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ON NEW BANACH SPACES OF ANALYTIC MAPPINGS

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Abstract

In infinite dimensional holomorphy most of the spaces encountered in the literature are not Banach spaces. Since there is a large number of Banach spaces of analytic functions in the one complex variable theory, we ask if these spaces can generate, in an interesting and natural way, Banach spaces of analytic functions in any complex Banach space. In this work, we present a method of construction of Banach spaces of analytic functions on the open unit ball of any complex Banach space from Banach spaces of analytic functions in the open unit disk. We also present some properties of these spaces.

1. Introduction

The study of analytic functions in infinite dimensions is as old an objective in mathematics as functional analysis: it dates back to the end of the last century. The theory was developed as a confluence of ideas and methods originating from one complex variable and functional analysis.

Let E be a complex Banach space and denote by B_E (resp. $\overline{B}_E$) the open (resp. closed) unit ball of E.

The Banach spaces of analytic functions on infinite dimensional spaces that have been studied are the following:

(1) $H^\infty(B_E)$, the set of all complex valued functions that are analytic and bounded on B_E, endowed with the sup norm;
(2) $A_u(B_E)$, the set of all functions in $H^\infty(B_E)$ that admit a uniformly continuous extension to $\overline{B}_E$, endowed with the sup norm;
(3) $A^\infty(B_E)$, the set of all functions in $H^\infty(B_E)$ that admit a continuous extension to $\overline{B}_E$, endowed with the sup norm.

Several complete locally convex spaces of analytic functions that are not normed spaces have also been extensively studied, for example, $H_b(E)$ [1], [3] and [10], $H_w(E)$ [2], $H_{wu}(E)$ [2], $H_K(E)$ [4], $H_N(E)$ [11] and [14], $H_S(E)$ [13] and [16].

It seems that in infinite dimensional holomorphy most of the spaces encountered in the literature are not Banach spaces but, since there are many Banach spaces of analytic functions in the one complex variable theory, we can ask if these spaces can generate, in an interesting and natural way, many Banach spaces of analytic functions in any complex Banach space.

The aim of this paper is to present a method of construction of Banach spaces of analytic functions on the open unit ball of any complex Banach space from Banach spaces of analytic functions in the open unit disk and to present some properties of these spaces.

2. Definition and examples

Throughout this paper, $\mathbf{N}$, $\mathbf{N}^*$ and $\mathbf{C}$ will denote the set of all non-negative integers, positive integers and complex numbers, respectively. Let E be a complex Banach space. For $s > 0$ and $a \in E$, we denote the open ball of E with centre a and radius s by $B_E(a, s)$ and we denote the closed ball of E with centre a and radius s by $\overline{B}_E(a, s)$. In the case where $a = 0$, we write $B_E(s)$ and $\overline{B}_E(s)$, respectively. In addition, if $s = 1$, we simply write B_E and $\overline{B}_E$, respectively. For simplicity we write $\Delta(a, s)$(resp. $\overline{\Delta}(a, s)$) instead of $B_{\mathbf{C}}(a, s)$(resp. $\overline{B}_{\mathbf{C}}(a, s)$). In the case where $a = 0$, we write $\Delta(s)$ and $\overline{\Delta}(s)$. Moreover, if $s = 1$, we simply write Δ and $\overline{\Delta}$. The sphere of E with centre $a = 0$ and radius s is denoted by $S_E(s)$. In the case where $s = 1$, we simply write S_E.

Given a function $f : B_E \to \mathbf{C}$ and a point $w \in S_E$, we define

$$f_w : \lambda \in \Delta \longmapsto f(\lambda w) \in \mathbf{C}.$$

Definition 1. Let $(H, \|\cdot\|)$ be a Banach space of holomorphic functions on Δ such that the topology generated by its norm is finer than the compact-open topology. We denote by $J_H(B_E)$ the set of all functions $f : B_E \to \mathbf{C}$ that satisfy the following conditions:

(a) f is analytic on B_E;
(b) for every $w \in S_E$, $f_w \in H$;
(c) the function $\Phi_f : w \in S_E \longmapsto f_w \in H$ is bounded and continuous.

Given $f, g \in J_H(B_E)$ and $\alpha \in \mathbf{C}$, we define:

$$(f + g)(x) = f(x) + g(x) \quad \text{for } x \in B_E,$$
$$(\alpha f)(x) = \alpha f(x) \quad \text{for } x \in B_E,$$
$$|||f||| = \sup\{\|f_w\| : w \in S_E\}.$$

Theorem 2. *$(J_H(B_E), |||\cdot|||)$ is a Banach space.*

Proof. It is easy to check that $(J_H(B_E), |||\cdot|||)$ is a normed vector space. Let us prove that it is complete. For this purpose, let (f_n) be a Cauchy sequence in $(J_H(B_E), |||\cdot|||)$.

It follows from the definition of $|||\cdot|||$ that, for each $w \in S_E$, $(f_{n,w})$ is a Cauchy sequence in H. Since H is a Banach space, for each $w \in S_E$, $(f_{n,w})$ converges to a function $g_w \in H$.

Fix $w_0 \in S_E$ and define $h : B_E \to \mathbf{C}$ by

$$h(x) = \begin{cases} g_{\frac{x}{\|x\|}}(\|x\|) & \text{if } x \neq 0 \\ g_{w_0}(0) & \text{if } x = 0. \end{cases}$$

Since the topology of H is finer than the compact-open topology, $f_{n,w} \longrightarrow g_w$ pointwise on Δ, for each $w \in S_E$. This implies that the definition of h does not depend on our choice of w_0, since, for all $w \in S_E$,

$$f_n(0) = f_{n,w}(0) \longrightarrow g_w(0) \quad \text{as} \quad n \to \infty.$$

In particular,

$$f_n(0) \longrightarrow h(0) \quad \text{as} \quad n \to \infty.$$

Also, for all $x \in B_E, x \neq 0$,

$$f_n(x) = f_{n,\frac{x}{\|x\|}}(\|x\|) \longrightarrow g_{\frac{x}{\|x\|}}(\|x\|) = h(x) \quad \text{as} \quad n \to \infty.$$

Therefore, f_n converges pointwise to h on B_E. Now, let $0 < s < 1$. By the hypothesis on the topology of H and the fact that (f_n) is bounded, there is a constant $C_s > 0$ such that

$$|f_{n,w}(\lambda)| \leq C_s \quad \text{for all} \quad w \in S_E,\ n \in \mathbf{N} \quad \text{and} \quad \lambda \in \overline{\Delta}(s).$$

This inequality implies that (f_n) is locally uniformly bounded on B_E. Hence, h is analytic on B_E (cf. [5, p. 204]).

To see that each $h_w \in H$, note that $h_w(\lambda) = g_w(\lambda)$ for all $\lambda \in (0, 1)$. Since h is analytic on B_E, h_w is analytic on Δ. Hence, by the principle of analytic continuation, we get that $h_w \equiv g_w$, which lies in H.

Since, for each $w \in S_E$, $f_{n,w} \longrightarrow h_w$ in H and (f_n) is a bounded sequence in $J_H(B_E)$, there is a constant $M > 0$ such that $\|h_w\| \leq M$ for all $w \in S_E$. In other words, the function Φ_h is bounded.

Now, given $\epsilon > 0$, there is an $n_0 \in \mathbf{N}$ such that

$$\|f_{n,w} - f_{m,w}\| \leq |||f_n - f_m||| < \epsilon \quad \text{for all} \quad w \in S_E \quad \text{whenever} \quad n, m \geq n_0.$$

By letting $m \longrightarrow \infty$, we see that

$$\|f_{n,w} - h_w\| \leq \epsilon \quad \text{for all} \quad w \in S_E \quad \text{whenever} \quad n \geq n_0. \tag{1}$$

By (1) and the fact that $\|\Phi_{f_n}(w) - \Phi_h(w)\| = \|f_{n,w} - h_w\|$ for $n \in \mathbf{N}$ and $w \in S_E$, we conclude that $\Phi_{f_n} \longrightarrow \Phi_h$ uniformly on S_E. Since each Φ_{f_n} is continuous, so is Φ_h. Therefore, $h \in J_H(B_E)$. Finally, (1) also implies that $|||f_n - h||| \longrightarrow 0$ as $n \to \infty$. This completes the proof of Theorem 2. ■

For each $a \in \overline{\Delta}$, let $h_a : \lambda \in \Delta \longmapsto \lambda a \in \Delta$. Let us now show that under certain conditions distinct H yield distinct $J_H(B_E)$.

Theorem 3. *Let $(H_1, \|\cdot\|_1)$ and $(H_2, \|\cdot\|_2)$ be two Banach spaces of holomorphic functions on Δ as in Definition 1, with $H_1 \neq H_2$. Assume the following conditions for both $H = H_1$ and $H = H_2$:*

(i) *the polynomials are contained in H;*

(ii) *there is a constant $M > 0$ such that $\|p\| \leq M\|p\|_\infty$ for every polynomial p, where $\|p\|_\infty = \sup\{|p(\lambda)| : \lambda \in \Delta\}$ and $\|p\|$ denotes the norm of p in H;*

(iii) *$\varphi \circ h_a \in H$ and $\|\varphi \circ h_a\| \leq \|\varphi\|$, for all $\varphi \in H$ and for all $a \in \overline{\Delta}$.*

If the polynomials are dense in both H_1 and H_2, then

$$J_{H_1}(B_E) \neq J_{H_2}(B_E) \quad \text{for all complex Banach spaces } E.$$

Proof. Let E be a complex Banach space. Fix $w_0 \in S_E$. By the Hahn–Banach Theorem, there is a continuous linear functional f on E such that $f(w_0) = 1$ and $\|f\| = 1$. Since $H_1 \neq H_2$, we can assume without loss of generality that there is $\varphi \in H_1 - H_2$. Now, define $g(x) = \varphi(f(x))$ for $x \in B_E$.

Clearly g is well defined and is analytic on B_E. Now, for all $w \in S_E$, $g_w(\lambda) = \varphi(\lambda f(w))$, $\lambda \in \Delta$. So, for all $w \in S_E$, $g_w \equiv \varphi \circ h_{f(w)}$, which lies in H_1 by hypothesis. The boundedness of Φ_g follows from (iii). To see that Φ_g is continuous, let $w \in S_E$ and $\epsilon > 0$. By the density of the polynomials in $(H_1, \|\cdot\|_1)$, there is a polynomial p such that

$$\|p - \varphi\|_1 \leq \frac{\epsilon}{3}. \tag{2}$$

Since p is uniformly continuous on $\overline{\Delta}$ and f is a continuous linear functional on E, there is $\delta > 0$ such that for all $w' \in S_E$ with $\|w - w'\| < \delta$, we get that

$$\|p \circ h_{f(w')} - p \circ h_{f(w)}\|_\infty < \frac{\epsilon}{3M}. \tag{3}$$

Now, for each $a \in \overline{\Delta}$, consider $T_a : \mu \in H_1 \longmapsto \mu \circ h_a \in H_1$. Clearly, T_a is linear. By condition (iii) we get that $\|T_a\| \leq 1$. By condition (ii), (3) implies

$$\|T_{f(w')}(p) - T_{f(w)}(p)\|_1 < \frac{\epsilon}{3}. \tag{4}$$

So, if $w' \in S_E$ with $\|w - w'\| < \delta$, then

$$\|\Phi_g(w) - \Phi_g(w')\|_1 = \|T_{f(w)}(\varphi) - T_{f(w')}(\varphi)\|_1 \leq 2\|\varphi - p\|_1 + \|T_{f(w)}(p) - T_{f(w')}(p)\|_1 < \epsilon,$$

by (2) and (4). This proves the continuity of Φ_g at w. Since w is arbitrary in S_E, Φ_g is continuous. This finally allows us to say that

$$g \in J_{H_1}(B_E). \tag{5}$$

Now, if g were in $J_{H_2}(B_E)$, then g_{w_0} would be in H_2. But, $g_{w_0} \equiv \varphi$, which is not in H_2. So $g \notin J_{H_2}(B_E)$, which proves the theorem. ■

Denote by A the disk algebra and for $1 \leq p < \infty$, denote by H^p and A^p the

Hardy and Bergman spaces, respectively. Recall that for all $f \in A$,

$$\|f\|_A = \sup_{|\lambda| \le 1} |f(\lambda)|,$$

for all $f \in H^p$,

$$\|f\|_{H^p} = \sup_{0<r<1} [\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta]^{1/p}$$

and for all $f \in A^p$,

$$\|f\|_{A^p} = [\int\int_{\Delta} |f(x+iy)|^p \, dx \, dy]^{1/p}.$$

These spaces are Banach spaces of holomorphic functions on Δ that verify the hypotheses of Theorem 3 (cf. [8] and [12]). Therefore, they provide many examples of 'J-spaces'. More precisely, we have the following corollary.

Corollary 4. *Let $1 \le p < q < \infty$. Then, for all complex Banach spaces E,*

$$J_{H^q}(B_E) \subsetneq J_{H^p}(B_E),$$

$$J_{A^q}(B_E) \subsetneq J_{A^p}(B_E),$$

$$J_A(B_E) \subsetneq J_{H^p}(B_E) \subsetneq J_{A^p}(B_E).$$

PROOF. The inclusions follow from the classical inclusions and Definition 1. That the inclusions are strict follows directly from Theorem 3. ■

For $f \in H^\infty(B_E)$, we denote $\|f\|_\infty = \sup\{|f(x)| : x \in B_E\}$. For simplicity, H^∞ will denote $H^\infty(\Delta)$. Note that $A^\infty(\Delta) = A$.

It is natural to ask whether Theorem 3 remains true if we do not assume that the polynomials are dense in both H_1 and H_2. The next result shows that this is not the case. Indeed, although $A \neq H^\infty$ and the polynomials are dense in A, we will see that $J_A(B_E) = J_{H^\infty}(B_E)$ for all complex Banach spaces E.

Theorem 5. *For all complex Banach spaces E,*

$$J_A(B_E) = J_{H^\infty}(B_E) = A^\infty(B_E).$$

PROOF. The proof is divided into two steps.

Step 1. $J_A(B_E) = A^\infty(B_E)$.

First, let us prove that $A^\infty(B_E) \subset J_A(B_E)$. For this purpose, let $f \in A^\infty(B_E)$. By the definition of $A^\infty(B_E)$, the only thing we have to show in order to have $f \in J_A(B_E)$ is that Φ_f is continuous. Let $w_0 \in S_E$ and let $\epsilon > 0$. For each $\lambda \in \overline{\Delta}$, since f is continuous at λw_0, there is $\delta_\lambda > 0$ such that

$$|f(x) - f(\lambda w_0)| < \frac{\epsilon}{2} \quad \text{for all} \quad x \in \overline{B}_E \quad \text{with} \quad \|x - \lambda w_0\| < \delta_\lambda. \tag{6}$$

By compactness, there are $\lambda_1, \ldots, \lambda_n \in \overline{\Delta}$ such that $\overline{\Delta} \subset \bigcup_{1\le i\le n} \Delta(\lambda_i, \frac{\delta_{\lambda_i}}{2})$. Let $\delta = \min\{\frac{\delta_{\lambda_i}}{2} : 1 \le i \le n\}$. Now, given $\lambda \in \overline{\Delta}$, there is $1 \le j \le n$ such that $\lambda \in \Delta(\lambda_j; \frac{\delta_{\lambda_j}}{2})$, and therefore

$$|f(\lambda w) - f(\lambda_j w_0)| < \frac{\epsilon}{2} \quad \text{whenever} \quad w \in S_E \quad \text{and} \quad \|w - w_0\| < \delta. \tag{7}$$

In particular,

$$|f(\lambda w_0) - f(\lambda_j w_0)| < \frac{\epsilon}{2}. \tag{8}$$

Finally, (7) and (8) lead us to conclude that

$$\|f_w - f_{w_0}\|_\infty < \epsilon \quad \text{whenever} \quad w \in S_E \quad \text{and} \quad \|w - w_0\| < \delta.$$

Now, let $f \in J_A(B_E)$. Since each f_w is defined on $\overline{\Delta}$, we can extend f to $\overline{B}_E$ by putting $f(w) = f_w(1), w \in S_E$. That f is bounded on B_E follows from the fact that $\|f\|_\infty = |||f|||$. To prove the continuity of f on $\overline{B}_E$, let $w_0 \in S_E$ and let $\epsilon > 0$. By the continuity of Φ_f, there is $\delta_1 > 0$ such that

$$\|f_w - f_{w_0}\|_\infty < \frac{\epsilon}{2} \quad \text{whenever} \quad w \in S_E \quad \text{and} \quad \|w - w_0\| < \delta_1. \tag{9}$$

Also, by the continuity of f_{w_0} on $\overline{\Delta}$, there is $\delta_2 > 0$ such that

$$|f_{w_0}(z) - f_{w_0}(1)| < \frac{\epsilon}{2} \quad \text{whenever} \quad z \in \overline{\Delta} \quad \text{and} \quad |z - 1| < \delta_2. \tag{10}$$

Let $\delta = \min\{\frac{\delta_1}{2}, \delta_2, 1\}$. Let $x \in \overline{B}_E$ be such that $\|x - w_0\| < \delta$. By (9) and (10),

$$|f(x) - f(w_0)| \le |f_{\frac{x}{\|x\|}}(\|x\|) - f_{w_0}(\|x\|)| + |f_{w_0}(\|x\|) - f_{w_0}(1)| < \epsilon.$$

This proves that $f \in A^\infty(B_E)$. Therefore, step 1 is done.

Step 2. $J_{H^\infty}(B_E) = A^\infty(B_E)$.

By step 1 and the fact that $J_A(B_E) \subset J_{H^\infty}(B_E)$, we get that $A^\infty(B_E) \subset J_{H^\infty}(B_E)$. The proof of the other inclusion is divided into two cases.

Case 1. $E = \mathbf{C}$.

Let $f \in J_{H^\infty}(\Delta)$. Define $\tilde{f}(w) = \lim_{\lambda\to 1^-} f_w(\lambda)$ for all $w \in S_{\mathbf{C}}$ for which this limit exists. Since $f \in H^\infty$, $\tilde{f}$ is defined a.e. on $S_{\mathbf{C}}$. Let us prove that $\tilde{f}$ is in fact defined everywhere on $S_{\mathbf{C}}$ and that it is continuous. For this purpose, let $w_0 \in S_{\mathbf{C}}$. Choose a sequence (w_n) in $S_{\mathbf{C}}$ such that $w_n \longrightarrow w_0$ and $\lim_{\lambda\to 1^-} f_{w_n}(\lambda)$ exists, say $a_n = \lim_{\lambda\to 1^-} f_{w_n}(\lambda)$. Since Φ_f is continuous at w_0,

$$\|f_{w_n} - f_{w_0}\|_\infty \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

By (11), (a_n) is a Cauchy sequence; therefore it converges to some point $a \in \mathbf{C}$. Again by (11) we get that $a = \lim_{\lambda \to 1^-} f_{w_0}(\lambda)$. This proves that $\tilde{f}(w_0)$ is defined and that $\tilde{f}$ is continuous at w_0.

Now since, for all $r < 1$ and $\theta \in [-\pi, \pi]$,

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(t) P_r(\theta - t)\, dt,$$

we get that $f \in A$ (cf. [12, p. 32]).

Case 2. E is any complex Banach space.

Let $f \in J_{H^\infty}(B_E)$. Since $\Phi_f : w \in S_E \longmapsto f_w \in H^\infty$ is continuous, we have that, for all $w \in S_E$, the function

$$\Phi_{f_w} : \lambda \in S_{\mathbf{C}} \longmapsto f_{\lambda w} \in H^\infty$$

is continuous. Therefore, for all $w \in S_E$, $f_w \in J_{H^\infty}(\Delta)$. By case 1, this means that $f_w \in A$, for all $w \in S_E$. This implies that $f \in J_A(B_E)$, which is $A^\infty(B_E)$ by step 1. ■

Remark 6. Step 2 of the above theorem gives us a characterisation of all functions on $H^\infty(B_E)$ that lie in $A^\infty(B_E)$. They are precisely the functions $f \in H^\infty(B_E)$ for which Φ_f is continuous. In particular, for $E = \mathbf{C}$, we have the following result. A function $f \in H^\infty$ lies in the disk algebra A if and only if

$$\|f(e^{i\theta}z) - f(z)\|_\infty \longrightarrow 0 \quad \text{as} \quad \theta \to 0.$$

3. Some properties of 'J-spaces'

In this section we will present some analytic properties and some Banach space properties of $J_H(B_E)$. Let us begin by showing the following proposition.

Proposition 7. *For every $f \in J_H(B_E)$, the Taylor series of f at 0 converges uniformly on $\overline{B}_E(r)$ for all $0 < r < 1$.*

PROOF. Since the topology of H is finer than the compact open topology, for all $0 < s < 1$ there is a constant $R_s > 0$ such that

$$|f(x)| \leq R_s |||f||| \quad \text{for all} \quad f \in J_H(B_E) \quad \text{and for all} \quad x \in \overline{B}_E(s). \tag{12}$$

The above inequality implies that $r_f(0)$ (the radius of boundness of f at 0) is 1. Since

$$r_f(0) = \min\{1, R_f(0)\},$$

where $R_f(0)$ denotes the radius of uniform convergence of the Taylor series of f at 0 (cf. [10, p. 219]), we have that $R_f(0) \geq 1$. Now, the proposition follows from the definition of $R_f(0)$. ■

This is a very good property, since in infinite dimensional holomorphy even an entire function may have a Taylor series around 0 that does not converge uniformly on $B_E(s)$ for some $0 < s < 1$. In fact, as a consequence of Josefson–Nissenzweig's Theorem [6, p. 219], if $\dim E = \infty$, then, for all $s > 0$, there is a complex-valued entire function on E such that $f(B_E(s))$ is unbounded (see also [5, p. 222]).

The next result is a version of Hurwitz's Theorem (cf. [15, p. 348]) for 'J-spaces'.

Theorem 8. *Let (f_n) be a sequence in $J_H(B_E)$ such that each f_n is zero-free in B_E and $f_n \longrightarrow f$ in $J_H(B_E)$. Then, either f is zero-free in B_E or f is identically zero in B_E.*

PROOF. Suppose that f is not zero-free in B_E. We have to show that f is identically zero in B_E. For this purpose, fix $x_0 \in B_E$ such that $f(x_0) = 0$.

Case 1. $x_0 = 0$.

Let $w \in S_E$ and define $g_n(\lambda) = f_{n,w}(\lambda)$ and $g(\lambda) = f_w(\lambda)$, for $\lambda \in \Delta$. Since $f_n \longrightarrow f$ in $J_H(B_E)$ and the topology of H is finer than the compact open topology, we get that (g_n) is a sequence of analytic functions from Δ into $\mathbf{C}$ that converges normally to the analytic function g. Now, since each g_n is zero-free on Δ, the classical Hurwitz's Theorem leads us to conclude that either g is zero-free or g is identically zero on Δ. But $g(0) = f(0) = 0$. Thus, g is identically zero, that is, $f_w \equiv 0$. Since $w \in S_E$ is arbitrary, we get $f \equiv 0$.

Case 2. $x_0 \neq 0$.

Let $w_0 = \frac{x_0}{\|x_0\|}$. By the same argument as above, we conclude that $f_{w_0} \equiv 0$. In particular, $f(0) = 0$. Now it follows from case 1 that f is identically zero. ■

For each $n \in \mathbf{N}$ we denote by $\mathscr{P}(^nE)$ the space of all continuous n-homogeneous polynomials from E into $\mathbf{C}$. The space of continuous linear maps from E into $\mathbf{C}$, $\mathscr{P}(^1E)$, will also be denoted by E^*. Recall that the space of all continuous polynomials from E into $\mathbf{C}$ is by definition $\bigoplus_{n\in\mathbf{N}} \mathscr{P}(^nE)$. This space will be denoted by $\mathscr{P}(E)$. Now, we have the following proposition.

Proposition 9. *Let H be as in Definition 1. If H contains all polynomials in one complex variable, then $\mathscr{P}(E) \subset J_H(B_E)$.*

PROOF. Since $J_H(B_E)$ is a vector space, it is enough to show that $\mathscr{P}(^nE) \subset J_H(B_E)$, for all $n \in \mathbf{N}$. For this, fix $n \in \mathbf{N}$ and let $P \in \mathscr{P}(^nE)$. We have that P is analytic on B_E. Now, for all $w \in S_E$, $P_w(\lambda) = \lambda^n P(w)$, for all $\lambda \in \Delta$. So, for all $w \in S_E$, $P_w \in H$. Moreover, if we define by $g_n(\lambda) = \lambda^n$, $\lambda \in \Delta$, we get that

$$\|P_w\| = \|g_n\| |P(w)| \leq \|g_n\| \|P\|_\infty \quad \text{for all} \quad w \in S_E,$$

which proves that Φ_P is bounded. The continuity of Φ_P follows from the fact that P is continuous on $\overline{B}_E$. ■

It is natural to ask which conditions on H guarantee that $J_H(\Delta) = H$. Note that this is not always the case. For instance, $J_{H^\infty}(\Delta) = A \neq H^\infty$ (see Theorem 5). However, we have the following proposition.

Proposition 10. *Let $(H, \|\cdot\|)$ be as in Definition 1. If the polynomials are dense in H and $|||f||| = \|f\|$ for all $f \in J_H(\Delta)$, then $J_H(\Delta) = H$.*

PROOF. Clearly $J_H(\Delta) \subset H$. On the other hand, by Proposition 9 and the hypotheses, we get

$$H = \overline{\mathscr{P}(\mathbf{C})}^{\|\cdot\|} \subset \overline{J_H(\Delta)}^{\|\cdot\|} = J_H(\Delta). \quad \blacksquare$$

Corollary 11. *$J_A(\Delta) = A$ and for all $1 \leq p < \infty$, $J_{H^p}(\Delta) = H^p$ and $J_{A^p}(\Delta) = A^p$.*

Proposition 12. *Let H be as in Definition 1 and suppose it contains all polynomials in one complex variable. For all $m \in \mathbf{N}$ and for all $\xi \in B_E$, the map*

$$\varphi_{m,\xi} : f \in J_H(B_E) \longmapsto \frac{1}{m!}\hat{d}^m f(\xi) \in J_H(B_E)$$

is a continuous linear operator.

PROOF. Let $m \in \mathbf{N}$ and $\xi \in B_E$. The linearity of $\varphi_{m,\xi}$ is clear. Now, by the proof of Proposition 9, we get that

$$|||\varphi_{m,\xi}(f)||| \leq \|g_m\| \|\varphi_{m,\xi}(f)\|_\infty \quad \text{for all} \quad f \in J_H(B_E). \tag{13}$$

Fix $\|\xi\| < r < 1$. Let $0 < s < 1$ such that $B_E(\xi, s) \subset \overline{B}_E(r)$. By the Cauchy integral formula and (12), there is a constant $R_r > 0$ such that for all $w \in S_E$

$$|\varphi_{m,\xi}(f)(w)| \leq \frac{1}{s^m} R_r |||f||| \quad \text{for all} \quad f \in J_H(B_E). \tag{14}$$

By (13) and (14) we conclude that

$$|||\varphi_{m,\xi}(f)||| \leq \frac{\|g_m\| R_r}{s^m} |||f||| \quad \text{for all} \quad f \in J_H(B_E),$$

which proves the proposition. $\blacksquare$

Corollary 13. *Let H be as in Proposition 12. Then, for all $m \in \mathbf{N}$, the vector space $\mathscr{P}(^m E)$ is a complemented subspace of $J_H(B_E)$. Consequently, if E is an infinite dimensional complex Banach space such that, for some $m \in \mathbf{N}^*$, $\mathscr{P}(^m E)$ is reflexive, then $J_H(B_E)$ does not have the Dunford–Pettis property* [8].

In particular, if E satisfies the hypothesis of Corollary 13, $A^\infty(B_E)$ does not have the Dunford–Pettis property. Information about the reflexivity of $\mathscr{P}(^m E)$ can be found in [10].

Proposition 14. *Let $(H, \|\cdot\|)$ be as in Theorem 3. Then, for all complex Banach spaces E, H is isometrically isomorphic to a complemented subspace of $J_H(B_E)$.*

Proof. Let E be a complex Banach space. Fix $w_0 \in S_E$. By the Hahn–Banach Theorem, there is $f \in E^*$ such that $f(w_0) = 1 = \|f\|$. Define the function

$$F : \varphi \in H \longmapsto \varphi \circ f_{|B_E} \in J_H(B_E).$$

The proof of Theorem 3 shows us that F is well defined (cf. (5)). Moreover, since F is linear and $|||F(\varphi)||| = \|\varphi\|$, we have that F is an isometry. Now, the function

$$\Pi_{w_0} : g \in J_H(B_E) \longmapsto F(g_{w_0}) \in J_H(B_E)$$

is a linear projection of $J_H(B_E)$ onto $F(H)$. ■

Corollary 15. *For all complex Banach spaces E, $A^\infty(B_E)$ does not have the Schur or the Radon–Nikodym properties.*

Let $(H, \|\cdot\|)$ be as in Definition 1. Suppose the polynomials are dense in H. What can we say about the density of the polynomials in $J_H(B_E)$ if $\dim E < \infty$? Thus we have the following theorem.

Theorem 16. *Let $(H, \|\cdot\|)$ be as in Definition 1. Suppose that:*

(i) *the polynomials are dense in H;*
(ii) *there is a constant $M > 0$ such that $\|p\| \leq M\|p\|_\infty$ for every polynomial p;*
(iii) *$\varphi \circ h_a \in H$ and $\|\varphi \circ h_a\| \leq \|\varphi\|$, for all $\varphi \in H$ and for all $a \in \overline{\Delta}$.*

If $\dim E < \infty$, *then $\mathscr{P}(E)$ is dense in $J_H(B_E)$.*

Proof. Let $g \in J_H(B_E)$ and let $\epsilon > 0$. We want to find $P \in \mathscr{P}(E)$ such that $|||P - g||| < \epsilon$. For this purpose define, for each $n \in \mathbf{N}^*$,

$$g_n(x) = g((1 - \frac{1}{n})x), \ x \in B_E.$$

Note that, for each $n \in \mathbf{N}^*$, $g_n \in A^\infty(B_E)$. Since H contains the polynomials, condition (ii) implies that $A \subset H$. Therefore, each $g_n \in J_H(B_E)$. The proof that $g_n \longrightarrow g$ in $J_H(B_E)$ is divided into two steps.

Step 1. For all $w \in S_E$, $g_{n,w} \longrightarrow g_w$ in H.

Fix $w \in S_E$ and let $\delta > 0$. By (i), there is a polynomial p such that

$$\|p - g_w\| < \frac{\delta}{3}. \tag{15}$$

Put $p_n(\lambda) = p((1 - \frac{1}{n})\lambda)$, $\lambda \in \Delta$. By the uniform continuity of p on $\overline{\Delta}$, there is $n_w > 0$ such that

$$\|p_n - p\|_\infty < \frac{\delta}{3M} \quad \text{for all} \quad n \geq n_w. \tag{16}$$

Note that by condition (iii) we get that, for all $n \in \mathbf{N}^*$,

$$\|g_{n,w} - p_n\| \leq \|g_w - p\|. \tag{17}$$

Now, (15), (16), (17) and (ii) lead us to conclude that $\|g_{n,w} - g_w\| < \delta$ for all $n \geq n_w$, which proves step 1.

Step 2. $g_n \longrightarrow g$ in $J_H(B_E)$.

Let $\delta > 0$. By step 1 and the fact that Φ_g is continuous, for all $w \in S_E$ there is $n_w \in \mathbf{N}^*$ and there is $\delta_w > 0$ such that

$$\|g_{n,w} - g_w\| < \frac{\delta}{3} \quad \text{for all} \quad n \geq n_w \quad \text{and} \quad \|g_w - g_{w'}\| < \frac{\delta}{3} \quad \text{whenever} \quad \|w - w'\| < \delta_w. \tag{18}$$

Note that condition (iii) implies that, for all $w, w' \in S_E$ and for all $n \in \mathbf{N}^*$,

$$\|g_{n,w} - g_{n,w'}\| \leq \|g_w - g_{w'}\|. \tag{19}$$

By the compactness of S_E, S_E is contained in a finite union of the form $\bigcup_{i=1}^{j} \Delta(w_i, \delta_{w_i})$. Now, let $N = \max\{n_{w_i} : 1 \leq i \leq j\}$. By (18) and (19) we get that

$$\|g_{n,w} - g_w\| < \delta \quad \text{for all} \quad n \geq N \quad \text{and for all} \quad w \in S_E,$$

which proves step 2.

Now, by step 2, there is $n_0 \in \mathbf{N}^*$ such that $|||g_{n_0} - g||| < \frac{\epsilon}{2}$. Since the Taylor series of g_{n_0} at 0 converges uniformly on $\overline{B}_E$, there is $P \in \mathscr{P}(E)$ such that $\|P - g_{n_0}\|_\infty < \frac{\epsilon}{2M}$. By (ii) we finally conclude that $|||P - g||| < \epsilon$. ■

Corollary 17. *Let* $(H, \|\cdot\|)$ *be as in Theorem 16. If* dim $E < \infty$, *then* $J_H(B_E)$ *is separable.*

We can get a more general result about the separability of $J_H(B_E)$ when dim $E < \infty$.

Theorem 18. *Let H be a separable Banach space of holomorphic mappings on* Δ *satisfying the property stated in Definition 1. If* dim $E < \infty$, *then* $J_H(B_E)$ *is separable.*

PROOF. Denote by $C(S_E, H)$ the set of all continuous functions from S_E into H, endowed with the sup norm. Define the function

$$\Phi : f \in J_H(B_E) \longmapsto \Phi_f \in C(S_E, H).$$

Clearly, Φ is an isometric isomorphism. Since S_E is compact and H is separable, it follows that $C(S_E, H)$ is separable (cf. [9, p. 263]). Since a subspace of a metrisable separable space is separable and Φ is a linear isometry, we conclude that $J_H(B_E)$ is separable. ■

Let H be as in Definition 1. The following proposition shows that if $\dim E = \infty$, then $J_H(B_E)$ is not necessarily separable.

Proposition 19. *If H contains the polynomials and E^* is not separable, then $J_H(B_E)$ is not separable.*

Proof. By Proposition 9, we have that $E^* \subset J_H(B_E)$. Now, with $m = 1$ and $f \in E^*$ in (14), we get that $\|f\|_\infty \leq C\ |||f|||$ for some constant $C > 0$. Since E^* is not separable, the above inequality shows that $(E^*, |||\cdot|||)$ is not separable. Therefore, neither is $J_H(B_E)$. ∎

For $1 < p < \infty$, R. M. Aron proved that $\mathscr{P}(^p l_p)$ is not separable when considered with its usual topology. Supposing that H contains the polynomials, (13) and (14) lead us to conclude that the topology of $J_H(B_{l_p})$ restricted to $\mathscr{P}(^p l_p)$ coincides with its usual topology. So, even when E^* is a separable infinite dimensional Banach space, $J_H(B_E)$ may not be separable.

If $\dim E = \infty$, $\mathscr{P}(E)$ is not necessarily dense in $J_H(B_E)$. For instance, $\overline{\mathscr{P}(E)} = A_u(B_E) \neq A^\infty(B_E) = J_A(B_E)$ whenever $\dim E = \infty$ (cf. [3, p. 90]). Let us now see that $\mathscr{P}(E)$ is not necessarily dense in $J_{H^2}(B_E)$, either.

Proposition 20. *$\mathscr{P}(l_2)$ is not dense in $J_{H^2}(B_{l_2})$.*

Proof. Consider $f(x) = \sum_{k=1}^\infty x_k^k,\ x \in B_{l_2}$. We have that $f \in H^\infty(B_{l_2})$, which implies that $f_w \in H^2$ for all $w \in S_{l_2}$ and that Φ_f is bounded. To see that Φ_f is continuous, let $w \in S_{l_2}$ and let $\epsilon > 0$. Since the function $x \in [0,1) \longmapsto \frac{x}{1-x} \in [0,\infty)$ tends to 0 as x tends to 0, there is $0 < \delta < 1$ such that

$$\frac{x^n}{1-x} < \frac{\epsilon}{4} \quad \text{for all} \quad n \geq 1, \quad \text{whenever} \quad 0 < x < \delta. \tag{20}$$

By the fact that $w \in l_2$, there is $n_0 \in \mathbf{N}^*$ such that $|w_n| < \frac{\delta}{4}$ for all $n \geq n_0$. This and (20) imply

$$|\sum_{k=n_0}^{\infty}(\lambda w_k)^k| \leq \frac{(\delta/4)^{n_0}}{1-\delta/4} < \frac{\epsilon}{4} \quad \text{for all} \quad \lambda \in \overline{\Delta}.$$

Now, if $w' \in S_{l_2}$ and $\|w' - w\| < \frac{\delta}{4}$, then by (20)

$$|\sum_{k=n_0}^{\infty}(\lambda w_k')^k| \leq \frac{(\delta/2)^{n_0}}{1-\delta/2} < \frac{\epsilon}{4} \quad \text{for all} \quad \lambda \in \overline{\Delta}.$$

Choose $0 < \delta_1 < \delta/4$ such that $\|w - w'\| < \delta_1$ implies $\sum_{k=1}^{n_0-1} |w_k^k - (w_k')^k| < \frac{\epsilon}{2}$. Therefore, for all $w' \in S_{l_2}$ with $\|w' - w\| < \delta_1$, the above inequalities imply $\|f_{w'} - f_w\|_\infty < \epsilon$. Since $\|f_{w'} - f_w\|_{H^2} \leq \|f_{w'} - f_w\|_\infty$, we conclude that Φ_f is continuous at w. Hence,

$$f \in J_{H^2}(B_{l_2}).$$

Let $0 < \epsilon < \frac{1}{\sqrt{32}}$. Suppose there is a polynomial P, say of degree m, such that $|||P - f||| < \epsilon$. This implies that, for all $k \in \mathbf{N}^*$,

$$\|P_{e_k} - f_{e_k}\|_{H^2} < \epsilon, \tag{21}$$

where $\{e_k : k \in \mathbf{N}^*\}$ denotes the standard basis of l_2. Denote by $\mathscr{P}_m$ the set of all complex polynomials with degree $\le m$. By (21) we get that (P_{e_k}) is a $\|\cdot\|_{H^2}$-bounded family of polynomials in $\mathscr{P}_m$. Hence, since $\mathscr{P}_m$ is a finite dimensional space, there are a polynomial $Q \in \mathscr{P}_m$ and a subsequence $(P_{e_{k_j}})$ of (P_{e_k}) such that $P_{e_{k_j}} \longrightarrow Q$ in the $\|\cdot\|_{H^2}$-norm. By this fact and (21), there is $j_0 \in \mathbf{N}^*$ such that

$$\|f_{e_{k_j}} - Q\|_{H^2} < 2\epsilon \quad \text{for all} \quad j \ge j_0. \tag{22}$$

By applying the triangle inequality on (22) and by using the canonical isometric imbedding of H^2 into L^2 (cf. [12, pp 38 and 39]), we get

$$\|e^{ik\theta} - 1\|_{L^2} < \frac{1}{\sqrt{2}}$$

for some $k \in \mathbf{N}^*$, because of our choice of ϵ. Since this is a contradiction, our theorem is proved. ■

Remark 21. The proof of Proposition 20 can easily be adapted to show that

$$\mathscr{P}(l_p) \quad \text{is not dense in} \quad J_{H^q}(B_{l_p})$$

for all $1 \le p < \infty$ and $1 \le q \le \infty$.

Let us now see that the Taylor coefficients of functions in $J_H(B_E)$ enjoy a good equicontinuity property.

Proposition 22. *Let $f \in J_H(B_E)$. Then, the family $\{\frac{1}{m!}\hat{d}^m f(0) : m \in \mathbf{N}\}$ is uniformly equicontinuous on $\overline{B}_E(s)$ for all $0 < s < 1$.*

PROOF. Let $0 < s < 1$ and let $\epsilon > 0$. Since f is uniformly continuous on $\overline{B}_E(s)$, there is $\delta > 0$ such that, if $z_1, z_2 \in \overline{B}_E(s)$ and $\|z_1 - z_2\| < \delta$, then $|f(z_1) - f(z_2)| < \epsilon$. Now, let $x, y \in \overline{B}_E(s)$ and $\|x - y\| < \delta$. By the Cauchy integral formula we get that

$$|\frac{1}{m!}\hat{d}^m f(0)(x) - \frac{1}{m!}\hat{d}^m f(0)(y)| = |\frac{1}{2\pi}\int_{|\lambda|=1} \frac{f(\lambda x) - f(\lambda y)}{\lambda^{m+1}} d\lambda|$$

$$\le \sup\{|f(\lambda x) - f(\lambda y)| : |\lambda| = 1|\} \le \epsilon,$$

for all $m \in \mathbf{N}$. This proves what we wanted. ■

The next proposition shows that the Taylor coefficients of functions in $J_{H^p}(B_E)$ are uniformly bounded.

Proposition 23. *Let $1 \leq p \leq \infty$ and let $f \in J_{H^p}(B_E)$. Then, for all $m \in \mathbf{N}$,*

$$|||\frac{1}{m!}\hat{d}^m f(0)||| \leq |||f|||.$$

Proof. Let $f \in J_{H^p}(B_E)$. Then, for all $w \in S_E$,

$$f_w(\lambda) = \sum_{m=0}^{\infty} \frac{1}{m!}\hat{d}^m f(0)(w)\lambda^m, \ \lambda \in \Delta.$$

Since, for each $w \in S_E$,

$$f_w(\lambda) = \sum_{m=0}^{\infty}(\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}_w(e^{i\theta})e^{-im\theta}d\theta)\lambda^m, \ \lambda \in \Delta,$$

where $\tilde{f}_w$ denotes the radial limit of f_w, it follows from the unicity of the coefficients of the Taylor series that, for all $w \in S_E$,

$$\frac{1}{m!}\hat{d}^m f(0)(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}_w(e^{i\theta})e^{-im\theta}d\theta. \tag{23}$$

Now the desired inequality follows. ■

If $f \in J_{H^p}(B_E)$, then the fact that Φ_f is continuous and (23) lead us to conclude that the Taylor coefficients of f verify the following type of equicontinuity on S_E: let $w_0 \in S_E$. Given $\epsilon > 0$, there is $\delta > 0$ such that, if $w \in S_E, \|w - w_0\| < \delta$, then $|\frac{1}{m!}\hat{d}^m f(0)(w) - \frac{1}{m!}\hat{d}^m f(0)(w_0)| < \epsilon$, for all $m \in \mathbf{N}$. A natural question is whether the Taylor coefficients are equicontinuous on $\overline{B}_E$. In this direction we have the following proposition.

Proposition 24. *Let E be a finite dimensional complex Banach space and let $1 \leq p \leq \infty$. If $f \in J_{H^p}(B_E)$, then the family $\{\frac{1}{m!}\hat{d}^m f(0) : m \in \mathbf{N}\}$ is equicontinuous on $\overline{B}_E$.*

Proof. Let $f \in J_{H^p}(B_E)$. For each $x \in \overline{B}_E$, we denote by f_x the function $f_x : \lambda \in \Delta \longmapsto f(\lambda x) \in \mathbf{C}$. Now, for each $0 < r < 1$, denote by Φ_{f_r} the function

$$\Phi_{f_r} : w \in S_E \longmapsto f_{rw} \in H^p.$$

Claim. *$\Phi_{f_r} \longrightarrow \Phi_f$ uniformly on S_E as $r \to 1$.*

Proof of Claim. Let $\epsilon > 0$. For each $w \in S_E$, there is $R_w > 0$ and $\delta_w > 0$ such that

$$\|f_{rw} - f_w\| < \frac{\epsilon}{3} \quad \text{if} \quad R_w < r < 1 \quad \text{and} \quad \|f_{w'} - f_w\| < \frac{\epsilon}{3} \quad \text{if} \quad w' \in S_E, \|w' - w\| < \delta_w. \tag{24}$$

Note that if $w' \in S_E$ and $\|w' - w\| < \delta_w$, then

$$\|f_{rw} - f_{rw'}\| < \frac{\epsilon}{3} \quad \text{for all} \quad R_w < r < 1. \tag{25}$$

By the compactness of S_E, $S_E \subset \bigcup_{i=1}^{j} \Delta(w_i, \delta_{w_i})$. Let $R = \max\{R_{w_i} : 1 \le i \le j\}$. By (24) and (25) we get that

$$\|f_{rw} - f_w\| < \epsilon \quad \text{for all} \quad w \in S_E, \quad \text{whenever} \quad R < r < 1,$$

which proves our claim.

Define the function

$$\Psi : x \in \overline{B}_E \longmapsto f_x \in H^p.$$

Note that $\Psi_{|S_E} \equiv \Phi_f$ and for $x \in B_E$, $f_x \in H^\infty$. By the claim and the fact that Φ_f is continuous, we get that Ψ is continuous on $\overline{B}_E$.

Now, let $\epsilon > 0$. The continuity of Ψ implies that there is $\delta > 0$ such that, if $x, y \in \overline{B}_E$ with $\|x - y\| < \delta$, then $\|f_x - f_y\| < \epsilon$. Since, for all $x \in \overline{B}_E$,

$$\frac{1}{m!}\hat{d}^m f(0)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}_x(e^{i\theta})e^{-im\theta}\,d\theta,$$

where $\tilde{f}_x$ denotes the radial limit of f_x, it follows that

$$\left|\frac{1}{m!}\hat{d}^m f(0)(x) - \frac{1}{m!}\hat{d}^m f(0)(y)\right| \le \|f_x - f_y\| < \epsilon,$$

whenever $x, y \in \overline{B}_E$ with $\|x - y\| < \delta$ and $m \in \mathbf{N}$. This proves the desired result. ■

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