

# PLANE GRAVITATIONAL WAVES AND BERTOTTI–ROBINSON SPACE-TIMES

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## ABSTRACT

We construct a solution of the vacuum Einstein–Maxwell field equations that is a simple example of a plane impulsive gravitational wave sharing its wave front with an electromagnetic shock wave.

## 1. Introduction

Plane impulsive gravitational waves sharing their wave fronts with electromagnetic shock waves appeared as a by-product of the work of Chandrasekhar and Xanthopoulos [2]. Their waves have the property of producing ‘a spray of gravitational and electromagnetic radiation’ [2] behind them, while propagating into a region devoid of electromagnetic or gravitational fields. Using a subfamily of the Bertotti–Robinson (BR) family of solutions of the Einstein–Maxwell vacuum field equations [1], [5] we demonstrate a construction of a plane impulsive gravitational wave sharing its wave front with an electromagnetic shock wave. The history of this wave is a null hyperplane and the space-time to the future of it is *conformally flat* while the space-time to the past of it is flat.

Our construction begins with a brief summary of the BR space-times in §2. Plane gravitational waves are introduced into a subfamily of these BR space-times in §3 and are specialised to a single impulsive plane gravitational wave. In §4 a further specialisation results in this latter wave sharing its wave front with a plane electromagnetic shock wave.

## 2. Bertotti–Robinson space-times

We take the line-element of the BR space-times in the form [4]

$$ds^2 = -\cos^2(au - bv) dx^2 - \cos^2(au + bv) dy^2 + 2 du dv, \quad (2.1)$$

where  $a, b$  are constants. A convenient half-null tetrad is then defined via the 1-forms

$$\mathfrak{g}^1 = \cos(au - bv) dx, \quad \mathfrak{g}^2 = \cos(au + bv) dy, \quad (2.2a)$$

$$\mathfrak{g}^3 = dv, \quad \mathfrak{g}^4 = du. \quad (2.2b)$$

Calculation of the Ricci tensor components on this tetrad results in

$$R_{ab} = \text{diag}(-2ab, +2ab, -2b^2, -2a^2). \quad (2.3)$$

This is the Ricci tensor of a vacuum Einstein–Maxwell space-time. The Maxwell field is derived from the potential 1-form given by

$$A = \cos \alpha \sin(au - bv) dx + \sin \alpha \sin(au + bv) dy, \quad (2.4)$$

with  $\alpha = \text{constant}$ . Then (2.3) reads (with our sign conventions)

$$R_{ab} = 2 E_{ab}, \quad (2.5)$$

where  $E_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{dc} F^{dc}$  is the electromagnetic energy–momentum tensor corresponding to the Maxwell field given by the 2-form

$$F = \frac{1}{2} F_{ab} \vartheta^a \wedge \vartheta^b = dA. \quad (2.6)$$

Calculating the components of this Maxwell field on the *orthonormal* tetrad given by the 1-forms,  $\vartheta^1, \vartheta^2, 2^{-\frac{1}{2}}(\vartheta^3 \pm \vartheta^4)$ , we find that the electric and magnetic 3-vectors  $\mathbf{E}$  and  $\mathbf{B}$  with respect to this frame (see [6]) read

$$\mathbf{E} = \left( -\frac{(a-b)}{\sqrt{2}} \cos \alpha, -\frac{(a+b)}{\sqrt{2}} \sin \alpha, 0 \right), \quad (2.7a)$$

$$\mathbf{B} = \left( -\frac{(a-b)}{\sqrt{2}} \sin \alpha, \frac{(a+b)}{\sqrt{2}} \cos \alpha, 0 \right). \quad (2.7b)$$

Thus if  $a = 0$  or  $b = 0$  then the electromagnetic field is null.

In the space-time with line-element (2.1) the hypersurface-orthogonal vector fields  $l = \frac{\partial}{\partial u}$  and  $k = \frac{\partial}{\partial v}$  are null. The shear and expansion of  $l$  are given by

$$\sigma_l = -\frac{a \sin 2bv}{2 \cos(au + bv) \cos(au - bv)}, \quad (2.8a)$$

$$\rho_l = \frac{a \sin 2au}{2 \cos(au + bv) \cos(au - bv)}, \quad (2.8b)$$

respectively, while the shear and expansion of  $k$  are given by

$$\sigma_k = -\frac{b}{a} \rho_l \quad \text{and} \quad \rho_k = -\frac{b}{a} \sigma_l, \quad (2.9)$$

respectively.

### 3. Plane gravitational waves on BR background

We will now restrict considerations to a BR space-time with  $b = 0$ . In such a space-time it is clear from (2.8 and 2.9) that the null hypersurfaces,  $u = \text{constant}$ , generated by the integral curves of the vector field  $k$ , are null hyperplanes since these generators are shear-free and expansion-free null geodesics. Thus  $u = \text{constant}$  are the histories of plane electromagnetic waves. We can insert plane gravitational

waves having  $u = \text{constant}$  as the histories of their wave fronts by first adjusting the BR line-element (2.1) with  $b = 0$  to read

$$ds^2 = -\cos^2 au (dx^2 + dy^2) + 2du (dv + c(u, x, y) du) , \quad (3.1)$$

where  $c(u, x, y)$  is a function to be determined. The Ricci tensor components for this line-element, which are calculated on the tetrad given by the 1-forms (2.2a) with  $b = 0$  and (2.2b) with  $\vartheta^3$  replaced by  $dv + c(u, x, y) du$ , all vanish except for

$$R_{44} = -2a^2 - \frac{1}{2} \sec^2 au \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) . \quad (3.2)$$

The Newman–Penrose components of the Weyl tensor  $\Psi_A$  with  $A = 0, 1, 2, 3, 4$  vanish except for

$$\Psi_4 = \frac{1}{4} \sec^2 au \left( \frac{\partial^2 c}{\partial x^2} - \frac{\partial^2 c}{\partial y^2} - 2i \frac{\partial^2 c}{\partial x \partial y} \right) , \quad (3.3)$$

indicating a Petrov type N-Weyl tensor with degenerate principal null direction  $k$ . Hence we see that we have plane gravitational waves on a BR background with  $b = 0$  if

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 , \quad (3.4)$$

and  $\Psi_4 \neq 0$ . We can specialise these gravitational waves to a single plane impulsive wave with history  $u = 0$ , say, on a BR background with  $b = 0$  by taking  $c(u, x, y) = \delta(u) f(x, y)$  with  $f$  harmonic in  $(x, y)$  and  $\delta(u)$ , the Dirac delta function singular on  $u = 0$ . In this case, putting  $\sqrt{2} \zeta = x + iy$ , the line-element (3.1) becomes

$$ds^2 = -2 \cos^2 au d\zeta d\bar{\zeta} + 2du (dv + \delta(u) f(\zeta, \bar{\zeta}) du) , \quad (3.5)$$

with

$$\frac{\partial^2 f}{\partial \zeta \partial \bar{\zeta}} = 0 . \quad (3.6)$$

We can remove the delta function from this line-element using the discontinuous coordinate transformation

$$u = U, \quad (3.7a)$$

$$v = V - \theta(U) f(Z, \bar{Z}) + |g(Z)|^2 \frac{\tan aU^+}{a} , \quad (3.7b)$$

$$\zeta = Z + \bar{g} \frac{\tan aU^+}{a} , \quad (3.7c)$$

$$\bar{\zeta} = \bar{Z} + g \frac{\tan aU^+}{a} , \quad (3.7d)$$

where  $\theta(U)$  is the Heaviside step function (which equals 1 if  $U > 0$  and equals 0 if  $U < 0$ ),  $U^+ = U\theta(U)$  and  $-\frac{\partial f}{\partial Z} = g(Z)$ , where  $g$  is an arbitrary analytic function of  $Z$  because  $f(Z, \bar{Z})$  is harmonic and the bar throughout denotes complex

conjugation. Substitution of (3.7) into (3.5) results in the line-element

$$ds^2 = -2 \cos^2 aU \left| dZ + \frac{\tan aU^+}{a} \bar{H}(\bar{Z}) d\bar{Z} \right|^2 + 2dU dV, \quad (3.8)$$

with  $H(Z) = \frac{dg}{dZ}$ . The limit  $a \rightarrow 0$  applied to (3.1)–(3.8) yields the corresponding formulae for the pure gravitational pp-waves. We emphasise that calculation of the Ricci tensor of the space-time with line-element (3.8) on the half-null tetrad given by the 1-forms

$$\omega^1 = \cos aU \left( dZ + \frac{\tan aU^+}{a} \bar{H}(\bar{Z}) d\bar{Z} \right), \quad (3.9a)$$

$$\omega^2 = \bar{\omega}^1, \quad (3.9b)$$

$$\omega^3 = dV, \quad (3.9c)$$

$$\omega^4 = dU \quad (3.9d)$$

results in  $R_{ab} = 0$  except for  $R_{44} = -2a^2$  and  $\Psi_A = 0$  except for  $\Psi_4 = H(Z) \delta(U)$ .

#### 4. Electromagnetic shock/gravity impulse wave

The null hyperplane  $U = 0$  in the space-time with line-element (3.8) is the history of a gravitational impulse wave propagating in a BR background (2.1) with  $b = 0$ . We can easily have a plane electromagnetic shock wave share the wave front with the gravitational impulse wave by simply replacing the factor  $\cos^2 aU$  in (3.8) by  $\cos^2 aU^+$  where, as above,  $U^+ = U \theta(U)$ . We then arrive at the line-element

$$ds^2 = -2 \cos^2 aU^+ \left| dZ + \frac{\tan aU^+}{a} \bar{H}(\bar{Z}) d\bar{Z} \right|^2 + 2dU dV. \quad (4.1)$$

With  $\cos aU$  in (3.9a,b) replaced by  $\cos aU^+$  we find that the components of the Ricci tensor of the space-time with line-element (4.1) on the resulting half-null tetrad all vanish except for  $R_{44} = -2a^2 \theta(U)$  and  $\Psi_A = 0$  except for  $\Psi_4 = H(Z) \delta(U)$ .

In (4.1) we constructed a space-time model of a plane impulsive gravitational wave sharing its wave front with a plane electromagnetic shock wave if  $a \neq 0$ . The history of the wave front is the null hyperplane  $U = 0$ . To the past of this hyperplane ( $U < 0$ ) the space-time is flat, while to the future ( $U > 0$ ) the space-time is a BR space-time (2.1) with  $b = 0$  and is conformally flat. The non-linear interaction between this wave and plane waves of different types is now a non-trivial challenge to be considered. The head-on collision between a homogeneous wave of the type (4.1) and a plane impulsive pure gravitational wave has recently been fully worked out in the sense that the space-time in the interaction region after the waves collide has been obtained by solving the Einstein–Maxwell vacuum field equations with the appropriate boundary conditions [3].

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