ALMOST OPENNESS IN TOPOLOGICAL VECTOR SPACES

By Hugo Arizmendi
IMATE, Universita Nacional Autonoma de Mexico

and

Robin Harte
Department of Mathematics, Trinity College, Dublin

(Communicated by T. T. West, M.R.I.A.)

[Received 2 April 1997. Read 23 April 1998. Published 30 September 1999.]

Abstract

We attempt to extend properties of the 'almost open' and 'bounded below' operators from normed to topological vector spaces and algebras.

It is familiar that the group of invertible elements in a Banach algebra is an open set whose boundary consists of topological zero divisors. The extension of these facts to bounded operators on incomplete normed spaces [1] and to normed algebras is perhaps not entirely predictable, but holds no great surprises; it might be hoped that incomplete normed algebras and spaces would foreshadow what happens in more general topological algebras and spaces.

Recall that a linear operator $T : X \to Y$ between topological vector spaces is *continuous* if and only if

$$\forall V \in \mathrm{Nbd}_Y(0)\ \exists U \in \mathrm{Nbd}_X(0) : \quad T(U) \subseteq V, \tag{0.1}$$

or equivalently

$$\forall V \in \mathrm{Nbd}_Y(0)\ \exists U \in \mathrm{Nbd}_X(0) : \quad U \subseteq T^{-1}(V). \tag{0.2}$$

We shall call the linear operator T *open* if and only if

$$\forall U \in \mathrm{Nbd}_X(0)\ \exists V \in \mathrm{Nbd}_Y(0) : \quad V \subseteq T(U); \tag{0.3}$$

more generally T is *almost open* if and only if

$$\forall U \in \mathrm{Nbd}_X(0)\ \exists V \in \mathrm{Nbd}_Y(0) : \quad V \subseteq \mathrm{cl}\ T(U). \tag{0.4}$$

For example if T is open then it must be *onto*: for if $y \in Y$ is arbitrary then for arbitrary $V \in \mathrm{Nbd}_Y(0)$ there is $k > 0$ for which

$$y \in kV, \tag{0.5}$$

and now applying (0.3) with $U = X \in \mathrm{Nbd}_X(0)$ gives

$$y \in kV \subseteq kT(X) = T(X).$$

The condition of being open is just the strengthened version of being onto, which gives a one–one operator a continuous inverse. Notice how the definition of 'open' has been derived from the continuity condition (0.1); if instead we work with the condition (0.2) we arrive at the concept of 'boundedness below': call $T : X \to Y$ *bounded below* or 'almost closed' (cf. [5, definition 5.19]) if and only if

$$\forall U \in \mathrm{Nbd}_X(0)\ \exists V \in \mathrm{Nbd}_Y(0) : \quad T^{-1}(V) \subseteq U. \tag{0.6}$$

Lemma 1. *If X is Hausdorff then*

$$T \text{ bounded below} \implies T \text{ one–one} ; \tag{1.1}$$

whether or not X is Hausdorff

$$T \text{ bounded below and onto} \implies T \text{ open}. \tag{1.2}$$

PROOF. If T is bounded below and $Tx = 0$ we claim

$$x \in \bigcap \{U \in \mathrm{Nbd}_X(0) : U\}, \tag{1.3}$$

for if $U \in \mathrm{Nbd}_X(0)$ is arbitrary then there is $V \in \mathrm{Nbd}_Y(0)$ for which $T^{-1}(V) \subseteq U$, and if $Tx = 0$ then $x \in T^{-1}(0) \subseteq T^{-1}(V)$. If T is bounded below and onto then for arbitrary $U \in \mathrm{Nbd}_X(0)$ there is $V \in \mathrm{Nbd}_Y(0)$ and $K \subseteq X$ with

$$T^{-1}(V) \subseteq U \text{ and } V = T(K),$$

giving

$$K \subseteq T^{-1}(V) \subseteq U$$

and then

$$V = T(K) \subseteq T(U). \quad \blacksquare$$

Between normed spaces [1] the sets of bounded below and of almost open operators form open subsets of the normed space of bounded operators, each disjoint from the boundary of the other. If we ask the same question for continuous linear operators between topological vector spaces we must also make a choice of the topology on the space of operators; it seems however rather difficult to adapt the normed space arguments [1, (0.7), theorem 1.2, (1.3.1)] to the topological vector space situation. One small fragment [1, (1.3.2)] transfers easily:

Theorem 2. *There is implication*

$$T \text{ bounded below and dense} \implies T \text{ almost open}; \tag{2.1}$$

in particular if $Z \subseteq Y$ *is a dense subspace*

$$\forall V \in \mathrm{Nbd}_Y(0) : \ \mathrm{int}(V) \subseteq \mathrm{cl}(V \cap Z). \tag{2.2}$$

PROOF. Here of course *dense* means 'has dense range'. For (2.2)

$$y_0 \in \mathrm{int}_Y(V) \Longrightarrow \exists W \in \mathrm{Nbd}_Y(0) : y_0 + W \subseteq V,$$

and take $z \in Z \cap (y_0 + W)$. Now if $U \in \mathrm{Nbd}_X(0)$ there is $V = \mathrm{int}(V) \in \mathrm{Nbd}_Y(0)$ with

$$T^{-1}(V) \subseteq U, \Longrightarrow V \cap (TX) \subseteq TU, \Longrightarrow V \subseteq \mathrm{cl}(V \cap TX) \subseteq \mathrm{cl}(TU). \quad \blacksquare$$

If A is a topological algebra, with identity 1 and invertible group A^{-1}, so that A is a topological vector space with jointly continuous multiplication, call the element $a \in A$ a *left topological zero divisor* if and only if L_a is not bounded below, and a *right topological zero divisor* if and only if R_a is not bounded below. Also call the element $a \in A$ *almost left invertible* if and only if R_a is almost open, and *almost right invertible* if and only if L_a is almost open. In contrast to continuous operators between topological vector spaces we now have a preferred topology, although the problem of whether the bounded belows or the almost opens form open sets is no easier. Evidently

$$R_a \text{ onto } \Longrightarrow a \text{ left invertible } \Longrightarrow R_a \text{ open } \Longrightarrow R_a \text{ almost open} \tag{2.3}$$

and

$$L_a \text{ onto } \Longrightarrow a \text{ right invertible } \Longrightarrow L_a \text{ open } \Longrightarrow L_a \text{ almost open} \tag{2.4}$$

while

$$R_a \text{ onto } \Longrightarrow L_a \text{ bounded below} \tag{2.5}$$

and

$$L_a \text{ onto } \Longrightarrow R_a \text{ bounded below.} \tag{2.6}$$

In normed algebras [1, theorem 3.2], if right multiplication has dense range then left multiplication is bounded below.

Problem 3. Is there implication

$$R_a \text{ dense } \Longrightarrow R_a \text{ almost open } \Longrightarrow L_a \text{ bounded below?} \tag{3.1}$$

Is there implication

$$L_a \text{ dense } \Longrightarrow L_a \text{ almost open } \Longrightarrow R_a \text{ bounded below?} \tag{3.2}$$

Recall that a subset of a topological vector space is said to be *bounded* if it is 'absorbed' by neighbourhoods of the origin:

Definition 4. The subset $K \subseteq X$ is said to be bounded if there is $k_U > 0$ for which

$$K \subseteq k_U U \text{ for each } U \in \mathrm{Nbd}_X(0). \tag{4.1}$$

We shall call the generalised sequence $(x_\lambda)_{\lambda \in \Lambda}$ in X bounded if for arbitrary $U \in \mathrm{Nbd}_X(0)$ there are $\alpha_U \in \Lambda$ and $k_U > 0$ for which

$$\alpha_U \prec \lambda \in \Lambda \Longrightarrow x_\lambda \in k_U U. \tag{4.2}$$

We shall call $T : X \to Y$ almost onto if for arbitrary $y \in Y$ there is a generalised sequence (x_λ) in X for which

$$(x_\lambda) \text{ is bounded and } Tx_\lambda \to y, \tag{4.3}$$

and we shall call T conditionally bounded below if there is implication, for arbitrary generalised sequences (x_λ) in X,

$$Tx_\lambda \to 0 \in Y \text{ and } (x_\lambda) \text{ bounded } \Longrightarrow x_\lambda \to 0 \in X. \tag{4.4}$$

Zelazko [6, (6)] calls the condition (4.2) ‘almost bounded’, while Vera [4, definition 2] calls it ‘ultimately bounded’.

Theorem 5. *There is implication*

$$T \textit{ almost open } \Longrightarrow T \textit{ almost onto} \tag{5.1}$$

and

$$T \textit{ bounded below } \Longrightarrow T \textit{ conditionally bounded below}. \tag{5.2}$$

Proof. If T is almost open then for each $U \in \mathrm{Nbd}_X(0)$ there is $V \in \mathrm{Nbd}_Y(0)$ for which

$$V \subseteq \mathrm{cl}\ T(U)$$

and then if $y \in Y$ there is $k_U > 0$ for which

$$y \in k_U V \subseteq \mathrm{cl}\ T(k_U U) :$$

thus there are $x_W \in X$ for each $W \in \mathrm{Nbd}_Y(0)$ for which

$$x_W \in k_U U \textit{ and } y - Tx_W \in W.$$

This gives (5.1): notice that the generalised sequence (x_W) is bounded in the simple sense that its whole set of terms $K = \{x_W\}$ satisfies (4.1). For (5.2) we have, if T is bounded below, implication for arbitrary (x_λ) in X

$$Tx_\lambda \to 0 \in Y \Longrightarrow x_\lambda \to 0 \in X : \tag{5.3}$$

simply argue

$$\alpha_U \prec \lambda \Longrightarrow Tx_\lambda \in V \Longrightarrow x_\lambda \in U. \quad \blacksquare$$

Of course (5.1) is trivial for bounded operators between normed spaces [2].

Theorem 6. *If $a \in A$ is an element of a topological algebra A there is implication*

$$R_a \text{ almost onto } \Longrightarrow R_a \text{ almost open} \Longrightarrow L_a \text{ conditionally bounded below} \tag{6.1}$$

and

$$L_a \text{ almost onto } \Longrightarrow L_a \text{ almost open} \Longrightarrow R_a \text{ conditionally bounded below.} \tag{6.2}$$

PROOF. If R_a is almost onto there is a generalised sequence (b_λ) in A for which

$$(b_\lambda) \text{ is bounded in } A \text{ and } b_\lambda a \to 1 \in A, \tag{6.3}$$

so that there are $k_U > 0$ for which, for arbitrary $U \in \mathrm{Nbd}_A(0)$, $1 \in \mathrm{cl}\{xa : x \in k_U U\}$ and hence

$$y \in A \Longrightarrow y \in \mathrm{cl}\{yxa : x \in k_U U\} \subseteq \mathrm{cl}\{xa : x \in k_U yU\}. \tag{6.4}$$

Thus $\{y \in A : k_U y \in U\} \subseteq \mathrm{cl}\{xa : x \in UU\}$ and finally

$$V = \{y \in A : k_{U'} y \in U'\} \text{ with } U'U' \subseteq U \tag{6.5}$$

gives $V \subseteq \mathrm{cl}\{xa : x \in U\}$.

For the second implication we remember (5.1) and claim

$$R_a \text{ almost onto } \Longrightarrow L_a \text{ conditionally bounded below.} \tag{6.6}$$

Indeed if $1 - b_\lambda a \to 0$ and $ax_\mu \to 0$ with bounded (b_λ) and (x_μ), then there are α_U, $k_U > 0$, β_U, and $h_U > 0$ for which, for arbitrary $U \in \mathrm{Nbd}_A(0)$,

$$\alpha_U \prec \lambda \Longrightarrow b_\lambda \in k_U U \text{ and } 1 - b_\lambda a \in U \tag{6.7}$$

and

$$\beta_U \prec \mu \Longrightarrow x_\mu \in h_U U \text{ and } ax_\mu \in U, \tag{6.8}$$

so that

$$\beta_U, \beta_{U'} \prec \mu \text{ and } \alpha_U, \alpha_{U'} \prec \lambda \Longrightarrow x_\mu = (1 - b_\lambda a)x_\mu + b_\lambda(ax_\mu) \in U'(h_U U) + (k_U U)U'. \tag{6.9}$$

Now if $U \in \mathrm{Nbd}_A(0)$ there is $U' \in \mathrm{Nbd}_A(0)$ for which $U'(h_U U) + (k_U U)U' \subseteq U$. This proves (6.1), and similarly (6.2). ■

We shall call $a \in A$ *approximately left invertible*, or 'topologically left invertible', if and only if R_a is dense, and *approximately right invertible*, or 'topologically right invertible', if and only if L_a is dense, and *boundedly topologically left, or right, invertible* if and only if R_a, or L_a, is almost onto: this is the terminology of Thatte and Bhatt [3]. We shall also call *$a \in A$ almost left, or right, invertible* [1] if and only if R_a, or L_a, is almost open: thus

$$a \text{ boundedly topologically left invertible } \Longleftrightarrow a \text{ almost left invertible} \tag{6.10}$$

and

$$a \text{ boundedly topologically right invertible} \iff a \text{ almost right invertible.} \qquad (6.11)$$

In fact neither the bounded below nor the almost open operators need form open sets, the implication (5.2) is not reversible, and the first part of both (3.1) and (3.2) may fail. Our example (cf. Zelazko [6]) is the algebra ℓ_∞, with pointwise operations and the 'c_0 topology', in which a generalised sequence x_λ converges if and only if the product $x_\lambda c$ converges in the usual ℓ_∞ norm for each $c \in c_0$.

Example 7. If A is the algebra ℓ_∞ with the c_0 topology and if $a \in A$ there is implication

$$R_a = L_a \text{ dense} \iff 0 \notin \{a_n : n \in \mathbf{N}\} \qquad (6.12)$$

and

$$R_a = L_a \text{ almost onto} \implies \inf_n|a_n| > 0 \implies R_a = L_a \text{ open.} \qquad (6.13)$$

Also

$$R_a = L_a \text{ one–one} \implies 0 \notin \{a_n : n \in \mathbf{N}\} \implies R_a = L_a \text{ conditionally bounded below} \qquad (6.14)$$

and

$$R_a = L_a \text{ bounded below} \implies \inf_n|a_n| > 0 \implies a \in A^{-1} \text{ invertible.} \qquad (6.15)$$

Finally the terminating sequences are dense

$$\mathrm{cl}_A(c_{00}) = A. \qquad (6.16)$$

Proof. If $a_n \neq 0$ for all $n \in \mathbf{N}$ then $L_a = R_a$ is dense: we claim that for arbitrary $c \in c_0$ and arbitrary $y \in \ell_\infty$ there is $x \in \ell_\infty$ for which

$$||(y - ax)c||_\infty \le 1.$$

Indeed take

$$x_n = 0 \textit{ if } |y_n c_n| \le 1 \ , \ = \frac{y_n}{a_n} \textit{ if } |y_n c_n| > 1. \qquad (6.17)$$

Conversely if $a_k = 0$ then, with $c = \delta_k \in c_0$ the Kronecker delta, we have, for arbitrary $b \in A$,

$$||(1 - ab)c||_\infty \ge |(1 - a_k b_k)c_k| = 1.$$

If $\inf_n|a_n| > 0$ then $ba = 1 = ab$ with $b \in A$ given by $b_n = 1/a_n$, so that certainly $R_a = L_a$ is open; conversely, for a contradiction, suppose $\inf_n|a_n| = 0$ and that also

$$(1 - b_\lambda a)c \to 0 \textit{ in } \ell_\infty \textit{ for arbitrary } c \in c_0, \qquad (6.18)$$

with

$$\sup_{\alpha_c < \lambda}\sup_k|b_{\lambda k}c_k| = K_c < \infty \textit{ for arbitrary } c \in c_0. \qquad (6.19)$$

Since $|a_n|$ can be arbitrarily small there is a subsequence $(a'_n) = (a_{\theta(n)})$ for which

$$a'_n \to 0 \textit{ with } |a'_{n+1}| \leq |a'_n| \leq 1 \ : \tag{6.20}$$

evidently both (7.7) and (7.8) hold with $\theta(k)$ in place of k. Thus if (7.7) holds and we take $c'_n = c_{\theta(n)} = a'_n$ and write $b'_{\lambda n} = b_{\lambda\theta(n)}$ we have

$$\alpha'_\epsilon \prec \lambda \Longrightarrow \sup_k |1 - b'_{\lambda k} a'_k||a'_k| \leq \epsilon,$$

while if (7.8) holds and we take $c'_n = c_{\theta(n)} = \sqrt{|a'_n|}$ we have

$$\sup_{\alpha_c \prec \lambda} \sup_k |b'_{\lambda k}| \sqrt{|a'_k|} \leq K_c < \infty.$$

Thus if $\alpha_c, \alpha'_\epsilon \prec \lambda$ and $k \in \mathbf{N}$ we have

$$(1 - K_c \sqrt{|a'_k|})|a'_k| \leq (1 - |b'_{\lambda k} a'_k|)|a'_k| \leq \epsilon \ :$$

but taking $4\epsilon = |a'_k| \leq 1/4K_c^2$ gives a contradiction.

If $a_k = 0$ then $L_a(\delta_k) = 0 \neq \delta_k$, so that L_a is not one–one. Conversely if $0 \notin \{a_n : n \in \mathbf{N}\}$ suppose $\|ax_\lambda c\| \to 0$ for each $c \in c_0$: taking in particular $c = \delta_k$ the Kronecker delta it follows

$$|x_{\lambda k}| \to 0 \text{ for each } k \in \mathbf{N}. \tag{6.21}$$

The argument is reminiscent of Lebesgue dominated convergence: if at the same time $(x_\lambda)_{\lambda \in \Lambda}$ is bounded in the sense (4.2) then for each $c \in c_0$ there is $\beta_c \in \Lambda$ and $K_c > 0$ for which

$$\sup_{\beta_c \prec \lambda} \sup_n |x_{\lambda n} c_n| = K_c < \infty.$$

Thus for arbitrary $c \in c_0$ and arbitrary $\epsilon > 0$ there is $N(\epsilon) \in \mathbf{N}$ for which

$$n \geq N(\epsilon) \Longrightarrow |c_n| \leq \epsilon$$

and then if $\epsilon' > 0$, $\alpha(\epsilon') \in \Lambda$ for which, by (7.10),

$$\sup_{\alpha(\epsilon') \prec \lambda} \max_{n \leq N(\epsilon)} |x_{\lambda n}| \leq \epsilon'.$$

Now if $\alpha(\epsilon') \prec \lambda$ and $\beta_c \prec \lambda$ then

$$\max(\max_{n \leq N(\epsilon)} |x_{\lambda n} c_n|, \sup_{n \geq N(\epsilon)} |x_{\lambda n} c_n|) \leq \max(\max_{n \leq N(\epsilon)} |c_n| \epsilon', K_c \epsilon),$$

so choose $\epsilon > 0$ and $\epsilon' > 0$ in such a way that

$$\max_{n \leq N(\epsilon)} |c_n| \epsilon' \leq K_c \epsilon \leq 1.$$

If $\inf_n |a_n| > 0$ then we have already seen that $a \in A^{-1}$ is invertible; conversely to see that if $\inf_n |a_n| = 0$ then L_a is not bounded below we may assume that $0 \notin$

$\{a_n : n \in \mathbf{N}\}$: take again a subsequence a' satisfying (7.9) and define $x = (x_n) = (x_{nk})$ by setting $x_{nm} = 0$ if $m \notin \theta(\mathbf{N})$ and

$$x'_{nm} = x_{n\theta(m)} \ = \frac{1}{a'_m} \text{ if } n \leq m \leq 2n, \ = 0 \text{ else.} \tag{6.22}$$

Evidently, for arbitrary $c \in c_0$,

$$||ax_nc||_\infty = \sup_m |a_mx_{nm}c_m| \leq \sup_{k\geq n}|c'_k| \to 0;$$

but if we take $c \in c_0$ with

$$c'_m = \sqrt{|a'_m|} \text{ for each } m \in \mathbf{N} \tag{6.23}$$

then $||x_nc||_\infty = ||x'_nc'||_\infty \geq \frac{1}{\sqrt{|a'_n|}} \to \infty$. Thus, in the topology of A, a_nx_n tends to 0 but not x_n.

Finally, for (7.5), we need for arbitrary $y \in \ell_\infty$ and $c \in c_0$ an element $x \in \ell_\infty$ for which $||(y-x)c||_\infty \leq 1$: so take

$$x_n = y_n \text{ if } |y_nc_n| > 1 \ , \ = 0 \text{ if } |y_nc_n| \leq 1. \ \blacksquare \tag{6.24}$$

Of course it is clear that all the implications in Example 7 are two-way. From (7.5) in particular it is clear that, unlike normed algebras, the sets of almost open and of bounded below multiplication operators have empty interiors, and are far from disjoint from one another's boundaries.

Some of this pathology survives both in Frechet algebras and in 'locally multiplicatively convex algebras'. For example in the algebra of entire functions [5, example 5.27] the elements that induce bounded below multiplications form an open set, but those whose multiplications are almost open have empty interior.

Example 8. If A is the algebra of entire functions on $\mathbf{C}$ with the topology given by the seminorms $||\cdot||_R$ for all $R > 0$, where

$$||f||_R = \sup_{|\lambda|=R}|f(\lambda)| = \sup_{|\lambda|\leq R}|f(\lambda)| \tag{6.25}$$

for each $R > 0$ and each $f \in A$, then for $f \in A$ there is implication

$$R_f = L_f \text{ one–one} \implies f \neq 0 \implies L_f = R_f \text{ bounded below} \tag{6.26}$$

and

$$R_f = L_f \text{ dense} \implies f^{-1}(0) = \emptyset \implies f \in A^{-1} \text{ invertible.} \tag{6.27}$$

Also

$$\text{int}_A A^{-1} = \emptyset. \tag{6.28}$$

Proof. Recall that the zeroes of non-trivial holomorphic functions have no accumulation points:

$$0 \neq f \in A \Longrightarrow f^{-1}(0) = \text{iso } f^{-1}(0). \tag{6.29}$$

Thus if $0 \neq f \in A$ and $R > 0$ there is $R' \geq R$ for which

$$f^{-1}(0) \cap \{|z| = R'\} = \emptyset : \tag{6.30}$$

now put

$$k_R = \inf_{|\lambda|=R'} |f(\lambda)|. \tag{6.31}$$

For arbitrary $g \in A$ it follows

$$k_R||g||_R \leq k_R||g||_{R'} \leq \sup_{|\lambda|=R'} |f(\lambda)g(\lambda)| = ||L_f(g)||_{R'}, \tag{6.32}$$

giving the second implication of (8.2). Of course the first implication of (8.3), and the second implication of (8.4), are clear: for the first implication of (8.1) argue that if $f(\lambda) = 0$ with $|\lambda| \leq R$ then for arbitrary $g \in A$

$$||R_f(g) - 1||_R \geq |g(\lambda)f(\lambda) - 1| = 1. \tag{6.33}$$

Finally, for (8.4) observe that for arbitrary $R > 0$

$$1 - \tfrac{1}{n}z \notin A^{-1} \textit{ and } ||1 - \tfrac{1}{n}z||_R \to 0. \quad \blacksquare \tag{6.34}$$

REFERENCES

[1] R.E. Harte, Almost open mappings between normed spaces, *Proceedings of the American Mathematical Society* **90** (1984), 243–9.

[2] R.E. Harte and W.Y. Lee, On the bounded closure of the range of an operator, *Proceedings of the American Mathematical Society* **125** (1997), 2313–18.

[3] A.D. Thatte and S.J. Bhatt, On topolizing invertibility, *Indian Journal of Pure and Applied Mathematics* **15** (1984), 1308–12.

[4] M.R. Vera, The (Γ, t)-topology on $L(E, E)$ and the spectrum of a bounded linear operator on a locally convex topological vector space, *Boletín de la Sociedad Matemática Mexicana* **3** (1997), 151–64.

[5] W. Zelazko, On ideal theory in Banach and topological algebras, *Monografias del Instituto de Matematicas* **15**, UNAM, 1981.

[6] W. Zelazko, A non m-convex algebra on which operate all entire functions, *Annales Polonici Mathematici* **46** (1985), 389–94.