

A BLOCH THEOREM FOR ALGEBROID MULTIFUNCTIONS

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ABSTRACT

We prove an analogue of Bloch's theorem for multifunctions of the form

$$F(z) = \{w \in \mathbf{C} : w^n + a_1(z)w^{n-1} + \cdots + a_{n-1}(z)w + a_n(z) = 0\},$$

where a_1, \dots, a_n are holomorphic functions. We give an example to show that the corresponding 'Landau constant' is no longer a constant, but a function of the diameter of $F(0)$.

1. Introduction and statement of results

A classical theorem of Bloch [1] asserts the existence of a constant $L > 0$ such that the image of each function f , holomorphic on a neighbourhood of the closed unit disc and satisfying $|f'(0)| = 1$, contains a disc of radius at least L . The optimal value of L is usually called Landau's constant: it is known to lie between 0.5 and 0.544. For an elementary proof of Bloch's theorem, see for example [2, chapter XII], where it is also shown how to deduce Picard's theorem for entire functions. Our aim in this paper is to extend Bloch's theorem to algebroid multifunctions.

Let D be a (plane) domain. An *algebroid multifunction of degree n* on D is a set-valued function F of the form

$$F(z) = \{w \in \mathbf{C} : w^n + a_1(z)w^{n-1} + \cdots + a_{n-1}(z)w + a_n(z) = 0\} \quad (z \in D), \quad (1)$$

where a_1, \dots, a_n are holomorphic functions on D . Its *image* $F(D)$ is defined as

$$F(D) = \bigcup_{z \in D} F(z).$$

The challenge is to estimate the *inradius* of $F(D)$, namely the supremum of the radii of all discs in $F(D)$.

In seeking to extend Bloch's theorem to this situation, we are faced with the problem of choosing an appropriate analogue of the normalisation condition $|f'(0)| = 1$. We shall work with a very general condition which seems wide enough to encompass all the likely possibilities. Given a family \mathcal{F} of algebroid multifunctions of degree n on a domain D , we say that \mathcal{F} is *normalised* if, whenever (F_k) is a sequence in \mathcal{F} which converges locally uniformly to another algebroid multifunction F , then F is non-constant.

Here, by *converges locally uniformly* we mean that $\rho(F_k(z), F(z)) \rightarrow 0$ locally uniformly on D , where ρ denotes the so-called matching distance for n -tuples:

$$\rho(\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}) = \min_{\sigma \in S_n} \max_{1 \leq j \leq n} |\alpha_j - \beta_{\sigma(j)}|.$$

This is equivalent to demanding that the n -tuples of coefficients $(a_1(z), \dots, a_n(z))$ corresponding to each F_k converge locally uniformly to the n -tuple corresponding to F .

The condition that \mathcal{F} be normalised, on its own, is insufficient, however, even for ordinary holomorphic functions. For example, the sequence $f_n(z) = n + z/n$ is a normalised family on the unit disc U , but the image $f_n(U)$ can be arbitrarily small. To exclude this rather artificial type of example, we shall impose another condition. We say that a family \mathcal{F} of algebroid multifunctions on D is *translation-invariant* if $F \in \mathcal{F} \Rightarrow F + c \in \mathcal{F}$ for all $c \in \mathbb{C}$, where

$$(F + c)(z) = \{w + c : w \in F(z)\} \quad (z \in D).$$

In view of the fact that the inradius of $(F + c)(D)$ is independent of c , this seems a very natural requirement.

We are now ready to state our main result.

Theorem 1. *Let D be a domain such that $0 \in D$. Let \mathcal{F} be a normalised, translation-invariant family of algebroid multifunctions of degree n on D . Then, for each $d \geq 0$,*

$$L_{\mathcal{F}}(d) := \inf\{\text{inradius of } F(D) : F \in \mathcal{F}, \text{diam } F(0) \leq d\} > 0.$$

If $n = 1$, then clearly $\text{diam } F(0) = 0$ for each $F \in \mathcal{F}$, so the function $L_{\mathcal{F}}$ is constant. In particular, taking D to be the unit disc and $\mathcal{F} = \{F : |a'_1(0)| = 1\}$, we recapture the classical Bloch theorem with $L = L_{\mathcal{F}}$.

On the other hand, if $n \geq 2$, then the function $L_{\mathcal{F}}(d)$ may well tend to zero as $d \rightarrow \infty$. For example, let D be the unit disc, and let \mathcal{F} be the family of F as in (1) such that

$$a'_1(0) = a'_2(0) = \dots = a'_{n-1}(0) = 0 \quad \text{and} \quad |a'_n(0)| = 1.$$

Then \mathcal{F} is a normalised, translation-invariant family. Also, for each $t > 1$, the multifunction

$$F_t(z) = \{w \in \mathbb{C} : w^n + (z + t) = 0\} \quad (z \in D)$$

belongs to \mathcal{F} , and its image $F_t(D)$ is a subset of the annulus $(t - 1)^{1/n} < |w| < (t + 1)^{1/n}$, so cannot contain any discs of radius larger than $(t + 1)^{1/n} - (t - 1)^{1/n}$. As

$$\text{diam}(F_t(0)) \leq 2t^{1/n},$$

this shows that, for this family \mathcal{F} ,

$$L_{\mathcal{F}}(d) \leq ((d/2)^n + 1)^{1/n} - ((d/2)^n - 1)^{1/n} = O\left(\frac{1}{d^{n-1}}\right) \quad \text{as } d \rightarrow \infty.$$

Our second result concerns the growth of the ‘Landau function’ $L_{\mathcal{F}}$ as the domain grows. We write U for the open unit disc, so rU is the disc $\{z : |z| < r\}$.

Theorem 2. *Let \mathcal{F} be a normalised, translation-invariant family of algebroid multifunctions of degree n on U . Suppose that each $F \in \mathcal{F}$ extends to an algebroid multifunction on \mathbb{C} . For $d \geq 0$ and $r \geq 1$, set*

$$L_{\mathcal{F}}(d, r) := \inf\{\text{inradius of } F(rU) : F \in \mathcal{F}, \text{diam } F(0) \leq d\}.$$

Then, for each $d \geq 0$,

$$\inf_{r \geq 1} \frac{L_{\mathcal{F}}(d, r)}{r^{1/n}} > 0.$$

The family $\overline{\mathcal{F}}$, consisting of the multifunction

$$F(z) = \{w : w^n - z = 0\} \quad (z \in \mathbf{C})$$

and its translates $F + c$ ($c \in \mathbf{C}$), shows that growth rate $r^{1/n}$ in Theorem 2 is sharp.

Notice that Theorem 2 has the consequence that the image $F(\mathbf{C})$ of each non-constant algebroid multifunction F on \mathbf{C} contains discs of arbitrarily large radius. However, it would be disingenuous to call this a corollary, since we shall actually use something even stronger in the course of the proof of Theorem 1 (see Lemma 4 below).

This prompts the question as to whether it is possible to find a more elementary, constructive proof of Theorem 1, at least for certain families \mathcal{F} , along the lines of the proof of Bloch’s theorem given in [2]. It is likely that such a proof would also yield explicit lower bounds for $L_{\mathcal{F}}(d)$.

To illustrate why such bounds might be desirable, consider the following problem, which we have been unable to resolve. Let \mathcal{F}_n be the family of multifunctions F of degree n as in (1), defined on the unit disc U , and satisfying $|a'_1(0)| = 1$. This family is normalised and translation-invariant, so by Theorem 1 we have $L_{\mathcal{F}_n}(d) > 0$ for all $d \geq 0$. However, we know of no example showing that $L_{\mathcal{F}_n}(d) \rightarrow 0$ as $d \rightarrow \infty$. It is thus conceivable that there is an absolute constant $L_n > 0$, namely $\lim_{d \rightarrow \infty} L_{\mathcal{F}_n}(d)$, such that the inradius of the image of each $F \in \mathcal{F}_n$ is at least L_n . Such a result, if true, would be the most direct generalisation to algebroid multifunctions of Bloch’s original theorem.

2. Proofs

We shall prove Theorem 1 using a normal-families argument. Such arguments usually work more smoothly with meromorphic functions than with holomorphic ones, and this is just as true for multifunctions as for ordinary functions. Accordingly, we shall work with the following meromorphic analogue of algebroid multifunction.

A *meromorphic algebroid multifunction of degree n* on a domain D is a set-valued function F of the form

$$F(z) = \{w \in \mathbf{C}_\infty : a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_{n-1}(z)w + a_n(z) = 0\} \quad (z \in D), \quad (2)$$

where a_0, a_1, \dots, a_n are holomorphic functions on D such that $\sum_{j=0}^n |a_j(z)| > 0$ for all $z \in D$. Here we adopt the convention that $\infty \in F(z)$ if and only if $a_0(z) = 0$, and the multiplicity of ∞ in $F(z)$ is the smallest integer k such that $a_k(z) \neq 0$. In this manner, each $F(z)$ becomes an n -tuple in the Riemann sphere \mathbf{C}_∞ .

If F is meromorphic algebroid of degree n , then so is $\varphi \circ F$ for each Möbius transformation φ . Indeed, this is clear if φ is a translation or a dilatation, and, because of the convention regarding ∞ , it is also true if $\varphi(w) = 1/w$. A general Möbius transformation can be expressed as a composition of functions of these three types.

We shall need some elementary properties of meromorphic algebroid multifunctions, beginning with a form of the principle of isolated zeros.

Lemma 1. *Let F be a meromorphic algebroid multifunction on a domain D , let $w_0 \in \mathbf{C}_\infty$ and set $Z = \{z \in D : w_0 \in F(z)\}$. Then either Z is the whole of D or it contains only isolated points.*

PROOF. Replacing F by $\varphi \circ F$ for a suitable Möbius transformation φ , we can reduce to the case where $w_0 = \infty$. In this case, writing F as in (2), we have

$$Z = \{z \in D : a_0(z) = 0\},$$

and the result therefore follows from the usual form of the principle of isolated zeros. ■

There are thus two types of points in the image $F(D)$ of F : those that lie in $F(z)$ for all $z \in D$, and those that do so only for a discrete set of $z \in D$. It will be helpful to distinguish between the two. Accordingly, we define the *reduced image* of F by

$$\check{F}(D) = \left(\bigcup_{z \in D} F(z) \right) \setminus \left(\bigcap_{z \in D} F(z) \right).$$

Note that $\check{F}(D)$ is empty if and only if F is constant on D . The next result is an analogue of the open mapping theorem.

Lemma 2. *Let F be a meromorphic algebroid multifunction on a domain D . Then its reduced image $\check{F}(D)$ is an open subset of \mathbf{C}_∞ .*

PROOF. It is enough to show that if $0 \in \check{F}(D)$ then $\check{F}(D)$ contains a disc about 0. The general case follows upon considering $\varphi \circ F$ for appropriate Möbius transformations φ .

Suppose, then, that $0 \in \check{F}(D)$, say $0 \in F(z_0)$. Then, writing F as in (2), we have $a_n(z_0) = 0$. As z_0 is an isolated zero, there exists $r > 0$ such that $a_n(z) \neq 0$ for all z with $0 < |z - z_0| \leq r$. By continuity, there exists $\delta > 0$ such that, whenever $|w| < \delta$,

$$\max_{|z - z_0| = r} |a_0(z) w^n + a_1(z) w^{n-1} + \dots + a_{n-1}(z) w| < \min_{|z - z_0| = r} |a_n(z)|.$$

Applying Rouché's theorem, it follows that, for each w with $|w| < \delta$, the equation

$$a_0(z) w^n + a_1(z) w^{n-1} + \dots + a_{n-1}(z) w + a_n(z) = 0$$

has a solution z with $|z - z_0| < r$. Hence $F(D)$ contains the disc $\{w : |w| < \delta\}$. Finally, as $F(D) \setminus \check{F}(D)$ is a finite set, we conclude that $\check{F}(D)$ contains a disc around 0, as required. ■

We say that a sequence (F_k) of meromorphic algebroid multifunctions of degree n converges spherically locally uniformly to another, F , if $\rho_s(F_k(z), F(z)) \rightarrow 0$ locally uniformly, where s is any metric on \mathbf{C}_∞ giving the usual topology, and ρ_s is the matching distance for n -tuples measured according to s :

$$\rho_s(\{\alpha_1, \dots, \alpha_n\}, \{\beta_1, \dots, \beta_n\}) = \min_{\sigma \in S_n} \max_{1 \leq j \leq n} s(\alpha_j, \beta_{\sigma(j)}).$$

The choice of s is irrelevant, since all such metrics are uniformly equivalent. The next result is an analogue of Hurwitz's theorem.

Lemma 3. *Let (F_k) be a sequence of meromorphic algebroid multifunctions of degree n on a domain D , converging spherically locally uniformly to another one, say F . Let K be a compact subset of $\check{F}(D)$. Then $K \subset \check{F}_k(D)$ for all large enough k .*

PROOF. Suppose, for a contradiction, that the result is false. Then, replacing (F_k) by a subsequence if necessary, there exists a sequence (w_k) in K such that $w_k \notin \check{F}_k(D)$ for all k . Taking a further subsequence, we can also suppose that w_k converges to some $w_0 \in K$.

Since $w_0 \in K \subset \check{F}(D)$, there exists $z_0 \in D$ such that $w_0 \in F(z_0)$. Replacing (F_k) by $(\varphi \circ F_k)$ for a suitable Möbius transformation φ , we may assume that $\infty \notin F(z_0)$. Hence, if D' is a small enough disc around z_0 , then $F_k \rightarrow F$ locally uniformly on D' (i.e. with respect to the Euclidean metric). Dropping the first few F_k if necessary, we can suppose that the F_k and F are all algebroid on D' (i.e. we can take $a_0 \equiv 1$). Then the coefficients $a_{1,k}(z), \dots, a_{n,k}(z)$ corresponding to F_k converge locally uniformly on D' to those of F , say $a_1(z), \dots, a_n(z)$.

Now, the fact that $w_k \notin \check{F}_k(D')$ means that the equation

$$w_k^n + a_{1,k}(z)w_k^{n-1} + \dots + a_{n,k}(z) = 0$$

holds either for all $z \in D'$ or for no $z \in D'$. Letting $k \rightarrow \infty$, and applying the classical Hurwitz theorem, we deduce that

$$w_0^n + a_1(z)w_0^{n-1} + \dots + a_n(z) = 0$$

holds either for all $z \in D'$ or for no $z \in D'$. Since $w_0 \in F(z_0)$, we must have $w_0 \in F(z)$ for all $z \in D'$. By Lemma 1, it follows that $w_0 \in F(z)$ for all $z \in D$. But this contradicts the fact that $w_0 \in K \subset \check{F}(D)$. ■

Let \mathcal{F} be a family of meromorphic algebroid multifunctions of degree n on a domain D . We say that \mathcal{F} is a *normal family* if every sequence in \mathcal{F} contains a subsequence which is spherically locally uniformly convergent on D . The following result is an analogue of Montel’s theorem, due to Dufresnoy [3]. Again s denotes any metric on \mathbf{C}_∞ giving the usual topology.

Lemma 4. *Let \mathcal{F} be a family of meromorphic algebroid multifunctions of degree n on a domain D . Suppose that there exists $\delta > 0$ such that, for each $F \in \mathcal{F}$, one can find $2n + 1$ points $w_1, \dots, w_{2n+1} \in \mathbf{C}_\infty \setminus F(D)$ satisfying $s(w_j, w_k) \geq \delta$ for all j, k with $j \neq k$. Then \mathcal{F} is a normal family.*

PROOF. See [3, §30]. Here is a sketch of an alternative approach. As the $(n + 1)$ -tuple $(a_0(z), \dots, a_n(z))$ in (2) is determined by $F(z)$ up to multiplication by a non-zero complex number, it is natural to consider it as an element of complex-projective n -space \mathbf{CP}^n . In this way, a meromorphic algebroid multifunction of degree n may be identified with a holomorphic function taking values in \mathbf{CP}^n . Viewed in these terms, Lemma 4 is then a special case of a result of Tu [4, corollary 2], which is based upon a projective version of Zalcman’s Bloch principle. ■

We are now ready to prove our main results.

PROOF OF THEOREM 1. Suppose, for a contradiction, that there exists $d \geq 0$ such that $L_{\mathcal{F}}(d) = 0$. This means that, for each $k \geq 1$, there exists $F_k \in \mathcal{F}$ with $\text{diam } F_k(0) \leq d$ and such that the inradius of $F_k(D)$ is less than $1/k$. Since \mathcal{F} is translation-invariant, we can suppose without loss of generality that $0 \in F_k(0)$, whence $F_k(0) \subset \{w : |w| \leq d\}$ for all k .

We next claim that the (F_k) form a normal family. To see this, let $\Delta_1, \dots, \Delta_{2n+1}$ be $2n+1$ disjoint closed discs in \mathbb{C} , each of (Euclidean) radius 1. Let s be a metric on \mathbb{C}_∞ giving the usual topology. By compactness, there exists $\delta > 0$ such that, if w, w' belong to two different Δ_j , then $s(w, w') \geq \delta$. Now, since each $F_k(D)$ has inradius less than 1, it contains no Δ_j and so, picking $w_j \in \Delta_j \setminus F_k(D)$ ($j = 1, \dots, 2n+1$), we obtain $2n+1$ points outside $F_k(D)$, each pair an s -distance at least δ apart. By Lemma 4, the (F_k) form a normal family, as claimed.

Thus, replacing (F_k) by a subsequence, we can suppose that F_k converges spherically locally uniformly on D to some meromorphic algebroid multifunction F .

Our next task is to show that this convergence is locally uniform, not just with respect to the spherical metric, but also to the Euclidean metric. We know that $\infty \notin F_k(D)$ for all k , so by Lemma 3 it follows that $\infty \notin \check{F}(D)$. Moreover, since

$$F_k(0) \subset \{w : |w| \leq d\}$$

for all k , we also have $\infty \notin F(0)$. Hence $\infty \notin F(D)$. As the Euclidean and spherical metrics are uniformly equivalent on each compact subset of \mathbb{C} , we conclude that $F_k \rightarrow F$ locally uniformly in the Euclidean sense, as desired.

Now, since \mathcal{F} is normalised, F must be non-constant. By Lemma 2, $\check{F}(D)$ contains a compact disk, say K . By Lemma 3, $\check{F}_k(D)$ also contains K for all large enough k . It follows that the inradius of $F_k(D)$ is at least the Euclidean radius of K for all large k . However, from the very choice of the (F_k) , the inradius of $F_k(D)$ tends to zero as $k \rightarrow \infty$. We have thus arrived at a contradiction. ■

PROOF OF THEOREM 2. Given $F \in \mathcal{F}$ and $r \geq 1$, define a multifunction F_r on U by

$$F_r(z) = r^{-1/n} F(rz) \quad (z \in U).$$

Evidently $\text{diam } F_r(0) \leq \text{diam } F(0)$, and the inradius of $F(rU)$ is $r^{1/n}$ times the inradius of $F(U)$. Hence, given $d \geq 0$,

$$\inf_{r \geq 1} \frac{L_{\mathcal{F}}(d, r)}{r^{1/n}} \geq L_{\tilde{\mathcal{F}}}(d),$$

where

$$\tilde{\mathcal{F}} = \{F_r : F \in \mathcal{F}, \text{diam } F(0) \leq d, r \geq 1\}.$$

Clearly $\tilde{\mathcal{F}}$ is translation-invariant. If we could show that it is also normalised, then by Theorem 1 we would have $L_{\tilde{\mathcal{F}}}(d) > 0$, and the proof would be complete.

So let us suppose, for a contradiction, that there exists a sequence (G_k) in $\tilde{\mathcal{F}}$ which is locally uniformly convergent on U to a constant multifunction. By translation-invariance, we can assume that $0 \in G_k(0)$ for each k . From the definition of $\tilde{\mathcal{F}}$, each G_k is of the form $(F_k)_{r_k}$, where $F_k \in \mathcal{F}$ with $\text{diam } F_k(0) \leq d$, and $r_k \geq 1$. As $0 \in G_k(0)$, it also follows that $0 \in F_k(0)$, and hence the sequence of sets $(F_k(0))$ is uniformly bounded.

Let $a_{1,k}(z), a_{2,k}(z), \dots, a_{n,k}(z)$ be the coefficients of F_k . Then the corresponding j th coefficient of G_k is $r_k^{-j/n} a_{j,k}(r_k z)$, and the assumption is that, as $k \rightarrow \infty$, this converges locally uniformly on U to a constant. Taking derivatives, and using the fact that $r_k \geq 1$, we deduce that $a'_{j,k} \rightarrow 0$ locally uniformly on U as $k \rightarrow \infty$. Further, since the sets $(F_k(0))$ are uniformly bounded, the sequence $(a_{1,k}(0), a_{2,k}(0), \dots, a_{n,k}(0))$ is bounded, and so has a convergent subsequence. The corresponding subsequence of (F_k) then converges locally uniformly on U to a constant multifunction, contradicting the assumption that \mathcal{F} is a normalised family on U . ■

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