Since $\mathscr{C}$ contains the generators of $C^*(\pi(I) V(S))$ it is enough to show that it is closed under multiplication. Fix $V_s^* \pi(a) V_t$ and $V_z^* \pi(b) V_r$ in $\mathscr{C}$. Since S is right-reversible, there exist u, v in S such that $w := ut = vz \in S$. By covariance and (1.1) we have that:

$$\begin{aligned} V_s^* \pi(a) V_t V_z^* \pi(b) V_r &= V_s^* \pi(a) V_u^* V_u V_t V_z^* V_v^* V_v \pi(b) V_r \\ &= V_s^* V_u^* \pi(\alpha_u(a)) V_w V_w^* \pi(\alpha_v(b)) V_v V_r \\ &= V_s^* V_u^* \pi(\alpha_u(a)) \overline{\pi}(\overline{\alpha}_w(1_{M(I)})) \pi(\alpha_v(b)) V_v V_r \\ &= V_{us}^* \pi(\alpha_u(a) \overline{\alpha}_w(1_{M(I)}) \alpha_v(b)) V_{vr} \in \mathscr{C}. \end{aligned} \tag{1.2}$$

■

Proposition 1.4. *Let S be an Ore semigroup. If the dynamical system with extendible endomorphisms (I, S, α) admits a non-zero covariant representation, then there exists a crossed product for the system, and it is unique up to isomorphism.*

Proof. The proof follows the paths from [1] and [9]. Recall from [1] that a covariant representation (π, V) on a Hilbert space H is called cyclic if the C^*-algebra $C^*(\pi, V)$ generated by $\pi(I) \cup V(S)$ has a cyclic vector. Two such representations are equivalent if there is a unitary intertwining their images.

If (π, V) is a covariant representation of (I, S, α) on a Hilbert space H, then for each h in H the subspace $\overline{\operatorname{span}}\{V_s^* \pi(a) V_t h \mid a \in I, s, t \in S\}$ of H is invariant under $\pi(A)$, $V^*(S)$ and, by the assumption on S, also under $V(S)$. Hence it is invariant under $C^*(\pi, V)$, so restricting (π, V) to it gives a cyclic covariant representation. By performing a standard Zorn's lemma argument, the space H can be decomposed into a direct sum of subspaces on which (π, V) is cyclic. Correspondingly, (π, V) is equivalent to a direct sum of cyclic covariant representations.

Fix a set Λ of cyclic covariant representations of (I, S, α) such that every cyclic covariant representation of this system is equivalent to an element of Λ (such a set exists if we consider covariant representations on a fixed Hilbert space of sufficiently large dimension, depending on the cardinality of I). Let

$$\iota_I(a) = \bigoplus_{(\pi,V)\in\Lambda} \pi(a) \text{ and } \iota_S(s) = \bigoplus_{(\pi,V)\in\Lambda} V_s \text{ for } a \in I, s \in S.$$

If we let B denote $C^*(\iota_I(I)\,\iota_S(S))$, then B equals $\overline{\operatorname{span}}\{\iota_S(s)^* \iota_I(a)\,\iota_S(t) \mid a \in I, s, t \in S\}$ by Lemma 1.3. We claim that (B, ι_I, ι_S) is a crossed product for (I, S, α). To check the properness of ι_I it suffices to show that $\iota_I(u_\lambda)\, b \to b$ for an approximate unit (u_λ) in I and b of the form $\iota_S(s)^* \iota_I(a)\, \iota_S(t)$. This convergence follows by covariance and the extendibility of α_s for all s, since we sum up over non-degenerate representations of I. Verifying covariance of (ι_I, ι_S) is routine, hence (i) in Definition 1.1 holds, (ii) follows since every covariant representation is equivalent to a direct sum of cyclic covariant representations from Λ, and finally (iii) is included in the definition of B. If (B', ι_I', ι_S') is another triple satisfying (i)–(iii) from Definition 1.1, it is immediate that $\phi = \iota_I' \times \iota_S' : B \to B'$ given by (ii) is an isomorphism such that $\phi \circ \iota_I = \iota_I'$ and $\overline{\phi} \circ \iota_S = \iota_S'$. ■

The assumption that the semigroup dynamical system admits a non-zero covariant representation is usually trivially fulfilled. All known interesting examples have such a covariant representation (in practice there are lots of them), although there are (uninteresting) systems which do not admit any non-zero covariant representations. For injective endomorphic actions of Ore semigroups on unital C^*-algebras it is proved in [12] that there is always a non-zero covariant representation. The same holds in the non-unital case [22].

Remark 1.5. If the systems (I, S, α) and (J, S, τ) with extendible endomorphisms admit crossed products and if $\sigma : I \to J$ is an isomorphism such that $\sigma^{-1} \circ \tau_s \circ \sigma = \alpha_s$ for all s in S, then by uniqueness of the crossed product we have that $I \rtimes_\alpha S \cong J \rtimes_\tau S$.

Definition 1.6 ([1]). Suppose that α is an extendible endomorphism of a C^*-algebra A.

Given an ideal I of A, let $\psi: A \to M(I)$ denote the canonical non-degenerate homomorphism defined by $\psi(a)a' = a \cdot a', a \in A, a' \in I$. Then I is called extendibly α-invariant if it is α-invariant, in the sense that $\alpha(I) \subset I$, and for an approximate unit $(u_\lambda)_\lambda \subset I$ we have $\alpha(u_\lambda) \to \overline{\psi}(\overline{\alpha}(1_{M(A)}))$ strictly in $M(I)$.

As an example, consider $A = C(\mathbb{N} \cup \infty)$ with the forward shift $\alpha(f)(n) = f(n-1)$ if $n \geqslant 1$ and $\alpha(f)(0) = 0$. Take $I = c_0(\mathbb{N})$. If $(u_\lambda)_\lambda$ is an approximate unit for I, then $\alpha(u_\lambda)$ converges strictly to $(0, 1, 1, \ldots)$, which equals $\psi(\alpha(1, 1, \ldots))$. Hence I is extendibly α-invariant [1]. Another example is provided in the last section.

Given a semigroup dynamical system (A, S, α) and an ideal I of A, there is a system $(A/I, S, \alpha^I)$ with endomorphisms given by $\alpha^I_s(a+I) = \alpha_s(a) + I$ for all a in A and s in S. By [1], α^I_s is extendible if α_s is.

Theorem 1.7. *Let S be an Ore semigroup. If (A, S, α) is a system with extendible endomorphisms, assume that $(A \rtimes_\alpha S, \iota_A, j_S)$ is a crossed product and I is an extendibly α_s-invariant ideal of A for all s in S. Then there is a short exact sequence*

$$0 \to I \rtimes_\alpha S \xrightarrow{\phi} A \rtimes_\alpha S \xrightarrow{\psi} A/I \rtimes_{\alpha^I} S \to 0$$

where ϕ is an isomorphism of $I \rtimes_\alpha S$ onto the ideal $D := \overline{span}\{j_S(s)^* \iota_A(a) j_S(t) \mid a \in I, s, t \in S\}$ of $A \rtimes_\alpha S$.

If ι_I, ι_S denote the maps $I \to I \rtimes_\alpha S, S \to M(I \rtimes_\alpha S)$ and similarly $\iota_{A/I}, h_S$ denote $A/I \to A/I \rtimes_{\alpha^I} S, S \to M(A/I \rtimes_{\alpha^I} S)$, then

$$\phi \circ \iota_I = \iota_{A|I}, \ \overline{\phi} \circ \iota_S = j_S \text{ and } \psi \circ \iota_A = \iota_{A/I} \circ q^I, \ \overline{\psi} \circ j_S = h_S. \tag{1.3}$$

PROOF. By computations similar to (1.2) we see that D is an ideal of $A \rtimes_\alpha S$.

Let $\iota_I: I \to D$ be the composition of the canonical homomorphism $A \rtimes_\alpha S \to M(D)$ with $\iota_{A|I}$ and $\iota_S: S \to M(D)$ be the composition of $M(A \rtimes_\alpha S) \to M(D)$ with $j_S: S \to M(A \rtimes_\alpha S)$.

We claim that (D, ι_I, ι_S) is a crossed product for $(I, S, \alpha|_I)$. To get ι_I proper it is crucial that I is extendibly α_s-invariant. Indeed, if $(u_\lambda) \subset I$ is an approximate unit, ψ is as in Definition 1.6, $s \in S$ and $i \in I$, then $\iota_A(\alpha_s(u_\lambda)\, i)$ converges in norm to $\iota_A(\overline{\psi}(\overline{\alpha_s}(1_{M(A)}))\, i)$, which equals $\overline{\iota_A}(\overline{\alpha_s}(1_{M(A)}))\, \iota_A(i)$. By (1.1) for the pair (ι_A, j_S) this limit is $j_S(s) j_S(s)^* \iota_A(i)$. Hence the product $\iota_I(u_\lambda) j_S(s)^* \iota_A(i) j_S(t)$, which by covariance is $j_S(s)^* \iota_A(\alpha_s(u_\lambda)\, i) j_S(t)$, converges in norm to $j_S(s)^* \iota_A(i) j_S(t)$. Since elements of the form $j_S(s)^* \iota_A(i) j_S(t)$ span D, it follows that $\iota_I(u_\lambda) \to 1_{M(D)}$ strictly.

It is routine to verify the covariance of (ι_I, ι_S). *If* (π, V) is a covariant representation of (I, S, α), let π' be the composition of $\overline{\pi}: M(I) \to \mathbb{B}(H_\pi)$ with $A \to M(I)$. Since I is extendibly invariant, (π', V) is a covariant representation of (A, S, α). Restricting $\pi' \times V$ to D gives a non-degenerate representation, which is readily seen to play the role of $\pi \times V$ from Definition 1.1 (ii). Condition (iii) is immediate. By Proposition 1.4 there is an isomorphism $\phi: I \rtimes_\alpha S \to D$ satisfying the first half of (1.3).

Pick a representation ρ of $A \rtimes_\alpha S$ with kernel D, and notice that $\ker(\rho \circ \iota_A) = I$. Hence there is a non-degenerate representation η of A/I which forms a non-zero covariant representation of $(A/I, S, \alpha^I)$ together with $\overline{\rho} \circ j_S$. Therefore there is a crossed product for $(A/I, S, \alpha^I)$. The pair $(\iota_{A/I} \circ q^I, h_S)$ satisfies covariance, hence by representing $A/I\rtimes_{\alpha^I}S$ non-degenerately on a Hilbert space we get a non-degenerate homomorphism $\psi: A\rtimes_{\alpha}S\to M(A/I\rtimes_{\alpha^I}S)$ such that $\psi\circ\iota_A=\iota_{A/I}\circ q^I$ and $\overline{\psi}\circ j_S=h_S$. The representation $\eta\times(\overline{\rho}\circ j_S)$ of $A/I\rtimes_{\alpha^I}S$ lifts to ρ and we get the non-trivial inclusion $\ker(\psi)\subset D$. ■

Remark 1.8. If the ideal is not extendibly invariant with respect to the action, one may not get a short exact sequence of crossed products, as illustrated below.

Let $A=C_0(\mathbb{Z})$. The action by automorphisms determined by the forward shift $\alpha_1(f)(n)=f(n-1), f\in A, n\in\mathbb{Z}$, restricts to an action by extendible endomorphisms on the ideal $I=\{f\in C_0(\mathbb{Z})\,|\,f(n)=0 \text{ for } n\leqslant 0\}\cong C_0(\mathbb{N})$. It was shown in [1] that I is not extendibly α_n-invariant for any $n\in\mathbb{N}$, although it is invariant (for any approximate unit $(u_\lambda)_\lambda$ of I, $\alpha_1(u_\lambda)\to(0,1,1,\ldots)$ strictly in $M(I)$, whereas $\overline{\psi}(\overline{\alpha_1}(1_{M(A)}))=\overline{\psi}(1_{M(A)})=(1,1,\ldots)$ in $M(I)$). The system $(I,\mathbb{N},\alpha)$ admits the covariant representation (π,V) on $l^2(\mathbb{N})$, where π is given by multiplication operators and V is the Toeplitz representation defined on the canonical basis $\{\epsilon_k\,|\,k\in\mathbb{N}\}\subset l^2(\mathbb{N})$ by $T_n(\epsilon_k)=\epsilon_{k+n}$. If $I\rtimes\mathbb{N}$ embeds as an ideal of $A\rtimes\mathbb{N}(\cong A\rtimes\mathbb{Z})$, then the non-degenerate representation $\pi\times V$ of $I\rtimes\mathbb{N}$ extends to a representation χ of $A\rtimes\mathbb{Z}$. This in turn corresponds to a covariant representation (ρ,W) of $(A,\mathbb{Z},\alpha)$ where W is a unitary representation, because covariance implies $W_kW_k^*=\rho(\alpha_k(1))=1$ for all $k\in\mathbb{Z}$. Since χ extends $\pi\times V$, its restriction to $\mathbb{N}$ is V by Definition 1.1 (ii). This is a contradiction because V acts by proper isometries.

2. Tensor products of non-unital crossed products

For semigroup dynamical systems (A,S,α) and (B,T,β) in which S,T are Ore semigroups, we aim to prove an isomorphism of the form

$$(A\otimes_{\max}B)\rtimes_{\alpha\otimes\beta}(S\times T)\cong(A\rtimes_{\alpha}S)\otimes_{\max}(B\rtimes_{\beta}T). \tag{2.1}$$

The maximal tensor product interacts well with crossed products because it too is characterised by a universal property: if $\phi: C\to L$ and $\psi: D\to L$ are homomorphisms of C^*-algebras with commuting ranges, then there is a unique homomorphism $\phi\otimes_{\max}\psi: C\otimes_{\max}D\to L$ such that $\phi\otimes_{\max}\psi(c\otimes d)=\phi(c)\psi(d)$ when $c\in C$ and $d\in D$ (see e.g. [24]). Thus (2.1) says, loosely speaking, that pairs of commuting covariant representations of (A,S,α) and (B,T,β) correspond to covariant representations of $(A\otimes_{\max}B)\rtimes_{\alpha\otimes\beta}(S\times T)$. It turns out that to get from a commuting pair of covariant representations to a commuting pair of representations of the crossed product C^*-algebras we need to know that the isometric components are $*$-commuting. To satisfy this condition we restrict attention to quasi-lattice ordered groups and covariant isometric representations as introduced by Nica in [20].

Recall that by [20] a partially ordered group (G,S) consisting of a group G and a subsemigroup S such that $S\cap S^{-1}=\{e\}$ is called *quasi-lattice ordered* if every finite subset of G with an upper bound in S has a least upper bound in S. If $s,t\in G$, let $s\vee t$ denote the least upper bound when s and t have a common upper bound in S. A representation W of S by isometries on a Hilbert space is *Nica-covariant* if

$$W_sW_s^*W_tW_t^*=\begin{cases}W_{s\vee t}W_{s\vee t}^* & \text{if } s\vee t \text{ exists}\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Then, by [14],

$$W_s^* W_t = W_{s^{-1}(s \vee t)} W^*_{t^{-1}(s \vee t)}. \tag{2.2}$$

Remark 2.1. If (G, S) is a quasi-lattice ordered group and (I, S, α) is a dynamical system with extendible endomorphisms such that

$$\overline{\alpha}_s(1_{M(I)})\overline{\alpha}_t(1_{M(I)}) = \begin{cases} \overline{\alpha}_{s \vee t}(1_{M(I)}) & \text{if } s \vee t \text{ exists} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

for all $s, t \in S$, then, given a covariant representation (π, V), it follows from (1.1) that V is Nica-covariant.

Remark 2.2. Given C^*-algebras A and B, there exist non-degenerate homomorphisms

$$i_A: A \to M(A \otimes_{\max} B) \text{ and } i_B: B \to M(A \otimes_{\max} B)$$

with commuting ranges such that $i_A(a)\, i_B(b) = a \otimes b$ if $a \in A$ and $b \in B$ (see [24, theorem B. 27]). Hence we have $\overline{i_A} \otimes_{\max} \overline{i_B}: M(A) \otimes_{\max} M(B) \to M(A \otimes_{\max} B)$, which is the identity on $A \otimes_{\max} B$.

Lemma 2.3. *If α in End(A) and β in End(B) are extendible, then there is an extendible endomorphism $\alpha \otimes \beta$ of $A \otimes_{max} B$ whose extension $\overline{\alpha \otimes \beta}$ to $M(A \otimes_{max} B)$ satisfies*

$$\overline{\alpha \otimes \beta} \circ (\overline{i_A} \otimes_{max} \overline{i_B}) = (\overline{i_A} \circ \overline{\alpha}) \otimes_{max} (\overline{i_B} \circ \overline{\beta}). \tag{2.4}$$

Notice that this lemma implies that if (A, S, α) and (B, T, β) are semigroup dynamical systems with extendible endomorphisms, then there is an action $\alpha \otimes \beta$ of $A \otimes_{\max} B$ by extendible endomorphisms.

PROOF. By [24, lemma B. 31], there is an endomorphism $\alpha \otimes \beta$ of $A \otimes_{\max} B$ such that $(\alpha \otimes \beta)(a \otimes b) = \alpha(a) \otimes \beta(b)$ for a in A, b in B. If $(u_\lambda)_\lambda$ and $(v_\mu)_\mu$ are approximate units for A and B, then by extendibility of α and β it follows that $(\alpha \otimes \beta)(u_\lambda \otimes v_\mu)$ converges strictly to $\overline{i_A}(\overline{\alpha}(1_{M(A)}))\,\overline{i_B}(\overline{\beta}(1_{M(B)}))$, which is a projection because $\overline{i_A}$ and $\overline{i_B}$ have commuting ranges. Since $(u_\lambda \otimes v_\mu)_{\lambda,\mu}$ is an approximate unit for $A \otimes_{\max} B$, $\alpha \otimes \beta$ is extendible. The equality (2.4) follows by extendibility of α and β and the defining properties of i_A and i_B. ■

A straightforward argument shows the following.

Lemma 2.4. *Given non-degenerate representations $\pi_1: A \to \mathbb{B}(H)$, $\pi_2: B \to \mathbb{B}(H)$ with commuting ranges, we have $\overline{\pi_1 \otimes_{max} \pi_2} \cdot (\overline{i_A} \otimes_{max} \overline{i_B}) = \overline{\pi_1} \otimes_{max} \overline{\pi_2}$.*

Remark 2.5. If S is an Ore semigroup with enveloping group G such that $S \cap S^{-1} = \{e\}$ and if any pair of elements in S have a least upper bound in S, then the pair (G, S) is quasi-lattice ordered by [5, theorem 2.3]. Let T be another Ore semigroup such that $H = T^{-1}T, T \cap T^{-1} = \{e'\}$, the unit of H, and any pair of elements of T have a least upper bound in T. Then $(G \times H, S \times T)$ with the product order, namely $(s_1, t_1) \leqslant (s_2, t_2)$ if and only if $s_i \leqslant t_i, i = 1,2$, is again a quasi-lattice ordered group.

Theorem 2.6. *Let (G, S) and (H, T) be pairs of groups and generating semigroups as in Remark 2.5. Suppose that the dynamical systems with extendible endomorphisms (A, S, α) and (B, T, β) admit a pair of commuting covariant representations. If the actions α and β satisfy (2.3), then*

$$(A \otimes_{max} B) \rtimes_{\alpha\otimes\beta} (S \times T) \cong (A \rtimes_{\alpha} S) \otimes_{max} (B \rtimes_{\beta} T).$$

Proof. Let $(A \rtimes S, \iota_A, \iota_S)$ and $(B \rtimes T, \iota_B, \iota_T)$ denote the crossed products that exist by Proposition 1.4. If (π_1, V_1), (π_2, V_2) are a pair of covariant representations of (A, S, α) and (B, T, β) with commuting ranges, then by setting $(V_1 \otimes V_2)(s, t) = V_1(s) V_2(t)$ for $s \in S, t \in T$, it follows that $(\pi_1 \otimes_{\max} \pi_2, V_1 \otimes V_2)$ is a covariant representation of $(A \otimes_{\max} B, S \times T, \alpha \otimes \beta)$. Hence there exists a crossed product $((A \otimes_{\max} B) \rtimes_{\alpha\otimes\beta}(S \times T), \iota_{A\otimes B}, \iota_{S\times T})$.

Using Remark 2.2, define $\iota'_{S\times T}\colon S \times T \to M((A \rtimes S) \otimes_{\max} (B \rtimes T))$ by the formula $\iota'_{S\times T}(s, t) = \overline{i_{A\rtimes S}}(\iota_S(s))\, \overline{i_{B\rtimes T}}(\iota_T(t))$. Setting $\iota'_{A\otimes B} := \iota_A \otimes_{\max} \iota_B$ gives a proper homomorphism $A \otimes_{\max} B \to (A \rtimes S) \otimes_{\max} (B \rtimes T)$. If we can show that $((A \rtimes S) \otimes_{\max} (B \rtimes T), \iota'_{A\otimes B}, \iota'_{S\times T})$ is another crossed product for the system $(A \otimes_{\max} B, S \times T, \alpha \otimes \beta)$, then the theorem follows by uniqueness.

It is easily verified that $(\iota'_{A\otimes B}, \iota'_{S\times T})$ satisfies covariance, hence Definition 1.1 (i) holds. To verify (ii), let (π, U) be a covariant representation of $(A \otimes_{\max} B, S \times T, \alpha \otimes \beta)$ on a Hilbert space H. We must show that there is a non-degenerate representation $\pi \times U$ of $(A \rtimes S) \otimes_{\max} (B \rtimes T)$ on H such that

$$\pi \times U \circ \iota'_{A\otimes B} = \pi \text{ and } \overline{\pi \times U} \circ \iota'_{S\times T} = U. \tag{2.5}$$

By setting $\pi_1 := \overline{\pi} \circ i_A$ and $\pi_2 := \overline{\pi} \circ i_B$, we get non-degenerate representations of A and B. The formulas $U_1(s) = U(s, e')$ and $U_2(t) = U(e, t)$ give isometric representations of S and T. Moreover, (π_1, U_1) and (π_2, U_2) are covariant representations of (A, S, α) and (B, T, β). We aim to show that $\pi_1 \times U_1(A \rtimes S)$ and $\pi_2 \times U_2(B \rtimes T)$ commute in $\mathbb{B}(H)$. Then, by setting $\pi \times U := (\pi_1 \times U_1) \otimes_{\max} (\pi_2 \times U_2)$, we obtain a representation of $(A \rtimes S) \otimes_{\max} (B \rtimes T)$ on H, which is non-degenerate because both $\pi_1 \times U_1$ and $\pi_2 \times U_2$ are.

The representations π_1 and π_2 commute because i_A, i_B do, and U_1 and U_2 are easily seen to commute. By (2.4), $\alpha \otimes \beta$ satisfies (2.3), since $\overline{\alpha}$ and $\overline{\beta}$ do by hypothesis. Hence U is Nica-covariant, and this implies that U_1 and U_2 are also $*$-commuting: indeed, by (2.2) we have

$$\begin{aligned} U_1(s)^* U_2(t) &= U(s, e')^* U(e, t) \\ &= U((s, e')^{-1}((s, e') \vee (e, t)))\, U((e, t)^{-1}((s, e) \vee (e, t)))^* \\ &= U((s^{-1}, e')(s, t))\, U((e, t^{-1})(s, t))^* = U(e, t)\, U(s, e')^* \\ &= U_2(t)\, U_1(s)^*. \end{aligned}$$

By Remark 1.2 in the second equality, (2.4) in the fourth equality and covariance for appropriate pairs we have that $U_1(S)$ and $\pi_2(B)$ commute, as follows:

$$\begin{aligned} U_1(s)\,\pi_2(b) &= U(s,e')\,\overline{\pi}\circ i_B(b) = \overline{\pi}(\overline{\alpha\otimes\beta}_{(s,e')}(i_B(b)))\,U(s,e') \\ &= \overline{\pi}(\overline{\alpha\otimes\beta}_{(s,e')}(\overline{i_A}(1_{M(A)})\,i_B(b)))\,U(s,e') \\ &= \overline{\pi}(\overline{i_A}(\overline{\alpha}_s(1_{M(A)}))\,i_B(\beta_{e'}(b)))\,U(s,e') \\ &= \overline{\pi}(i_B(b))\,\overline{\pi}\circ\overline{i_A}(\overline{\alpha}_s(1_{M(A)}))\,U_1(s) \\ &= \overline{\pi}(i_B(b))\,\overline{\pi}_1(\overline{\alpha}_s(1_{M(A)}))\,U_1(s) \\ &= \pi_2(b)\,U_1(s)\,\overline{\pi}_1(1_{M(A)}) \\ &= \pi_2(b)\,U_1(s). \end{aligned}$$

Similar computations apply to U_2 and π_1, hence we obtain a non-degenerate representation $\pi\times U$ of $(A\rtimes S)\otimes_{\max}(B\rtimes T)$ on H. The equations in (2.5) follow from Lemma 2.4 and the defining properties of $\pi_i\times U_i$, $i=1,2$.

Finally, since $A\rtimes S = C^*(\iota_A(A)\cdot\iota_S(S))$ and $B\rtimes T = C^*(\iota_B(B)\cdot\iota_T(T))$, we have that $(A\rtimes S)\otimes_{\max}(B\rtimes T) = C^*(\iota'_{A\otimes B}(A\otimes_{\max}B), \iota'_{S\times T}(S\times T))$, hence (iii) in Definition 1.1 is fulfilled. ■

3. Ideals in tensor products

In this section we describe a composition series of ideals in a tensor product of semigroup crossed products by an Ore semigroup S. We use the following notation: if (C,S,α) and (D,S,β) are semigroup dynamical systems and $\phi\colon C\to D$ is a homomorphism such that $\phi\circ\alpha=\beta\circ\phi$, then the induced homomorphism $C\rtimes_\alpha S\to D\rtimes_\beta S$ will be denoted $\phi\rtimes S$. All tensor products are maximal and in order to avoid heavy notation we write only $\otimes$.

Theorem 3.1. *Given C^*-algebras A and B and ideals I of A and J of B, assume that we have actions γ and δ of an Ore semigroup S by extendible endomorphisms of A and B such that I is extendibly γ_s-invariant and J is extendibly δ_s-invariant for all s in S. Consider $\gamma\otimes\delta\colon S\to End(A\otimes B)$ and the actions $\gamma^I\colon S\to End(A/I)$ and $\delta^J\colon S\to End(B/J)$ induced by γ and δ.*

If (A,S,α) and (B,S,β) admit covariant pairs then there is a composition series

$$0\trianglelefteq I_1\trianglelefteq I_2\trianglelefteq (A\otimes B)\rtimes_{\gamma\otimes\delta}S \tag{3.1}$$

such that:

(i) *the embedding $I\otimes J\subseteq A\otimes B$ induces an isomorphism of $I\otimes J\rtimes_{\gamma\otimes\delta}S$ onto I_1;*

(ii) *$I_2/I_1\cong(A/I\otimes J)\rtimes_{\gamma^I\otimes\delta}S\oplus(I\otimes B/J)\rtimes_{\gamma\otimes\delta^J}S$;*

(iii) *if q denotes the map $q^I\otimes q^J\colon A\otimes B\to(A/I)\otimes(B/J)$, then $q\rtimes S$ induces an isomorphism of $(A\otimes B\rtimes_{\gamma\otimes\delta}S)/I_2$ onto $(A/I\otimes B/J)\rtimes_{\gamma^I\otimes\delta^J}S$.*

The proof will rely on a couple of lemmas.

Lemma 3.2. *Given C^*-algebras C and D, suppose that $\alpha\in End(C)$, $\beta\in End(D)$, and suppose that K is an extendibly α-invariant ideal of C and L is an extendibly β-invariant ideal of D. Then $K\otimes L$ is extendibly $\alpha\otimes\beta$-invariant in $C\otimes D$.*

PROOF. Composing the canonical homomorphisms $C \to M(K), D \to M(L)$ with $\overline{i_K}\colon M(K) \to M(K \otimes L)$ and $\overline{i_L}\colon M(L) \to M(K \otimes L)$ given as in Remark 2.2 gives rise to homomorphisms $C \to M(K \otimes L)$ and $D \to M(K \otimes L)$ with commuting ranges. Direct computations show that the induced homomorphism $\eta\colon C \otimes D \to M(K \otimes L)$ is the canonical one. If $(u_\lambda)_\lambda \subset K$ and $(v_\mu)_\mu \subset L$ are approximate units, then it follows from the assumption on K and L that $i_K(\alpha(u_\lambda))\, i_L(\beta(v_\mu)) \to \overline{\eta}(\overline{\alpha \otimes \beta}(1_{M(C \otimes D)}))$ strictly in $M(K \otimes L)$. But $i_K(\alpha(u_\lambda))\, i_L(\beta(v_\mu))$ equals $\alpha \otimes \beta(u_\lambda \otimes v_\mu)$, hence the conclusion follows. ■

Lemma 3.3. *With the assumptions from Theorem 3.1 we have*

$$\ker(q \rtimes S) \cong (A \otimes J) \rtimes_{\gamma \otimes \delta} S + (I \otimes B) \rtimes_{\gamma \otimes \delta} S.$$

PROOF. Since short exact sequences are preserved under the maximal tensor product with an arbitrary C^*-algebra [24, proposition B. 30], we obtain by successive applications of Lemma 3.2 and Theorem 1.7 the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (I \otimes J) \rtimes S \arrow[r] \arrow[d] & (I \otimes B) \rtimes S \arrow[r] \arrow[d] & (I \otimes B/J) \rtimes S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (A \otimes J) \rtimes S \arrow[r] \arrow[d] & (A \otimes B) \rtimes S \arrow[r] \arrow[d] \arrow[dr, "q \rtimes S"] & (A \otimes B/J) \rtimes S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (A/I \otimes J) \rtimes S \arrow[r] \arrow[d] & (A/I \otimes B) \rtimes S \arrow[r] \arrow[d] & (A/I \otimes B/J) \rtimes S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By a diagram chase, any element f of $(A \otimes B) \rtimes S$ such that $q \rtimes S(f) = 0$ is seen to be the sum of an element from $(I \otimes B) \rtimes_{\gamma \otimes \delta} S$ and an element from $(A \otimes J) \rtimes_{\gamma \otimes \delta} S$. ■

PROOF OF THEOREM 3.1. As in the proof of Lemma 3.3, $I \otimes J$ is extendibly $(\gamma \otimes \delta)_s$-invariant in $A \otimes B$ by Lemma 3.2; hence, by Theorem 1.7, $(I \otimes J) \rtimes S$ is isomorphic to an ideal of $A \otimes B \rtimes S$, which we denote I_1. This gives (i).

Set $I_2 := (A \otimes J) \rtimes_{\gamma \otimes \delta} S + (I \otimes B) \rtimes_{\gamma \otimes \delta} S$. This is an ideal of $A \otimes B \rtimes S$. By Remark 1.5 for the systems $((A \otimes J)/(I \otimes J), S, (\gamma \otimes \delta)^{I \otimes J}), (A/I \otimes J, S, \gamma^I \otimes \delta)$ and $((I \otimes B)/(I \otimes J), S, (\gamma \otimes \delta)^{I \otimes J}), (I \otimes B/J, S, \gamma \otimes \delta^J)$ respectively, we obtain (ii) as follows:

$$\begin{aligned}
I_2/I_1 &= ((A \otimes J) \rtimes_{\gamma \otimes \delta} S + (I \otimes B) \rtimes_{\gamma \otimes \delta} S)/(I \otimes J) \rtimes_{\gamma \otimes \delta} S \\
&= ((A \otimes J) \rtimes_{\gamma \otimes \delta} S)/((I \otimes J) \rtimes_{\gamma \otimes \delta} S) \oplus ((I \otimes B) \rtimes_{\gamma \otimes \delta} S)/((I \otimes J) \rtimes_{\gamma \otimes \delta} S) \\
&\cong (A \otimes J)/(I \otimes J) \rtimes_{\gamma \otimes \delta^{I \otimes J}} S \oplus (I \otimes B)/(I \otimes J) \rtimes_{\gamma \otimes \delta^{I \otimes J}} S \\
&\cong (A/I \otimes J) \rtimes_{\gamma^I \otimes \delta} S \oplus (I \otimes B/J) \rtimes_{\gamma \otimes \delta^J} S.
\end{aligned}$$

Finally, (iii) follows from Lemma 3.3. ■

4. Extendibly invariant ideals of abelian C^*-algebras

For a class of endomorphisms of an abelian C^*-algebra we describe the extendibly invariant ideals, thus providing a setting in which Theorem 1.7 can be applied. Examples of such algebras and endomorphisms are studied in [15; 4; 5; 17].

Given a compact Hausdorff space X, assume that $\theta: X \to X$ is continuous and injective, and that $\theta(X)$ is open in X. Notice that this implies that $\theta(X)$ must be clopen in X.

Proposition 4.1. *The formula*

$$\alpha(f)(x) = \begin{cases} f \circ \theta^{-1}(x), & \text{if } x \in \theta(X) \\ 0, & x \in X \backslash \theta(X) \end{cases} \tag{4.1}$$

defines an endomorphism of $C(X)$.

PROOF. We need only note that $\alpha(f) \in C(X)$ for $f \in C(X)$ because $\theta(X)$ is open. ■

Any ideal I of $C(X)$ has the form $C_0(Y)$ for some open subset Y of X. We determine when I is extendibly α-invariant according to Definition 1.6.

Remark 4.2. If $\theta(Y) \subset Y$, then I is α-invariant. Indeed, pick f in $C_0(Y)$ and x in $X \setminus Y$. In particular, $x \in X \backslash \theta(Y)$. If $x \in \theta(X) \backslash \theta(Y)$, then $\theta^{-1}(x) \notin \theta^{-1}(\theta(Y)) = Y$ by injectivity of θ. Hence $\alpha(f)(x) = f(\theta^{-1}(x)) = 0$. If $x \notin \theta(X)$ then $\alpha(f)(x) = 0$ by the definition of α.

Theorem 4.3. *If α is an endomorphism of $C(X)$ given by* (4.1) *and $C_0(Y)$ is an ideal of $C(X)$, then the following conditions are equivalent:*

(i) *$C_0(Y)$ is extendibly α-invariant;*
(ii) *$\theta(Y) = \theta(X) \cap Y$;*
(iii) *$\theta(Y) \subset Y$ and $\theta(X \setminus Y) \subset X \setminus Y$.*

PROOF. The map $\psi: C(X) \to M(C_0(Y)) = C_b(Y)$ from Definition 1.6 is simply the restriction to Y. We note that $I = C_0(Y)$ being extendibly α-invariant is equivalent to the fact that I is α-invariant, $\alpha|_I$ is extendible and $\overline{\alpha|}_I(1_{M(I)}) = \overline{\psi}(\overline{\alpha}(1_{C(X)}))$. We readily compute that $\overline{\psi}(\overline{\alpha}(1_{C(X)})) = \psi(\alpha(1_{C(X)})) = \chi_{\theta(X)|Y}$, where $\chi_{\theta(X)}$ denotes the characteristic function of $\theta(X)$.

Let $(f_\lambda)_\lambda$ be an approximate unit for I and assume that (ii) holds. By Remark 4.2, I is α-invariant. We claim that $\alpha(f_\lambda)$ converges strictly to $\chi_{\theta(Y)}$ in $M(C_0(Y))$. Recall that strict convergence on bounded subsets of $M(C_0(Y))$ is the same as uniform convergence on compact subsets of Y. If K is compact in Y, then by the assumption (ii) it follows that $K \cap \theta(Y) = K \cap \theta(X)$, which is compact in X. Since θ is injective, $K' := \theta^{-1}(K) \cap Y$ is compact in X, and $\theta(K')$ is compact in $\theta(Y)$. But $\theta: Y \to Y$ is a homeomorphism onto its image, hence K' is compact in Y. The claim follows now because $\alpha(f_\lambda)|_{K \cap \theta(Y)} = f_\lambda|_{\theta^{-1}(K) \cap Y} = f_\lambda|_{K'}$ and $\alpha(f_\lambda)|_{K \backslash \theta(Y)} = 0 = \chi_{\theta(Y)}|_{K \backslash \theta(Y)}$. Therefore $\alpha|_I$ is extendible and $\overline{\alpha|_I}(1_{M(I)}) = \chi_{\theta(Y)}$. Hence (ii) implies (i). Conversely, we have that $\alpha(f_\lambda) \to \chi_{\theta(X)}|_Y$ strictly in $M(C_0(Y))$. But $\alpha(f_\lambda)$ converges pointwise to $\chi_{\theta(Y)}$, thus (ii) follows.

Because θ is injective, it is straightforward that (ii) and (iii) are equivalent. ■

Remark 4.4. We have shown in [17] that there is an action α of the Ore semigroup $\mathbb{N}^k, k \in \mathbb{N} \cup \infty$, by endomorphisms of $C^*(\Gamma)$ for an abelian discrete group Γ; namely, if $\beta\colon \mathbb{N}^k \to \mathrm{End}(\Gamma)$ is an action by surjective endomorphisms, than α is characterised on the canonical unitaries $\delta_r \in C^*(\Gamma)$ by

$$\alpha_m(\delta_r) = \frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{\{s \mid \beta_m(s) = r\}} \delta_s, \text{ for } r \in \Gamma, m \in \mathbb{N}^k. \tag{4.2}$$

If $\hat{}\colon C^*(\Gamma) \to C(\hat{\Gamma})$ is the Fourier transform, we can regard $\alpha\colon \mathbb{N}^k \to \mathrm{End}(C(\hat{\Gamma}))$ as $\alpha_m(\hat{f}) = \widehat{\alpha_m(f)}$ for $f \in C^*(\Gamma)$. Let $\hat{\beta}$ be the action of $\mathbb{N}^k$ by injective endomorphisms of $\hat{\Gamma}$ given by $\hat{\beta}_m(\gamma) = \gamma \circ \beta_m$, for all $m \in \mathbb{N}^k$. We now show that the action $\mathbb{N}^k \to \mathrm{End}(C(\hat{\Gamma}))$ is by endomorphisms of the form (4.1).

Proposition 4.5. *If $\hat{f} \in C(\hat{\Gamma})$, $\gamma \in \hat{\Gamma}$ and m in $\mathbb{N}^k$, then we have*

$$\alpha_m(\hat{f})(\gamma) = \begin{cases} \hat{f}(\hat{\beta}_m^{-1}(\gamma)) & \text{if } \gamma \in \hat{\beta}_m(\hat{\Gamma}), \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. We claim that $\alpha_m(\hat{f})(\hat{\beta}_m(\chi)) = \hat{f}(\chi)$ if $\chi \in \hat{\Gamma}$ and $\alpha_m(\hat{f})(\gamma) = 0$ if $\gamma \notin \hat{\beta}_m(\hat{\Gamma})$. Then (4.3) follows.

Since $\{\delta_r \mid r \in \Gamma\}$ spans a dense subspace of $C^*(\Gamma)$, it suffices to pick $f = \delta_r$ for a fixed r. For $\chi \in \hat{\Gamma}$ we obtain the first half of the claim as follows:

$$\begin{aligned} \alpha_m(\hat{\delta}_r)(\hat{\beta}_m(\chi)) &= \left(\frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{\{s \mid \beta_m(s) = r\}} \delta_s\right)^{\wedge} (\hat{\beta}_m(\chi)) \\ &= \frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{\{s \mid \beta_m(s) = r\}} \hat{\beta}_m(\chi)(s) \\ &= \frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{\{s \mid \beta_m(s) = r\}} \chi(\beta_m(s)) \\ &= \chi(r) = \hat{\delta}_r(\chi). \end{aligned}$$

Assume that $\gamma \notin \hat{\beta}_m(\hat{\Gamma})$. Since the solutions of the equation $\beta_m(s) = r$ are the elements of the form $s_0 + s'$ for s_0 a particular solution and $s' \in \ker(\beta_m)$, we have

$$\begin{aligned} \alpha_m(\hat{\delta}_r)(\gamma) &= \frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{\{s \mid \beta_m(s) = r\}} \gamma(s) \\ &= \frac{1}{|\{s \mid \beta_m(s) = r\}|} \sum_{s' \in \ker(\beta_m)} \gamma(s_0)\,\gamma(s') \\ &= \gamma(s_0)\left(\frac{1}{|\ker(\beta_m)|} \sum_{s' \in \ker(\beta_m)} \gamma(s')\right). \end{aligned} \tag{4.4}$$

The annihilator of $\ker(\beta_m)$ in $\hat{\Gamma}$ is isomorphic to $(\Gamma/\ker(\beta_m))\hat{}$, hence to $\beta_m(\Gamma)\hat{}$. Since $\beta_m(\Gamma)\hat{} = \{\chi \circ \beta_m \mid \chi \in \hat{\Gamma}\} = \{\hat{\beta}_m(\chi) \mid \chi \in \hat{\Gamma}\} = \hat{\beta}_m(\hat{\Gamma})$, we see that γ is not identically 1 on $\ker(\beta_m)$. Hence $\gamma(\ker(\beta_m))$ is a non-trivial subgroup of $\mathbb{T}$, and therefore the sum in (4.4) is zero. ■

Example 4.6. Let p be a prime number and let G_p denote the group $\{(r/p^k) + \mathbb{Z} \mid r, k \in \mathbb{Z}, k \geqslant 0\}$. There is an action $\beta\colon \mathbb{N} \to \mathrm{End}(G_p)$ such that β_m is multiplication by p^m modulo $\mathbb{Z}$. Since G_p is the inductive limit $\overset{\lim}{\rightarrow} (1/p^k)\,\mathbb{Z}/\mathbb{Z}$ with inclusion maps, $\hat{G}_p$ will be isomorphic to the inverse limit of $((1/p^k)\,\mathbb{Z}/\mathbb{Z})\hat{}$. This turns out to equal the additive group $\mathscr{Z}_p$ of p-adic integers, which are formal sums $x = \Sigma_{k=0}^{\infty} x_k p^k$ with $0 \leqslant x_k < p$ for all k. If $x \in \mathscr{Z}_p$, then $\hat{\beta}_m(x)$ is obtained by multiplying p^m on x. Consider the augmentation homomorphism $\varepsilon\colon C^*(G_p) \to \mathbb{C}$ defined on the generating unitaries $\{\delta_g \mid g \in G_p\}$ by $\delta_g \mapsto 1$. Since $\varepsilon(\delta_g)$ is evaluation of the trivial character in $\widehat{G_p}$ at g from G_p, it follows that $\varepsilon(f) = \hat{f}(0)$ for $f \in C^*(G_p)$, thus $\ker(\varepsilon) \cong C_0(\mathscr{Z}_p \backslash \{0\})$. The action α on $C^*(G_p)$ given by (4.2) (see also [15]) is of the form described in Proposition 4.5. Hence $\ker(\varepsilon)$ is extendibly α-invariant by Theorem 4.3 (iii).

In [18] we will study an action of $\mathbb{N}^2$ on a group $G_{p,q}$ similar to G_p but involving two primes p and q. We will use the results of this paper to write down a composition series of ideals for a semigroup crossed product $C^*(G_{p,q}) \rtimes_{\alpha} \mathbb{N}^2$ in which we can identify the subquotients in terms of interesting number-theoretic dynamical systems.

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