ON THE EIGENVALUES OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH FRACTIONAL TRANSITION POINTS

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Abstract

We introduce a method of approximating the eigenvalues associated with the linear, second order equation

$$y'' + (\mu^2 x^\alpha - q)y = 0,$$

where $-\infty < a < 0 < b < \infty$, q is a real-value member of $L^1[a,b]$ and μ^2 is a real parameter, which may be positive or negative and $\alpha > 0$. The approximations have an error term of the form $0(\mu^{-R})$ as $|\mu| \to \alpha$, where R can be made arbitrarily large.

1. Introduction

We consider the problem of calculating the form of solutions of the linear, second-order differential equation

$$y'' + (\mu^2 x^\alpha - q)y = 0 \text{ on } [a,b], \tag{1.1}$$

where $-\infty < a < 0 < b < \infty$, q is a real-valued member of $L^1[a,b]$ and μ^2 is a real parameter, which may be positive or negative. We suppose that $\alpha > 0$ and is such that x^α changes sign as x passes through 0. In particular we are concerned with the case when (1.1), augmented by the boundary conditions

$$y(a)\cos\gamma + y'(a)\sin\gamma = 0 \tag{1.2}$$

$$y(b)\cos\beta + y'(b)\sin\beta = 0, \tag{1.3}$$

is used to define an eigenvalue problem. It was shown in [2] that, for an equation more general than (1.1), there is a doubly infinite set of eigenvalues of (1.1)–(1.3) with $\dots\lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \dots$ where $\lambda_{\pm n} \to \pm\infty$ as $n \to \infty$. Also in [2] the asymptotic form of $\lambda_{\pm n}$ was derived to within a $o(1)$ error term. The analysis of [2] used a Prüfer transformation of (1.1) and a partition of $[a,b]$ into intervals on which q has one sign.

Another approach to approximating the form of solutions of (1.1), and hence the form of eigenvalues of (1.1)–(1.3), was followed in [8] and [9]. The main idea here was to approximate solutions of (1.1) in terms of solutions of the equation

$$y'' + \mu^2 x^\alpha y = 0. \tag{1.4}$$

Equation (1.4), with $\alpha > -1$, is solvable in terms of Bessel Functions and this led, in

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[9], to a two-term approximation of solutions of (1.1) with an error term that went to zero at a computable rate as $\mu^2 \to \pm\infty$.

The analysis of [9], like that of [2], does not seem to admit refinements that would enable the asymptotic expansion of a solution of (1.1) (with $q \not\equiv 0$) in decreasing powers of $|\mu|$. Such an expansion would enable the eigenvalues of (1.1)–(1.3) to be represented in the form

$$\lambda_{\pm n} = F(n,k) + O(n^{-k}) \text{ as } n \to \infty \tag{1.5}$$

for any natural number k. In this paper we use a method of analysis of (1.1), introduced in [5] that synthesises the approaches of [2] and [9] and leads to a representation of the form of (1.5).

It is possible by a simple change of variables to reduce an equation of the form

$$y'' + (\mu^2(x - x_0)^\alpha - q)y = 0 \tag{1.6}$$

to the form (1.1). It is also possible to extend the analysis of the present paper to include (1.1) on the intervals $[a,0]$ and $[0,b]$. These observations enable us to deal with equations of the form

$$y'' + (\lambda w - q)y = 0 \text{ on } [a,b],$$

where $w(x)$ has finitely many zeros of the form

$$w(x) = \begin{cases} (x - x_0)^{\alpha_1} & x < x_0 \\ (x - x_0)^{\alpha_2} & x_0 \le x \end{cases}$$

in the intervals $[a,b]$. The computation of the connecting constants in these circumstances is routine but lengthy, and we refer to [10] for an examination of some of the details.

The cases where (1.1) has a singularity of the form $(x - x_0)^\alpha$ for $-1 < \alpha < 0$ are covered in [9] and are also amenable to our analysis, see also [4]. The details are different and will be pursued elsewhere.

It is also possible, under certain circumstances, to transform more general equations of the form

$$y'' + (\mu^2 f(x) + g(x))y = 0$$

to the form (1.6). This may be done, supposing sufficient differentiability, by means of the Liouville transformation. We refer to [2], [7] and [9] for details.

There are two possibilities involving μ to consider, namely that with $\mu = \lambda^{1/2}$ and that with $\mu = i\lambda^{1/2}$ for $\lambda^{1/2}$ real and positive. The analysis of the two cases is similar and we give detailed proofs for the former.

2. Results

We define a sequence of functions $\{r_n(x,\mu)\}$ for $x \in [a,b]$ where $\mu = \lambda^{1/2}$ or $i\lambda^{1/2}$. We set

$$r_0(x,\mu) := \mu x^{\alpha/2} \frac{H^{(1)}_{\nu-1}(k^{-1}\mu x^k)}{H^{(1)}_\nu(k^{-1}\mu x^k)},$$

where $\nu = \dfrac{1}{\alpha+2}$, $k = \dfrac{\alpha+2}{2}$ and $H_\nu^{(1)}$ is the Hankel–Bessel function of order ν.

In the case $\mu = \lambda^{1/2}$ we set

$$\begin{aligned} r_1(x,\lambda) &:= -\int_x^b e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt \\ &= \frac{-1}{xH_\nu^{(1)}(k^{-1}\lambda^{1/2}x^k)^2}\int_x^b tH_\nu^{(1)}(k^{-1}\lambda^{1/2}t^k)^2 q(t)dt, \end{aligned} \tag{2.1}$$

and for $n \geq 1$

$$r_{n+1}(x,\lambda) := \frac{1}{xH_\nu^{(1)}(k^{-1}\lambda^{1/2}x^k)^2}\int_x^b tH_\nu^{(1)}(k^{-1}\lambda^{1/2}t^k)^2\{2\sum_{j=1}^{n-1} r_j(t,\lambda) + r_n(t,\lambda)\} r_n(t,\lambda)dt. \tag{2.2}$$

In the case $\mu = i\lambda^{1/2}$

$$\begin{aligned} r_1(x,\lambda) &:= \int_a^x e^{-2\int_t^x r_0(s,\lambda)ds} q(t)dt \\ &= xH_\nu^{(1)}(ik^{-1}\lambda^{1/2}x^k)^2 \int_a^x \frac{q(t)}{tH_\nu^{(1)}(ik^{-1}\lambda^{1/2}t^k)^2}dt, \end{aligned} \tag{2.3}$$

and for $n \geq 1$

$$\begin{aligned} r_{n+1}(x,\lambda) = xH_\nu^{(1)}(ik^{-1}\lambda^{1/2}x^k)^2 &\int_a^x \frac{1}{tH_\nu^{(1)}(ik^{-1}\lambda^{1/2}t^k)^2} \\ &\times\{2\sum_{j=1}^{n-1} r_j(t,\lambda) + r_n(t,\lambda)\} r_n(t,\lambda)dt. \end{aligned} \tag{2.4}$$

We set

$$r(x,\lambda) := \sum_{n=0}^{\infty} r_n(x,\lambda) =: S(x,\lambda) + iT(x,\lambda) \tag{2.5}$$

for $x \in [a,b]$, where S and T are real-valued.

Theorem 1. *There exists λ_0 such that any real-valued solution of* (1.1) *can be expressed as*

$$Z(x,\lambda) = c_1 e^{\int_a^x S(t,\lambda)dt}\cos\{c_2 + \int_a^x T(t,\lambda)dt\}$$

for $x \in [a,b]$ and $|\lambda| \geq \lambda_0$, where c_1 and $c_2 \in \mathbb{R}$. If Z satisfies (1.2) then it may be shown, as in [6], *that*

$$\begin{aligned} c_2 =: c_2^\alpha &= \frac{\pi}{2} \ \textit{if}\ \gamma = 0 \\ &= \tan^{-1}\left(\frac{1}{T(a,\lambda)}\{S(a,\lambda) + \cot\gamma\}\right) \ \textit{if}\ \gamma \neq 0. \end{aligned} \tag{2.6}$$

Similarly, if Z satisfies (1.3) then

$$\begin{aligned} c_2 =: c_2^b &= m\pi + \frac{\pi}{2} \text{ if } \beta = 0 \\ &= m\pi + \tan^{-1}\left(\frac{1}{T(b,\lambda)}\{S(b,\lambda) + \cot\beta\}\right) \text{ if } \beta \neq 0 \end{aligned} \tag{2.7}$$

for integral m. It follows from (2.6) and (2.7) that the eigenvalues of (1.1)–(1.3) are the values of λ for which

$$c_2^a + \int_a^b T(t,\lambda)dt = c_2^b. \tag{2.8}$$

Theorem 2. *There exists a function* $E(\lambda)$ *such that*

(i) $E(\lambda) \to 0$ *as* $|\lambda| \to \infty$;

(ii) $|r_j(x,\lambda)| \leq C_j E(\lambda)^j$ *for* $x \in [a,b]$ *and* $|\lambda| \geq \lambda_0$.

Theorems 1 and 2 give a means of developing expansions for real-valued solutions of (1.1) and, combined with (2.8), give a means of obtaining the asymptotic expansion of eigenvalues of (1.1)–(1.3).

3. The main idea

We consider the equation

$$y'' + (\mu^2 x^\alpha - q)y = 0 \text{ on } [a,b], \tag{3.1}$$

where y denotes a strictly complex-valued solution in the sense that neither the real nor the imaginary part of y is identically zero. Since μ^2 and q are real-valued there exist non-trivial, real-valued solutions y_1 and y_2 of (3.1) with

$$y(x,\lambda) =: y_1(x,\lambda) + iy_2(x,\lambda). \tag{3.2}$$

We may also write

$$y(x,\lambda) =: R(x,\lambda)e^{i\theta(x,\lambda)}, \tag{3.3}$$

where R and θ are real-valued.

Lemma 1. *If there exists* $x_0 \in [a,b]$ *with* $r(x_0,\lambda)^2\theta'(x_0,\lambda) \neq 0$, *then* y_1 *and* y_2 *are linearly independent.*

Proof. This follows from an examination of the Wronskian. ■

We note from (3.2) and (3.3) that

$$\frac{y'}{y} = \frac{R'}{R} + i\theta' \text{ for } x \in [a,b]. \tag{3.4}$$

If the conditions of Lemma 1 are satisfied, then the solution y of (3.2) is without zeros in $[a, b]$ since it is a complex linear combination of real, linearly independent solutions.

Lemma 2. *If there is an* x_0 *satisfying the conditions of Lemma 1, then any real-valued solution of (3.1) may be written as*

$$z(x, \lambda) = \rho(x, \lambda) \cos(\phi(x, \lambda)),$$

where ρ *and* ϕ *are real-valued functions such that for all* $x \in [a, b]$

$$\frac{\rho'}{\rho} = \frac{R'}{R} \text{ and } \phi' = \theta' \tag{3.5}$$

with R *and* θ *defined in (3.3).*

PROOF. This is [6, lemma 2]. ■

Guided by (3.4) we now seek a complex-valued solution, y, of (3.1) so that we may set

$$\frac{R'}{R} = Re\left\{\frac{y'}{y}\right\} =: S(x, \lambda) \text{ and } \theta' = Im\left\{\frac{y'}{y}\right\} =: T(x, \lambda).$$

The determination of S and T from any one complex-valued solution of (3.1) enables us, by Lemma 2, to describe all real-valued solutions of (3.1) up to additive contrasts.

We note that if y is a solution of (3.1) then the quotient, $\frac{y'}{y}$, satisfies the Riccati equation

$$v' = -\mu^2 x^\alpha + q - v^2. \tag{3.6}$$

It may be verified, as in [5], that

$$v(x, \lambda) := \sum_{n=0}^{\infty} r_n(x, \lambda) \tag{3.7}$$

is a series solution of (3.6), where the r_n functions are given in §2.

4. Bounds for r_n

Lemma 3. *If* $\mu = \lambda^{1/2}$ *there is a constant,* C*, independent of* λ, x *and* t *such that*

$$|e^{2\int_x^t r_0(s,\lambda)ds}| \le C \text{ for } a \le x < t \le b.$$

PROOF. This is, essentially, contained in [5, lemmas 4, 5 and 6]. The main idea is to write

$$[a, b] = [a, -A\lambda^{-\nu}] \cup [-A\lambda^{-\nu}, -B\lambda^{-\nu}] \cup [-B\lambda^{-\nu}, B\lambda^{-\nu}] \cup [B\lambda^{-\nu}, A\lambda^{-\nu}] \cup [A\lambda^{-\nu}, b]$$

and to bound the real part of $r_0(s,\lambda)$, the quotient of Bessel functions, on each interval. We omit the details. ■

We now need a result analogous to a uniform form of the Riemann–Lebesgue Lemma.

Lemma 4. *There exists a function* $E(\lambda)$ *so that in both cases* $\mu = \lambda^{1/2}$ *and* $\mu = i\lambda^{1/2}$

(i) $E(\lambda) \to 0$ *as* $|\lambda| \to \infty$;

(ii) $|r_1(x,\lambda)| \leq E(\lambda)$ *for all* $x \in [a,b]$.

Proof. In the case $\mu = \lambda^{1/2}$ the outline of the proof of this result is given in §6. The case $\mu = i\lambda^{1/2}$ is proved similarly. ■

Lemma 5. *If* C *is the constant of Lemma 3 and* λ_0 *is so large that* $8C(b-a)E(\lambda) \leq 1$ *for all* $\lambda \geq \lambda_0$, *then*

$$|r_n(x,\lambda)| \leq \frac{E(\lambda)}{2^{n-1}} \text{ for } x \in [a,b] \text{ and } \lambda \geq \lambda_0.$$

Proof. This follows by induction and Lemmas 3 and 4. ■

Lemma 6. *There exist constants* $\{c_n\}$, *independent of* λ, *with* $|r_n(x,\lambda)| \leq c_n E(\lambda)^n$ *for* $x \in [a,b]$ *and* λ *sufficiently large for* $n = 1, 2, \ldots$.

Proof. This follows from (2.2, 2.4) and Lemma 5 by induction. ■

It may also be shown inductively that

$$|r_{n+j}(x,\lambda)| \leq \frac{C_n}{2^J} E(\lambda)^n. \tag{4.1}$$

5. Applications to Eigenvalue problems

In order to use the previous results to derive estimates of the eigenvalues of (1.1)–(1.3) it is necessary to approximate $\int_a^b T(t,\lambda)dt$, where

$$T(t,\lambda) = Im\{r(t,\lambda)\} = Im\{r_0(t,\lambda)\} + \sum_{n=1}^{\infty} Im\{r_n(t,\lambda)\}. \tag{5.1}$$

In the light of Lemma 6 and (4.1) the error in truncating the series of (5.1) after N terms is less than

$$\sum_{n=N+1}^{\infty} |r_n(t,\lambda)| \leq c_{N+1} E(\lambda)^{N+1} \sum_{n=N+1}^{\infty} \frac{1}{2^{n-N}} = O(E(\lambda)^{N+1}). \tag{5.2}$$

We recall that, in the case $\mu = \lambda^{1/2}$, $r_0(x,\lambda) = \dfrac{\phi'}{\phi}$ where $\phi(x,\lambda) = x^{1/2}H_\nu^{(1)}(k^{-1}\lambda^{1/2}x^k)$, so

$$Im\left\{\int_a^b r_0(t,\lambda)dt\right\} = \arg\{\phi(b)\} - \arg\{\phi(a)\} = \arg\{b^{1/2}H_\nu^{(1)}(k^{-1}\lambda^{1/2}b^k)\}$$
$$- \arg\{i(-a)^{1/2}H_\nu^{(1)}(-ik^{-1}\lambda^{1/2}(-a)^k)\}. \tag{5.3}$$

From [1, 9.2.29] the first term on the right hand side of (5.3) has asymptotic expansion

$$2\nu\lambda^{1/2}b^k - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi + \left(\frac{4\nu^2-1}{16\nu\lambda^{1/2}b^k}\right) + \dots . \tag{5.4}$$

From [1, 9.2.27] the second term, for $\lambda \geq \lambda_0$, is equal to $\dfrac{\pi}{2}(2-\nu)$. Then $\displaystyle\int_a^b Im\{r_0(t,\lambda)\}dt$ has the asymptotic expansion

$$2\nu\lambda^{1/2}b^k - \frac{\pi}{2}\left(\frac{5}{2}-\nu\right) + \frac{4\nu^2-1}{16\nu b^k}\lambda^{-1/2} + \dots . \tag{5.5}$$

The estimation of $\displaystyle\int_a^b Im\{r_j(t,\lambda)\}dt$ depends on the particular function $q(x)$ of (1.1). An example of such a calculation is given in §7 below.

6. Outline of the proof of Lemma 4

We give an outline of the proof of Lemma 4 in the case where $\mu = \lambda^{1/2}$. The details are given in [10] using results from [3] and [11]. The proof for $\mu = i\lambda^{1/2}$ is similar.

We recall that

$$r_1(x,\lambda) = -\int_x^b e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt$$

and from Lemma 3 that

$$|e^{2\int_x^t r_0(s,\lambda)ds}| \leq C \text{ for } a \leq x \leq t \leq b. \tag{6.1}$$

It may be shown that for $s \geq B\lambda^{-\nu}$

$$r_0(s,\lambda) = i\lambda^{1/2}s^{\alpha/2} - \frac{(1-2\nu)k}{2s}(1+\epsilon(s,\lambda)) \tag{6.2}$$

and for $s \leq -B\lambda^{-\nu}$

$$r_0(s,\lambda) = -\lambda^{1/2}(-s)^{\alpha/2}\left[1 - \frac{(1-2\nu)}{2(-s)^k\lambda^{1/2}}(1+\epsilon(s,\lambda))\right], \tag{6.3}$$

where $|\epsilon(s,\lambda)| \leq \frac{1}{16}$ if B is sufficiently large.

We now choose σ so that $0 < \sigma < v$ and thus

$$\lambda^{-\sigma} > \lambda^{-v} \text{ and } 0 < k\sigma < kv = \frac{1}{2}. \tag{6.4}$$

We write

$$[a,b] = [a, -\lambda^{-\sigma}] \cup [-\lambda^{-\sigma}, \lambda^{-\sigma}] \cup [\lambda^{-\sigma}, b] =: J_1 \cup J_2 \cup J_3.$$

For $x \in J_3$ we approximate q by $f \in C^1[a,b]$ with $\displaystyle\int_a^b |f - q| dt < \epsilon$ and

$$r_1(x,\lambda) = -\int_x^b e^{2\int_x^t r_0(s,\lambda)ds} f(t)dt + \int_x^b e^{2\int_x^t r_0(s,\lambda)ds}(f(t) - q(t))dt. \tag{6.5}$$

The second term of (6.5) is less than $C\epsilon$ by (6.1) and the first term, by an integration by parts argument, may be shown, using (6.2), to be $o(1)$ as $\lambda \to \infty$.

For $x \in J_2$

$$r_1(x,\lambda) = -\int_x^{\lambda^{-\sigma}} e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt - e^{2\int_x^{\lambda^{-\sigma}} r_0(s,\lambda)ds} \int_{\lambda^{-\sigma}}^t e^{2\int_{\lambda^{\sigma}}^t r_0(s,\lambda)ds} q(t)dt. \tag{6.6}$$

The analysis of the second term of (6.6) is similar to that for $x \in J_3$ and, by (6.1), the first term is less than $C \displaystyle\int_{-\lambda^{-\sigma}}^{\lambda^{-\sigma}} |q(t)| dt = o(1)$ as $\lambda \to \infty$, since $q \in L^1[a,b]$.

For $x \in J_1$

$$r_1(x,\lambda) = -\int_x^{\lambda^{-\sigma}} e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt - \int_{-\lambda^{-\sigma}}^{\lambda^{-\sigma}} e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt - \int_{\lambda^{-\sigma}}^b e^{2\int_x^t r_0(s,\lambda)ds} q(t)dt. \tag{6.7}$$

The last two terms of (6.7) may be handled in the same way as for $x \in J_2$ and J_3. The first term may be shown to be $O(e^{-\lambda^{\epsilon}})$ for some $\epsilon > 0$ by (6.3).

7. An example

We consider the example of $q(t) = |t|^{-1/2}$, $a = -1$, $b = 1$ with Dirichlet boundary conditions at a and b. We first derive a suitable function, $E(\lambda)$. In the notation of §6 we have that, for $x \in J_3$,

$$\begin{aligned} |r_1(x,\lambda)| &\le C \sup_{t \ge \lambda^{-\sigma}} \left| \frac{t^{-1/2}}{r_0(t,\lambda)} \right| + C \sup_{x \ge \lambda^{-\sigma}} \int_x^1 \frac{t^{-3/2}}{|r_0(t,\lambda)|} + t^{-1/2} \frac{|r_0'(t,\lambda)|}{|r_0(t,\lambda)|^2} dt \\ &\le C \sup_{t \ge \lambda^{-\sigma}} \left| \lambda^{-1/2} t^{-1/2-\alpha/2} \right| + C \sup_{x \ge \lambda^{-\sigma}} \int_x^1 \lambda^{-1/2} t^{-3/2-\alpha/2} + \lambda^{-1/2} t^{k-1/2} dt \\ &\le C \sup_{x \ge \lambda^{-\sigma}} \lambda^{-1/2} x^{-1/2-\alpha/2} \\ &= C\lambda^{-1/2+\frac{\sigma}{2}(\alpha+1)}. \end{aligned} \tag{7.1}$$

For $x \in J_2$, we have from (6.6) and (7.1) that

$$
\begin{aligned}
|r_1(x,\lambda)| &\le C\lambda^{-1/2+\frac{\sigma}{2}(\alpha+1)} + C\int_{-\lambda^{-\sigma}}^{\lambda^{-\sigma}} |t|^{-1/2}dt \\
|r_1(x,\lambda)| &\le C\lambda^{-1/2+\frac{\sigma}{2}(\alpha+1)} + C\lambda^{-\sigma/2}.
\end{aligned} \tag{7.2}
$$

For $x \in J_1$ we get the bound of (7.2) together with an exponentially small error term. It follows from (7.1) and (7.2) that for λ sufficiently large we may take $E(\lambda)$ to be the term on the right hand side of (7.2), and the two exponents are equal if $\sigma = v$. This is precluded by the restriction that $0 < \sigma < v$ so we take

$$E(\lambda) = \lambda^{-v/2+\frac{\epsilon}{4}} \text{ for } \epsilon > 0.$$

From (5.2)

$$T(t,\lambda) = \sum_{n=0}^{3} Im\{r_n(t,\lambda)\} + O(\lambda^{-2v+\epsilon})$$

whence, from (5.5) and [1, 9.2.20],

$$\int_{-1}^{1} Im\{r_0(x,\lambda)\}dx = 2v\lambda^{-1/2} - \frac{\pi}{2}\left(\frac{5}{2} - v\right) + \left(\frac{4v^2-1}{16v}\right)\lambda^{-1/2} + O(\lambda^{-1}). \tag{7.3}$$

It may be shown that

$$
\begin{aligned}
\int_{-1}^{1} r_1(x,\lambda)dx &= -\int_{-1}^{1} \frac{1}{xH_v^{(1)}(k^{-1}\lambda^{1/2}x^k)^2}\int_x^1 tH_v^{(1)}(k^{-1}\lambda^{1/2}t^k)^2|t|^{-1/2}dtdx \\
&= (2v)^{2-3v}\lambda^{-\frac{3v}{2}}\left\{\int_0^\infty \frac{1}{vH_v^{(1)}(v)^2}\int_v^\infty u^{3v-1}H_v^{(1)}(-iu)^2dudv\right. \\
&\quad + \left(\int_0^\infty \frac{1}{vH_v^{(1)}(-iv)^2}v\right)\left(\int_0^\infty u^{3v-1}H_v^{(1)}(u)^2du\right) \\
&\quad + \left(\int_0^\infty \frac{1}{vH_v^{(1)}(-iv)^2}dv\right)\left(\int_0^\infty u^{3v-1}H_v^{(1)}(u)^2du\right) \\
&\quad \left. + \int_0^\infty \frac{1}{vH_v^{(1)}(-iv)^2}\int_0^v u^{3v-1}H_v^{(1)}(-iu)^2dudv\right\} + o(\lambda^{-\frac{3v}{2}}) \\
&=: (A+iB)\lambda^{-\frac{3v}{2}} + o(\lambda^{-\frac{3v}{2}}).
\end{aligned} \tag{7.4}
$$

It may further be shown that

$$\int_{-1}^{1} r_2(x,\lambda)dx = O(\lambda^{-3v}) \text{ and } \int_{-1}^{1} r_3(x,\lambda)dx = O(\lambda^{-\frac{9}{2}v}).$$

We thus have the result that

$$\int_{-1}^{1} T(t,\lambda)dt = 2\nu\lambda^{1/2} - \frac{\pi}{2}\left(\frac{5}{2}-\nu\right) + \left(\frac{4\nu^2-1}{16\nu}\right)\lambda^{-1/2} + B\lambda^{\frac{-3\nu}{2}} + O(\lambda^{-1}) + O(\lambda^{-3\nu}). \tag{7.5}$$

Since $0 < \nu < \frac{1}{2}$ because $\alpha > 0$, the $B\lambda^{-\frac{3\nu}{2}}$ term is more significant than the λ^{-1} term for all α and $\lambda^{-3\nu} = o(\lambda^{-1})$ if $\nu > \frac{1}{3}$, which occurs when $\alpha < 1$. In the case $\alpha \geq 1$ the error term would be improved by taking more terms from the expansion (5.5). Higher order terms may be added by taking more terms from the series for $T(t,\lambda)$, which may be calculated from the formulae in §2.

In the case of the Sturm–Liouville problem $y'' + (\lambda x^\alpha + |x|^{-1/2})y = 0$ on $[-1, 1]$ with $0 < \alpha < 1$ and Dirichlet boundary conditions at -1 and 1, we see from (2.8) that the positive eigenvalues satisfy the relation

$$m\pi = 2\nu\lambda_m^{1/2} - \frac{\pi}{2}\left(\frac{5}{2}-\nu\right) + \left(\frac{4\nu^2-1}{16\nu}\right)\lambda_m^{-1/2} + B\lambda_m^{-\frac{3\nu}{2}} + O(\lambda_m^{-1})$$

and may thus be approximated by reversion. Other boundary conditions may be considered by means of the formulae (2.6) and (2.7).

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