TOPICS IN PROBABILITY USING GENERALISED RIEMANN INTEGRATION

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Abstract

The theory of probability is classically based on measure theory. This paper treats a number of topics in the classical theory from the viewpoint of generalised Riemann integration. The terminology used is intended to be suggestive of the corresponding terminology in classical theory.

1. Random spaces

Let Ω be a sample space and Q a function defined on a class of subsets of Ω. We call (Ω, Q) a *random space*. A set $I \subseteq \Omega$ for which $Q(I)$ is defined is called an *outcome*. We call Q a *quasi-probability*, or *q-probability*, and give three examples.

1. If the class $\mathscr{I}$ of outcomes is a σ-field containing Ω, and if Q is a probability measure, then $(\Omega, \mathscr{I}, Q)$ is a probability space.
2. Let $\Omega = \mathbf{R}^{[0,\infty)}$; let the class of outcomes contain the cylinder sets

   $$I = \{x : u_j \leq x(t_j) < v_j,\ 0 = t_0 < t_1 < \ldots < t_n < \infty\};$$

   and let $Q(I)$ be

   $$\int_{u_1}^{v_1} \ldots \int_{u_n}^{v_n} \exp\left(\frac{c}{2}\sum_{j=1}^{n}\frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}\right)\prod_{j=1}^{n}\left(\frac{2\pi}{-c}(t_j - t_{j-1})\right)^{-\frac{1}{2}} dx_1 \ldots dx_n,$$

   where $c = a + \imath b,\ a \leq 0,\ b \geq 0,\ c \neq 0$. This set function is discussed in detail in [5]. In the case $c = \imath := \sqrt{-1}$, the function Q is called a *probability amplitude* by R.P. Feynman in [2]. When $c = -1$, Q is the volume function induced on the cylinder sets by Brownian motion, and Q generates a probability measure (the Wiener measure) on the σ-algebra of Borel sets in $\mathbf{R}^{[0,\infty)}$.
3. Take I to be a cylinder set as before, and take $Q(I)$ to be the following function:

   $$\int_{u_1}^{v_1} \ldots \int_{u_n}^{v_n} \exp\left(-\frac{1}{2}\sum_{j=1}^{n}\frac{(\ln x_j - \ln x_{j-1})^2}{t_j - t_{j-1}}\right)\prod_{j=1}^{n}\left(\left(2\pi(t_j - t_{j-1})\right)^{-\frac{1}{2}}\frac{dx_j}{x_j}\right).$$

   This is the volume function induced on the cylinder sets by a geometric Brownian process such as that arising in the Black–Scholes theory of the prices of derivative assets [4].

Classical probability has a theory of random variables. In the random space structure we talk of *observables*, in order to avoid confusion between the two concepts. An *observable* is a function $X : \Omega \longmapsto \mathbf{R}$ such that, for any real interval I, the set

$$[X \in I] := \{\omega : X(\omega) \in I\}$$

is an outcome. Thus if $(\Omega, \mathbf{I}, Q)$ is a probability space, X is a random variable. The *distribution function* F_X of an observable X is defined as

$$F_X(x) = Q[X < x],\ (x \in \mathbf{R}), \quad F_X(u,v) = Q[u \leq X < v],\ (u < v),$$

or, for any interval I,

$$F_X(I) := Q[X \in I].$$

The *expectation* of X is

$$E(X) := \int_{\mathbf{R}} xF_X(I),$$

where the integral, if it exists, is the generalised Riemann integral (also called the gauge integral or Henstock integral). See [3], [5].

2. The generalised Riemann integral

For quick reference, we give a definition of the generalised Riemann integral in $\mathbf{R}$. Let $\delta(x)$ be a positive function defined for $x \in \overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$. The function δ is called a *gauge*. Given $x \in [-\infty, \infty]$, and u, $v \in (-\infty, \infty)$ we say that the real interval I is *attached* to x if

$$\begin{aligned} I &= [u,v) \text{ and } x = u \text{ or } v; \text{ or} \\ I &= (-\infty, v) \text{ and } x = -\infty; \text{ or} \\ I &= [u, \infty) \text{ and } x = \infty. \end{aligned}$$

If I is attached to x we say the point-interval pair (x, I) is *δ-fine* if

$$v - u < \delta(x);\ v < -\frac{1}{\delta(x)}; \quad \text{or} \quad u > \frac{1}{\delta(x)}, \text{ respectively.}$$

A finite collection of attached point-interval pairs $\mathscr{E} = \{(x, I)\}$ is a *division* of $\mathbf{R}$ if the finite set of intervals I forms a non-intersecting cover for $\mathbf{R}$. The division $\mathscr{E}$ is δ-fine if each (x, I) in $\mathscr{E}$ is δ-fine. Note that, as $\delta(x)$ decreases, the number of partition points of $\mathbf{R}$ increases.

Now suppose h is a real- or complex-valued function of point-interval pairs (x, I), with $h(x, I) := 0$ if $x = -\infty$ or ∞. For example, $h(x, [u, v))$ could be a point function $f(x)$ multiplied by the interval length $v - u$. Then h is integrable over $\mathbf{R}$ in the generalised Riemann sense, and has integral α if, given $\varepsilon > 0$, there exists a gauge δ such that the Riemann sum of h over $\mathscr{E}$ satisfies

$$\left| (\mathscr{E}) \sum h(x, I) - \alpha \right| < \varepsilon$$

for every δ-fine division $\mathscr{E}$ of $\mathbf{R}$. We use the notation $(\mathscr{E})\sum h(x,[u,v))$ to denote the Riemann sum given by the division $\mathscr{E}$, and $\int_{\mathbf{R}} h(x,I)$ to denote the generalised Riemann integral of h. Note that the generalised Riemann integral over the domain $(-\infty,\infty)$ does not require use of the Cauchy extension.

Suppose h has the form $f(x)m([u,v))$, and m is a measure. Whenever f is finite, real-valued, m-measurable and Lebesgue integrable over $\mathbf{R}$ with respect to m, then $f(x)m([u,v))$ is generalised Riemann integrable [1], and

$$\int_{\mathbf{R}} f(x)m([u,v)) = \int_{-\infty}^{\infty} f(x)dm,$$

where the latter integral is the Lebesgue, and the former is the generalised Riemann integral.

So, if we are dealing with a probability space and if X is a random variable, then the expectation of X as defined above has the same value as that given by the classical Lebesgue definition.

3. Variance and Chebyshev's inequality

Suppose X is an observable and h is a real- or complex-valued function of the range of X, then the expectation of h is $E(h(X)) := \int_{\mathbf{R}} h(x)F_X([u,v))$ if the generalised Riemann integral exists. If $E(X)$ and $E(X^2)$ exist, the variance of X is

$$\mathrm{var}(X) := E\left((X-E(X))^2\right) = E(X^2) - (E(X))^2.$$

While the complete classical theory is given by the special case where $(\Omega, \mathbf{I}, Q)$ is a probability space, the theorems of classical probability theory have more general counterparts in the random space structure. To illustrate this, we prove a version of Chebyshev's Inequality.

If $E(X)$ and $E(X^2)$ exist, and if Q is real-valued, non-negative and finitely additive, then

$$Q\left[|X-E(X)| \geq \varepsilon\right] \leq \frac{\mathrm{var}(X)}{\varepsilon^2}.$$

PROOF. $Q \geq 0$ implies $F_X(I) \geq 0$. Let S denote $\{x \in \mathbf{R} : |x-E(X)| \geq \varepsilon\}$. The existence of

$$\int_{\mathbf{R}} (x-E(X))^2 F_X(I)$$

implies the existence of

$$\int_{S} (x-E(X))^2 F_X(I)$$

by [3, theorem 5.1, p. 51]. As Q is finitely additive, each Riemann sum approximation of $\int_S F_X(I)$ yields $Q[|X - E(X)|] \geq \varepsilon$. So

$$\begin{aligned}\operatorname{var}(X) &= \int_{\mathbf{R}} (x - E(X))^2 F_X(I) \\ &\geq \int_S (x - E(X))^2 F_X(I), \text{ since } F_X(I) \geq 0, \\ &\geq \varepsilon^2 \int_S F_X(I) \text{ by definition of } S, \\ &= \varepsilon^2 Q[|X - E(X)| \geq \varepsilon]. \quad \blacksquare\end{aligned}$$

As the following discussion shows, the theory of finitely and infinitely many observables is also analogous to the corresponding theory of random variables.

4. Independent observables

Let (Ω, Q) and observables $X_1, \dots, X_n$, be given, so that, writing $\mathbf{X} = (X_1, \dots, X_n)$,

$$\mathbf{X} : \Omega \longmapsto \mathbf{R}^n.$$

We say that $\mathbf{X}$ is an *observable vector* if

$$[X_1 \in I_1] \cap \ldots \cap [X_n \in I_n]$$

is an outcome.

If $I := I_1 \times \ldots \times I_n$ is any rectangular interval or parallelepiped in $\mathbf{R}^n$, the distribution function $F_{\mathbf{X}}$ of $\mathbf{X}$ is defined as

$$F_{X_1,\dots,X_n}(I) := Q([X_1 \in I_1] \cap \ldots \cap [X_n \in I_n]).$$

We say that the observables $X_1, \dots, X_n$ are *independent* if $F_{\mathbf{X}}(I)$ is the product of the n distribution functions $F_{X_1}(I_1), \dots, F_{X_n}(I_n)$. Given a function h of $\mathbf{X}$, the expectation of h is given by the generalised Riemann integral

$$E(h) := \int_{\mathbf{R}^n} h(x_1, \dots, x_n) F_{\mathbf{X}}(I),$$

where this integral is defined as in [5, pp 11–12].

To address the case of infinitely many observables, let Λ be an indexing set, and $\{X_\lambda : \lambda \in \Lambda\}$ a family of observables. We say that $\mathbf{X} := (X_\lambda)_{\lambda \in \Lambda}$ is an observable vector if, for each finite set $N := \{\lambda_1, \dots, \lambda_n\} \subseteq \Lambda$, $(X_{\lambda_1}, \dots, X_{\lambda_n})$ is an observable vector.

Writing F_N for $F_{X_{\lambda_1},\dots,X_{\lambda_n}}$ the distribution of $\mathbf{X}$ is the family of interval functions defined on $\mathbf{R}^N$,

$$\{F_N(I_{\lambda_1} \times \ldots \times I_{\lambda_n}) : N \subset \Lambda\}.$$

If h is a function of $\mathbf{X}$ then the expectation of h is

$$E(h) = \int_{\mathbf{R}^\Lambda} h(\mathbf{X}) F_N(I(N)),$$

where the integral is a generalised Riemann integral, the definition and meaning of which are as follows.

5. The generalised Riemann integral in infinite dimensions

We start by considering cylindrical intervals in $\mathbf{R}^\Lambda$. Let $I(N)$ be any n-dimensional interval, or parallelepiped, in $\mathbf{R}^N$,

$$I(N) := I_{\lambda_1} \times \ldots \times I_{\lambda_n},$$

where each I_{λ_j} is a real interval of the form $[u, v)$, $(-\infty, v)$ or $[u, +\infty)$; and $I(N)$ is attached to $x(N) := (x(\lambda_1), \ldots, x(\lambda_n))$ if $x(\lambda_j) = u$ or v, $-\infty$, or ∞, respectively. Let

$$I = I[N] := I(N) \times \mathbf{R}^{\Lambda \setminus N}.$$

We write $I[N]$ instead of I whenever we wish to emphasise that N is the set of dimensions in which the cylindrical interval I is restricted. We say that $I[N]$ is attached to x if $I(N)$ is attached to $x(N)$.

If f is a functional of $x \in \mathbf{R}^\Lambda$ and m is a volume function defined on cylindrical intervals $I[N]$, we consider Riemann sums

$$\sum f(x) m(I[N]),$$

which will, in some sense, approximate an integral in $\mathbf{R}^\Lambda$, which we wish to define. Each $I[N]$ is attached to the corresponding x in the Riemann sum; $f(x)$ is zero by definition if x is a point at infinity. The cylindrical intervals $I[N]$ in the Riemann sum form a disjoint finite cover for $\mathbf{R}^\Lambda$. Note that the dimension sets N are variable in the Riemann sum.

More generally, we consider functionals $h(x, I[N])$, and Riemann sums

$$\sum h(x, I[N])$$

approximating the integral $\int_{\mathbf{R}^\Lambda} h$, which we have yet to define.

To obtain convergence of the Riemann sums, we must examine sequences of Riemann sums in which the cylindrical intervals $I[N]$ 'shrink' according to some rule. Suppose the cylindrical interval $J[M] \subseteq I[N]$ has $M \supseteq N$ and each restricted edge of $J[M]$ is no greater than the corresponding edge of $I[N]$. Then $J[M]$ is a proper subset of $I[N]$ if M contains N as a proper subset, or each restricted edge of $J[M]$ is strictly shorter than the corresponding edge of $I[N]$, or both.

Guided by this, we obtain a 'shrinking rule' or gauge for the Riemann sums as follows. For each $x \in \overline{\mathbf{R}^\Lambda}$ (that is, $\mathbf{R}^\Lambda$ with points at infinity adjoined), let $\delta(x)$ be a positive number. If Λ is uncountable, let A be a countable, subset of Λ. If Λ is countable, take $A = \Lambda$. Let $\mathbf{L}_A$ be the family of finite subsets of A and, for each $x \in \overline{\mathbf{R}^\Lambda}$, let $L_A(x)$ be a member of $\mathbf{L}_A$. So

$$\delta : \overline{\mathbf{R}^\Lambda} \longmapsto \mathbf{R}^+,$$
$$L_A : \overline{\mathbf{R}^\Lambda} \longmapsto \mathbf{L}_A.$$

A *gauge* γ is defined as

$$\gamma := (\delta,\, L_A).$$

Suppose $I[N]$ is attached to x. We say that $(x, I[N])$ is γ-*fine* if N contains $L(x)$ and if, taking $\delta(x(\lambda)) = \delta(x)$, the one-dimensional intervals I_λ in $\mathbf{R}$ are δ-fine for each $\lambda \in N$.

Thus, any gauge γ depends on a choice of positive numbers $\delta(x)$, one for each $x \in \overline{R^\Lambda}$, a choice of a countable dimension set $A \subset \Lambda$, and a choice of finite subsets $L_A(x)$ of A, one for each $x \in \overline{R^\Lambda}$.

(If Λ were a finite set of order n, then we would take $L_A(x) = A = \Lambda$ for every x, and likewise $N = \Lambda$, so that every cylindrical interval $I[N]$ and every parallelepiped $I(N)$ is just the usual rectangular interval in the fixed n-dimensional space $\mathbf{R}^n$; and a gauge γ reduces to the familiar gauge δ defined at each point of $\overline{\mathbf{R}^n}$.)

The role of the countable set A in the definition of a gauge is to ensure that $\mathbf{R}^\Lambda$ can be covered by a finite set of mutually exclusive γ-fine cylindrical intervals $I[N]$. See [6] for a proof of this fundamental result.

So, given a gauge γ we can find a finite collection of attached point-interval pairs

$$\mathscr{E} = \{(x, I[N])\}$$

in which each $(x, I[N])$ is γ-fine, and the cylindrical intervals $I[N]$ are mutually exclusive and have union $\mathbf{R}^\Lambda$. We call $\mathscr{E}$ a γ-fine *division* of $\mathbf{R}^\Lambda$ and denote the Riemann sum by $(\mathscr{E})\sum h(x, I[N])$.

We say that h is integrable in $\mathbf{R}^\Lambda$ with generalised Riemann integral α if given $\varepsilon > 0$ we can choose a gauge γ so that, for every γ-fine division $\mathscr{E}$ of $\mathbf{R}^\Lambda$, we have

$$\left|(\mathscr{E})\sum h(x, N, I[N]) - \alpha\right| < \varepsilon,$$

and we write

$$\alpha = \int_{\mathbf{R}^\Lambda} h.$$

We take h to be zero by definition if x is a point at infinity. Note that the functions (or functionals) h may depend explicitly on the dimension sets N, as illustrated in Examples 2 and 3 in Section 1. The integrability of such functions is demonstrated in [5, chapters 3 and 4].

For proofs of Fubini's Theorem, the dominated convergence theorem and other properties of the integral, see [5, chapter 2].

6. Kolmogorov Extension Theorem

Theorem 1. *For $N \subset \Lambda$ and $M \subseteq N$, let the functions $F_N(I(N))$ on $\mathbf{R}^N$ satisfy the consistency condition*

$$F_M(I(M)) = F_N(\mathbf{R}^{N\setminus M} \times I(M)).$$

Then there exists a random space (Ω, Q) and a family $\mathbf{X}$ of observables $\{X_\lambda : \lambda \in \Lambda\}$ for which $\{F_N : N \subseteq \Lambda\}$ is the distribution.

PROOF. Take $\Omega = \mathbf{R}^\Lambda$, so $\omega \in \Omega$ is identified with $x \in \mathbf{R}^\Lambda$, and let $X_\lambda(\omega) := x(\lambda)$. Let

$$Q(I[N]) := F_N(I(N)).$$

The consistency condition ensures that Q is well defined. Then

$$Q[\mathbf{X} \in I[N]] = F_N(I(N)),$$

so $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ is an observable for each N, and $\{F_N : N \subset \Lambda\}$ is the distribution of the family of observables $\{X_\lambda : \lambda \in \Lambda\}$. ■

Theorem 1 is a generalisation of the Kolmogorov Extension Theorem and is valid under broader conditions. For instance, Theorem 1 holds for Example 2 in Section 1, where the classical version of Kolmogorov's Extension Theorem fails.

Given any functional f of $(X_\lambda)_{\lambda\in\Lambda}$, the expectation $E(f)$ of f is given by the generalised Riemann integral

$$\int_{\mathbf{R}^0\Lambda} f(x)Q(I[N]) \quad \text{or} \quad \int_{\mathbf{R}^\Lambda} f(x)F_N(I(N)).$$

In classical probability theory the expectations are obtained by proving the existence of a probability measure on the σ-algebra $\mathscr{B}$ generated by the cylindrical intervals $I[N]$. Using the terminology of [7, p. 259], let $\mathscr{C}$ denote the semi-algebra of cylindrical intervals I, along with $\mathbf{R}^\Lambda$ and $\emptyset$ (the empty set).

The following argument was contributed by V. A. Skvortsov.

Let μ be a probability pre-measure on $\mathscr{C}$ satisfying the following conditions:

1. μ is continuous in the sense that $\mu(I) \to 0$ if the maximum of the lengths of the bounded edges of I tends to zero,
2. μ is countably subadditive; that is, if $I_j \in \mathscr{C}$ and $A = \cup_{j=1}^\infty I_j$, then

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(I_j)$$

whenever $\mu(A)$ is defined.

Conditions 1 and 2 above are satisfied if we take μ to be the Q given in examples 1, 3 and 2 (with $c = -1$) in Section 1).

Use the Lebesgue construction of outer measure μ^* generated by μ on $\mathscr{C}$. The resulting measure on the Borel σ-algebra $\mathscr{B}$ generated by $\mathscr{C}$ coincides with μ^* (see [7, theorem 8, p. 257, and proposition 9, p. 260]) and with the probability measure generated according to the Kolmogorov Extension Theorem [4, p. 50]. So μ^* is a probability measure on $(\mathbf{R}^A, \mathscr{B})$.

Lemma 2. *If S is μ^*-measurable then, for every $\varepsilon_0 > 0$, there exist $\{I^{(j)}\}_{j=1}^{\infty}$, each $I^{(j)}$ being an open cylindrical interval, such that*

$$S \subseteq \cup_{j=1}^{\infty} I^{(j)}, \;\; \mu^*(\cup I^{(j)}) \leq \sum \mu(I^{(j)}) \leq \mu^*(S) + \varepsilon_0.$$

Proof. Follows from the definition of μ^* and the continuity of the pre-measure. ■

7. Lebesgue integrability implies generalised Riemann integrability

To show that the theory of observables is a generalisation of the theory of random variables, we prove that, if f is a function of random variables X_λ, the existence of the expectation $E(f)$ in the Lebesgue sense implies the existence of $E(f)$ in the Henstock sense, and the two are equal.

We show that if m is a measure on $(\mathbf{R}^A, \mathscr{B})$ and f is a finite, real-valued, m-measurable function of $x \in \mathbf{R}^A$, and if f is integrable with respect to m in the Lebesgue sense with integral α, then $f(x)m(I)$ is integrable in the generalised Riemann sense and the two integrals are equal. The proof follows that in [1]. Let $\overline{J}$ denote the closure of J in the Cartesian product topology in $\mathbf{R}^A$. We use the following lemma.

Lemma 3. *Under the above conditions, given $\varepsilon > 0$, for each $x \in \mathbf{R}^A$ there is an interval $I(x)$ of $\mathbf{R}^A$ such that, whenever $J^{(1)}, J^{(2)}, \ldots$ are disjoint sets in $\mathscr{B}$ with*

$$m(\mathbf{R}^A \setminus \cup_{j=1}^{\infty} J^{(j)}) = 0,$$

and $x^{(1)}, x^{(2)}, \ldots$ are points satisfying

$$x^{(j)} \in \overline{J^{(j)}} \subseteq I(x^{(j)}), \;\; j = 1, 2, 3, \ldots,$$

then

$$\left| \sum_{j=1}^{\infty} f(x^{(j)}) m(J^{(j)}) - \alpha \right| < \varepsilon.$$

Proof. Choose η so that $Y \in \mathscr{B}$ and $m(Y) < \eta$ imply

$$\int_Y |f| \, dm < \frac{\varepsilon}{3}.$$

Let $\xi = \varepsilon/(3\eta + 3)$, and let

$$S_k = \{x : (k-1)\xi < f(x) \leq k\xi\},$$

where k is any integer. Apply Lemma 2 with $S = S_k$ and

$$\varepsilon_0 = \frac{\eta}{2^{|k|+2}(|k|+1)},$$

and find $I^{(j,k)}$ such that

$$m\left((\cup_{j=1}^{\infty} I^{(j,k)}) \setminus S_k\right) = m(\cup_j I^{(j,k)}) - m(S_k) \leq \varepsilon_0.$$

Suppose $x \in S_k$. Then $x \in I^{(j,k)}$ for some $j = j(x)$. Let

$$I(x) := I^{(j(x),k)}.$$

Then

$$\begin{aligned}\left|\sum_{j=1}^{\infty} f(x^{(j)})m(J^{(j)}) - \alpha\right| &= \left|\sum_{j=1}^{\infty} \int_{J^{(j)}} (f(x^{(j)}) - f(x))dm\right| \\ &\leq \sum_{j=1}^{\infty} \int_{J^{(j)}} |f(x^{(j)} - f(x)|dm \\ &\leq p + q + r\end{aligned}$$

where, with $x^{(j)} \in S_{k(j)}$,

$$p = \sum_{j=1}^{\infty} \int_{J^{(j)} \cap S_{k(j)}} |f(x^{(j)}) - f(x)|dm,$$

$$q = \sum_{j=1}^{\infty} \int_{J^{(j)} \setminus S_{k(j)}} |f(x^{(j)})|dm,$$

and

$$r = \sum_{j=1}^{\infty} \int_{J^{(j)} \setminus S_{k(j)}} |f(x)|dm.$$

Now $x \in J^{(j)} \cap S_{k(j)}$ implies $(k(j) - 1)\xi < f(x)$ and $f(x^{(j)}) \leq k(j)\xi$, so

$$|f(x^{(j)}) - f(x)| \leq \xi,$$

and

$$p \leq \sum_{j=1}^{\infty} \int_{J^{(j)} \cap S_{k(j)}} \xi dm \leq \xi \sum_{j=1}^{\infty} m(J^{(j)}) = \xi < \frac{\varepsilon}{3}.$$

Since $x^{(j)} \in S_{k(j)}$ and $k(j) = k$ imply $f(x_j) \leq k\xi$, $|f(x^{(j)})| \leq |k|\xi \leq (|k| + 1)\xi$, giving

$$\begin{aligned} q &= \sum_{k=-\infty}^{\infty} \sum_{k(j)=k} \int_{J^{(j)}\setminus S_k} |f(x^{(j)})| dm \leq \sum_{k=-\infty}^{\infty} \sum_{k(j)=k} (|k|+1)\xi m(J^{(j)}\setminus S_k) \\ &\leq \sum_{k=-\infty}^{\infty} (|k|+1)\xi m(\cup_{j=1}^{\infty} I^{(j,k)}\setminus S_k) = \sum_{k=-\infty}^{\infty} (|k|+1)\xi \frac{\eta}{2^{|k|+2}(|k|+1)} \\ &= \xi\eta \frac{1}{2^2}\left(\sum_{-\infty}^{0} \frac{1}{2^{|k|}} + \sum_{1}^{\infty} \frac{1}{2^{|k|}}\right) = \frac{\varepsilon\eta}{3\eta+3}\left(\frac{3}{4}\right) < \frac{\varepsilon}{3}. \end{aligned}$$

The sets $J^{(j)}\setminus S_{k(j)}$ are disjoint; their union Y satisfies

$$\begin{aligned} m(Y) &= \sum_{k=-\infty}^{\infty} \sum_{k(j)=k} m(J^{(j)}\setminus S_k) \\ &\leq \sum_{k=-\infty}^{\infty} m(\cup_{j=1}^{\infty} I^{(j,k)}\setminus S_k) \\ &< \eta \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|+2}(|k|+1)} \\ &< \eta. \end{aligned}$$

So

$$r = \int_Y |f(x)| dm < \frac{\varepsilon}{3},$$

giving the required result. ■

Theorem 4. *If f is finite, real-valued, m-measurable and Lebesgue integrable with respect to m with integral α, then $f(x)m(I[N])$ is generalised Riemann integrable, and the two integrals are equal.*

Proof. Denote by N_I the dimension set N in which an interval $I = I[N]$ is restricted. Define a gauge γ as follows. In Lemma 3 above let $A = \cup_{j,k} N_{I^{(j,k)}}$. For $x \in S_k$, let

$$N(x) = N_{I(x)} = N_{I^{(j(x),k)}},$$

and choose $L_A(x) \subset A$ so that $L_A(x) \supseteq N(x)$. Let $\delta(x)$ be less than the length of each restricted edge of $I(x)$. Thus, if $I(N)$ is δ-fine in $\mathbf{R}^N$ and $N \supseteq L(x)$, we have $I[N] \subseteq I(x)$. Then, for each γ-fine division $\mathscr{E}$ of $\mathbf{R}^A$, the cylindrical intervals $I[N]$ involved in $\mathscr{E}$ satisfy the conditions of Lemma 3, giving

$$\left|(\mathscr{E})\sum f(x)F_N(I(N)) - \alpha\right| < \varepsilon,$$

so f is generalised Riemann integrable, with integral α. ■

8. Brownian motion

A collection $\{X_t : t \in (t', t'')\}$ of random variables is called a stochastic process. If the X_t are observables, then to avoid confusion we use the term *random process* for such a collection.

A *Brownian motion* is a random process $\{X_t : 0 \leq t < \infty\}$ defined on a random space (Ω, Q) satisfying the following conditions.

1. If $0 = t_0 < t_1 < \ldots < t_n < \infty$, the observables $Z_j := X_{t_j} - X_{t_{j-1}}$ $(1 \leq j \leq n)$ are independent.
2. If $0 \leq s < t < \infty$, the observable $Z := X_t - X_s$ is normally distributed with variance $t - s$.
3. $Q[X_0 = 0] = 1$.
4. The sample paths are almost surely continuous.

Take $\Lambda = [0, \infty)$. For $0 = t_0 < t_1 < \ldots < t_n < \infty$, take $N = \{t_1, \ldots, t_n\}$ and, for $x \in \mathbf{R}^{[0,\infty)}$, write $x_j := x(t_j)$. Define F as follows.

$$\begin{aligned} F_{\{t_0\}}(x_0) &= 0, \;\; x_0 < 0, \\ &= 1, \;\; x_0 \geq 0, \end{aligned}$$

and, for $n \geq 1$ and for any parallelepiped $I(N) = [u_1, v_1) \times \ldots \times [u_n, v_n)$ in $\mathbf{R}^N$, $F_N(I(N))$ is

$$\int_{u_1}^{v_1} \ldots \int_{u_n}^{v_n} \exp\left(-\frac{1}{2}\sum_{j=0}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}\right) \prod_{j=0}^{n} \left(2\pi(t_j - t_{j-1})\right)^{-\frac{1}{2}} dx_1 \ldots dx_n.$$

Take $\Omega = \mathbf{R}^{[0,\infty)}$ with ω and x identified, and take as quasi-probability $Q'(I[N]) := F_N(I(N))$. Then, by familiar proofs [4], the first three conditions for Brownian motion are satisfied by the random process X_t where $X_t(\omega) := x(t),\ t \in [0, \infty)$.

Classically, the fourth condition is achieved by constructing a continuous modification of a system analogous to the one just described (see [4]). The corresponding proof in generalised Riemann integration runs as follows.

Theorem 5. *Brownian motion exists as a random process.*

PROOF. Let $C \subset \mathbf{R}^{[0,\infty)}$ denote the set of continuous functions on $[0, \infty)$. Let $J[M]$ be a cylindrical interval restricted in the finite dimension set M. Let C_M denote the set of x in $\mathbf{R}^{[0,\infty)}$ that are continuous at each t in M and let $\mathbf{1}_{C_M}$ denote the indicator functional of C_M. From [5, proposition 46, p. 61], the integral

$$\int_{J[M]} Q'(I[N] \cap C_M) := \int_{J[M]} \mathbf{1}_{C_M} Q'(I[N])$$

exists and we have

$$\int_{J[M]} Q'(I[N]) = \int_{J[M]} Q'(I[N] \cap C_M).$$

Let $Q(J[M]) := \int_{J[M]} Q'(I[N] \cap C_M)$, and let

$$B\,(I[N] \cap C) := Q(I[N]), \text{ and } B\,\left(I[N] \cap (\mathbf{R}^{[0,\infty)} \backslash C)\right) := 0.$$

Then the quasi-probability $B\left((I[N] \cap C) \cup \left(I[N] \cap (\mathbf{R}^{[0,\infty)} \backslash C)\right)\right)$ is defined as

$$B(I[N]) := B(I[N] \cap C) + B\left(I[N] \cap (\mathbf{R}^{[0,\infty)} \backslash C)\right)$$

and equals $Q(I[N])$. By [5, proposition 10, p. 28, and proposition 18, p. 32], the distributions determined by Q and Q' are the same, so the distribution determined by B is also $\{F_N : N \subset [0, \infty)\}$. Therefore we have a random space $(\mathbf{R}^{[0,\infty)}, B)$ and a random process $\{X_t : 0 \leq t < \infty\}$, as defined above, satisfying all four conditions for a Brownian motion. ■

Theorem 4 shows that the classical definition of Brownian motion is consistent with the one given here, and that B generates the Wiener measure on $\mathscr{B}$. For alternative proofs, and for a more detailed discussion of the 'everywhere continuous, nowhere differentiable' properties of Brownian motion, see [5, chapter 3].

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