A NOTE ON THE TOPOLOGICAL STRUCTURE OF THE SOLUTION SET OF ABSTRACT VOLTERRA EQUATIONS

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Abstract

This paper discusses the topological structure of the set of solutions of $\frac{dx}{dt} = Vx$ a.e. on $[0, T]$, $x(0) = x_0$.

1. Introduction

In this paper we examine the solution set of the abstract Volterra equation

$$\begin{cases} \dfrac{dx(t)}{dt} = V\,x(t) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{1.1}$$

here V is the abstract Volterra operator. By placing mild conditions on the operator V we will show that the set of solutions of (1.1) is a R_δ set [6] (so in particular non-empty, compact and connected). In the literature the solution set of differential equations has received a lot of attention (see [4], [6] and [11] and their references). Only [2] and [3] examine the general case (1.1). In this paper we improve and complement the theory presented in the literature [2], [3], [6] and [11]. Our results rely on results in [8] and [9] together with a well known result in the literature [6, p. 161] (or see [11]). For convenience we recall the result in [6] and [11].

Theorem 1.1. *Let $E = C([0,T],\mathbf{R}^n)$ and suppose that $F : E \to E$ is a continuous, compact map such that the following conditions hold:*

(i) *there exists $x_0 \in \mathbf{R}^n$ with $F(u)(0) = x_0$ for all $u \in E$;*

(ii) *for every $\epsilon \in (0, T]$ and every $u, v \in E$, if $u(t) = v(t)$ for each $t \in [0,\epsilon]$ then $F(u)(t) = F(v)(t)$ for each $t \in [0,\epsilon]$.*

Then $Fix(F)$ is a R_δ set.

2. Solution set

We begin with the general problem (1.1). Solutions are sought in $AC([0,T],\mathbf{R}^n)$. We assume that the following conditions are satisfied:

$$\begin{aligned} &V \text{ is an abstract Volterra operator, i.e. if } x(t) = y(t) \text{ for } t \in [0,\epsilon],\ \epsilon \le T, \\ &\text{then } V\,x(t) = V\,y(t) \text{ for a.e. } t \in [0,\epsilon], \end{aligned} \tag{2.1}$$

$$V : C([0,T],\mathbf{R}^n) \to L^p([0,T],\mathbf{R}^n),\ 1 \le p \le \infty, \text{ is a continuous operator,} \tag{2.2}$$

$$\begin{aligned}&\text{for any constant } r>0 \text{ there exists } \mu_r \in L^p[0,T] \text{ such that for any}\\ &y\in C([0,T],\mathbf{R}^n) \text{ with } |y|_0=\sup_{[0,T]}|y(t)|\le r, \text{ we have } |V\,y(t)|\le \mu_r(t)\\ &\text{for a.e. } t\in[0,T]\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\text{there exists a constant } M>|x_0| \text{ with } |y|_0<M \text{ for any possible solution}\\ &y\in AC([0,T],\mathbf{R}^n) \text{ to (1.1).}\end{aligned} \tag{2.4}$$

Let $\epsilon > 0$ be given and let $\tau_\epsilon : \mathbf{R}^n \to [0,1]$ be the Urysohn function for

$$\left(\overline{B}(0,M)\,,\,\mathbf{R}^n\backslash B(0,M+\epsilon)\right),$$

such that $\tau_\epsilon(x)=1$ if $|x|\le M$ and $\tau_\epsilon(x)=0$ if $|x|\ge M+\epsilon$. Let $V_\epsilon(x)=\tau_\epsilon(x)\,V(x)$ and we will consider the problem

$$\begin{cases}\dfrac{d\,x(t)}{d\,t} &= V_\epsilon(x(t)) \quad \text{a.e. } t\in[0,T]\\ x(0) &= x_0.\end{cases} \tag{2.5}$$

Let $S_V(x_0;\mathbf{R}^n)$ denote the solution set of (1.1) and $S_{V_\epsilon}(x_0;\mathbf{R}^n)$ the solution set of (2.5).

Theorem 2.1. *Suppose* (2.1), (2.2), (2.3) *and* (2.4) *hold. Let* $\epsilon>0$ *be given and assume*

$$|w|_0<M \ \ \textit{for any possible solution} \ \ w\in AC([0,T],\mathbf{R}^n) \ \ \textit{to (2.5).} \tag{2.6}$$

Then $S_V(x_0;\mathbf{R}^n)$ *is a* R_δ *set.*

Proof. Notice that (2.4) and (2.6) imply $S_V(x_0;\mathbf{R}^n)=S_{V_\epsilon}(x_0;\mathbf{R}^n)$. Next we define the operator $N_\epsilon : C([0,T],\mathbf{R}^n)\to C([0,T],\mathbf{R}^n)$ by

$$N_\epsilon y(t) = x_0 + \int_0^t V_\epsilon(x(s))\,ds.$$

We first show $N_\epsilon : C([0,T],\mathbf{R}^n)\to C([0,T],\mathbf{R}^n)$ is continuous. To see this let $y_n\to y$ in $C([0,T],\mathbf{R}^n)$. Then (2.2) and (2.3) imply

$$\begin{aligned}|N_\epsilon y_n - N_\epsilon y|_0 \le & \int_0^T |\tau_\epsilon(y_n(s))-\tau_\epsilon(y(s))|\,|Vy(s)|\,ds\\ & + T^{\frac{1}{q}}\left(\int_0^T |V\,y_n(s)-V\,y(s)|^p\,ds\right)^{\frac{1}{p}}\\ & \to 0 \quad \text{if } \ 1\le p<\infty \ \ (\text{here } \ \frac{1}{p}+\frac{1}{q}=1)\end{aligned}$$

and

$$\begin{aligned}|N_\epsilon y_n - N_\epsilon y|_0 \le & \int_0^T |\tau_\epsilon(y_n(s)) - \tau_\epsilon(y(s))|\,|Vy(s)|\,ds \\ & + T \ ess\, sup\ |V\, y_n(s) - V\, y(s)| \\ & \to 0 \quad \text{if} \quad p = \infty.\end{aligned}$$

Thus $N_\epsilon : C([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is continuous. Next we show that N_ϵ is compact. Now (2.3) and the definition of τ_ϵ imply that there exists $\mu \in L^p[0,T]$ with

$$|\tau_\epsilon(y(t))\, V\, y(t)| \le \mu(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and all} \quad y \in C([0,T],\mathbf{R}^n).$$

Now $N_\epsilon(C([0,T],\mathbf{R}^n))$ is uniformly bounded since if $y \in C([0,T],\mathbf{R}^n)$ then

$$|N_\epsilon y|_0 \le |x_0| + \int_0^t \mu(s)\,ds.$$

Also $N_\epsilon(C([0,T],\mathbf{R}^n))$ is equicontinuous on $[0,T]$ since if $y \in C([0,T],\mathbf{R}^n)$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ then

$$|N_\epsilon y(t_1) - N_\epsilon y(t_2)| \le \int_{t_1}^{t_2} \mu(s)\,ds \to 0 \quad \text{as} \quad t_1 \to t_2.$$

The Arzela–Ascoli theorem now guarantees that $N_\epsilon : C([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is compact.

Theorem 1.1 implies that $S_{V_\epsilon}(x_0;\mathbf{R}^n) = Fix(N_\epsilon)$ is a R_δ set. ∎

Theorem 2.2. *Suppose that* (2.1), (2.2) *and* (2.3) *hold and assume that the following conditions are satisfied:*

$$\begin{aligned}&\text{there exists } \alpha \in L^1[0,T] \text{ and } g : [0,\infty) \to (0,\infty) \text{ a Borel measurable} \\ &\text{function such that for a.e. } t \in [0,T] \text{ and all } y \in C([0,T],\mathbf{R}^n) \\ &\text{we have } \langle y(t), V(y(t))\rangle \le \alpha(t)|y(t)|g(|y(t)|)\end{aligned} \tag{2.7}$$

and

$$\int_0^T \alpha(s)\,ds < \int_{|x_0|}^\infty \frac{dx}{g(x)}; \tag{2.8}$$

here $\langle .,.\rangle$ *denotes the Euclidean scalar product. Then* $S_V(x_0;\mathbf{R}^n)$ *is a* R_δ *set.*

PROOF. Let $\epsilon > 0$ be given,

$$M_0 = I^{-1}\left(\int_0^T \alpha(s)\,ds\right), \quad \text{where} \quad I(z) = \int_{|x_0|}^z \frac{dx}{g(x)}, \quad \text{and} \quad M = M_0 + 1.$$

We will show that any possible solution u of (1.1) satisfies $|u|_0 \le M_0$ and any possible solution y of (2.5) satisfies $|y|_0 \le M_0$. If this is true then Theorem 2.1 implies that $S_V(x_0;\mathbf{R}^n)$ is a R_δ set.

Suppose that u is a possible solution of (1.1). Then we have

$$|u(t)|' \leq \alpha(t)\, g(|u(t)|) \quad \text{a.e. on} \quad \{t \in [0,T] \,:\, |u(t)| > 0\}\,.$$

Lemma 3.2 in [5], applied with $R = |x_0|$, $\psi(x) = g(x)$ and $z(t) = |u(t)|$, implies $|u(t)| \leq M_0$ for all $t \in [0,T]$. Thus any possible solution u of (1.1) satisfies $|u|_0 \leq M_0$.

Next let y be a possible solution of (2.5). Suppose there exists $t \in (0,T]$ with $|y(t)| > M_0$. Then there exists $[0,t_0] \subseteq [0,T]$ with $0 \leq |y(s)| < M_0$ for $s \in [0,t_0]$ and $|y(t_0)| = M_0$. Also for a.e. $s \in [0,t_0]$ we have $y'(s) = V_\epsilon(y(s)) = V(y(s))$. Thus we have

$$|y(s)|' \leq \alpha(s)\, g(|y(s)|) \quad \text{a.e. on} \quad \{s \in [0,t_0] \,:\, |y(s)| > 0\}\,.$$

Now Lemma 3.2 in [5] will yield a contradiction (see the argument in [5, p. 44]). Thus $|y|_0 \leq M_0$ [of course we could show $|y|_0 \leq M_0$ directly also, since $\langle y(t), V_\epsilon(y(t))\rangle = \tau_\epsilon(y(t))\langle y(t), V(y(t))\rangle \leq \alpha(t)|y(t)|g(|y(t)|)$ and so $|y(t)|' \leq \alpha(t)g(|y(t)|)$ a.e. on $\{t \in [0,T] \,:\, |y(t)| > 0\}$]. ■

A special case of (1.1) is the initial value problem

$$\begin{cases} \dfrac{d\,x(t)}{d\,t} &= f(t,x(t)) \quad \text{a.e.} \;\; t \in [0,T] \\ x(0) &= x_0. \end{cases} \tag{2.9}$$

We suppose that the following conditions are satisfied:

$f \,:\, [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ is a L^1-Carathéodory function; by this we mean that
(i) the map $t \mapsto f(t,z)$ is measurable for all $z \in \mathbf{R}^n$,
(ii) the map $z \mapsto f(t,z)$ is continuous for almost all $t \in [0,T]$,
(iii) for each $r > 0$ there exists $\mu_r \in L^1[0,T]$ such that
$|z| \leq r$ implies $|f(t,z)| \leq \mu_r(t)$ for almost all $t \in [0,T]$ (2.10)

and

there exists a constant $M > |x_0|$ with $|y|_0 < M$ for any possible solution $y \in AC([0,T],\mathbf{R}^n)$ to (2.9). (2.11)

Let $\epsilon > 0$ be given and let τ_ϵ be the Urysohn function as described previously. Consider the problem

$$\begin{cases} \dfrac{d\,x(t)}{d\,t} &= \tau_\epsilon(x(t))\, f(t,x(t)) \quad \text{a.e.} \;\; t \in [0,T] \\ x(0) &= x_0. \end{cases} \tag{2.12}$$

Let $S_f(x_0;\mathbf{R}^n)$ denote the solution set of (2.9).

Theorem 2.3. *Suppose that* (2.10) *and* (2.11) *hold. Let* $\epsilon > 0$ *be given and assume that*

$|w|_0 < M$ *for any possible solution* $w \in AC([0,T],\mathbf{R}^n)$ *to* (2.12). (2.13)

Then $S_f(x_0;\mathbf{R}^n)$ *is a* R_δ *set.*

Theorem 2.4. *Suppose that* (2.10) *holds and assume that the following conditions are satisfied:*

there exists $\alpha \in L^1[0,T]$ *and* $g : [0,\infty) \to (0,\infty)$ *a Borel measurable function such that for a.e.* $t \in [0,T]$ *and all* $y \in \mathbf{R}^n$ *we have* $\langle y, f(t,y)\rangle \le \alpha(t)|y|g(|y|)$ (2.14)

and

$$\int_0^T \alpha(s)\,ds < \int_{|x_0|}^{\infty} \frac{dx}{g(x)}; \tag{2.15}$$

here $\langle .\,,.\rangle$ *denotes the Euclidean scalar product. Then* $S_f(x_0;\mathbf{R}^n)$ *is a* R_δ *set.*

Next we discuss the Volterra integral equation, namely

$$y(t) = h(t) + \int_0^t k(t,s)\,f(s,y(s))\,ds \quad \text{for} \quad t \in [0,T]; \tag{2.16}$$

here $k : [0,T] \times [0,t] \to \mathbf{R}$ and $f : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$. Suppose that the following conditions are satisfied:

$$h \in C([0,T],\mathbf{R}^n), \tag{2.17}$$

$f : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ is a L^q-Carathéodory function (here $q > 1$ is a constant); by this we mean that
(i) the map $t \mapsto f(t,z)$ is measurable for all $z \in \mathbf{R}^n$,
(ii) the map $z \mapsto f(t,z)$ is continuous for almost all $t \in [0,T]$,
(iii) for each $r > 0$ there exists $\mu_r \in L^q[0,T]$ such that
$|z| \le r$ implies $|f(t,z)| \le \mu_r(t)$ for almost all $t \in [0,T]$, (2.18)

$k_t(s) \in L^p[0,t]$ for each $t \in [0,T]$; here $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and for each $t \in [0,T]$, $k_t : [0,t] \to \mathbf{R}$ is defined by $k_t(s) = k(t,s)$ (2.19)

for any $t_1, t_2 \in [0,T]$ we have that $\displaystyle\int_0^{t_3} |k_{t_1}(s) - k_{t_2}(s)|^p\,ds \to 0$ as $t_1 \to t_2$, where $t_3 = \min\{t_1,t_2\}$ (2.20)

and

there exists a constant $M > |h|_0$ with $|y|_0 < M$ for any possible solution y to (2.16). (2.21)

Let $\epsilon > 0$ be given and let τ_ϵ be the Urysohn function as described previously. Consider the problem

$$y(t) = h(t) + \int_0^t k(t,s)\,\tau_\epsilon(y(s))\,f(s,y(s))\,ds \quad \text{for} \quad t \in [0,T]. \tag{2.22}$$

Let $S(\mathbf{R}^n)$ denote the solution set of (2.16) and $S_\epsilon(\mathbf{R}^n)$ the solution set of (2.22).

Theorem 2.5. *Suppose that* (2.17), (2.18), (2.19), (2.20) *and* (2.21) *hold. Let* $\epsilon > 0$ *be given and assume*

$$|w|_0 < M \quad \textit{for any possible solution} \quad w \in C([0,T],\mathbf{R}^n) \quad \textit{to } (2.22). \tag{2.23}$$

Then $S(\mathbf{R}^n)$ *is a* R_δ *set.*

Proof. Notice that $S(\mathbf{R}^n) = S_\epsilon(\mathbf{R}^n)$. Define the operator $N_\epsilon : C([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ by

$$N_\epsilon\, y(t) = h(t) + \int_0^t k(t,s)\,\tau_\epsilon(y(s))\,f(s,y(s))\,ds.$$

A slight modification of the argument in [9] and [10] implies that $N_\epsilon : C([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is continuous. Next we show that N_ϵ is compact. Now (2.18) and the definition of τ_ϵ imply that there exists $\mu \in L^q[0,T]$ with

$$|\tau_\epsilon(y(t))\,f(t,y(t))| \le \mu(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and all} \quad y \in C([0,T],\mathbf{R}^n).$$

Now $N_\epsilon(C([0,T],\mathbf{R}^n))$ is uniformly bounded since if $y \in C([0,T],\mathbf{R}^n)$ and $t \in [0,T]$ then

$$|N_\epsilon\, y(t)| \le |h|_0 + \left(\sup_{t\in[0,T]} \left(\int_0^t |k_t(s)|^p\,ds\right)^{\frac{1}{p}}\right) \left(\int_0^T \mu^q(s)\,ds\right)^{\frac{1}{q}}.$$

Also $N_\epsilon(C([0,T],\mathbf{R}^n))$ is equicontinuous on $[0,T]$ since if $y \in C([0,T],\mathbf{R}^n)$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ then

$$\begin{aligned} |N_\epsilon\, y(t_1) - N_\epsilon\, y(t_2)| \le |h(t_1) - h(t_2)| &+ \left(\int_0^{t_1} |k_{t_1}(s) - k_{t_2}(s)|^p\,ds\right)^{\frac{1}{p}} \left(\int_0^T \mu^q(s)\,ds\right)^{\frac{1}{q}} \\ &+ \left(\sup_{t\in[0,T]} \left(\int_0^t |k_t(s)|^p\,ds\right)^{\frac{1}{p}}\right) \left(\int_{t_1}^{t_2} \mu^q(s)\,ds\right)^{\frac{1}{q}}. \end{aligned}$$

The Arzela–Ascoli theorem now guarantees that $N_\epsilon : C([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is compact.

Theorem 1.1 implies that $S_\epsilon(\mathbf{R}^n) = Fix\,(N_\epsilon)$ is a R_δ set. ■

Theorem 2.6. *Suppose that* (2.17), (2.18), (2.19) *and* (2.20) *hold and assume that the* *following conditions are satisfied:*

> *there exists* $\alpha \in L^1[0,T]$ *and* $g : [0,\infty) \to (0,\infty)$ *a non-decreasing continuous function such that* $|k(t,s)\, f(s,u)| \leq \alpha(s)\, g(|u|)$ *for a.e.* $s \in [0,t]$, *a.e.* $t \in [0,T]$ *and all* $u \in \mathbf{R}^n$ (2.24)

and

$$\int_0^T \alpha(s)\, ds < \int_{|h|_0}^{\infty} \frac{dx}{g(x)}. \tag{2.25}$$

Then $S(\mathbf{R}^n)$ *is a* R_δ *set.*

PROOF. Let $\epsilon > 0$ be given,

$$M_0 = I^{-1}\left(\int_0^T \alpha(s)\, ds\right), \quad \text{where } I(z) = \int_{|h|_0}^{z} \frac{dx}{g(x)}, \quad \text{and } M = M_0 + 1.$$

We will show that any possible solution u of (2.16) satisfies $|u|_0 \leq M_0$ and any possible solution y of (2.22) satisfies $|y|_0 \leq M_0$. If this is true then Theorem 2.5 implies that $S(\mathbf{R}^n)$ is a R_δ set.

Suppose that u is a possible solution of (2.16). Then

$$|u(t)| \leq |h|_0 + \int_0^t \alpha(s)\, g(|u(s)|)\, ds \equiv w(t) \quad \text{for } t \in [0,T].$$

Now $w'(t) = \alpha(t)\, g(|u(t)|) \leq \alpha(t)\, g(w(t))$ a.e. and so

$$\int_{|h|_0}^{w(x)} \frac{ds}{g(s)} = \int_0^x \frac{w'(s)}{g(w(s))}\, ds \leq \int_0^x \alpha(s)\, ds \leq \int_0^T \alpha(s)\, ds$$

for $x \in [0,T]$. Thus $w(x) \leq M_0$ for any $x \in [0,T]$ and so $|u(x)| \leq M_0$ for all $x \in [0,T]$.

Next let y be a possible solution of (2.22). For $t \in [0,T]$, since $\tau_\epsilon : \mathbf{R}^n \to [0,1]$, we have

$$\begin{aligned} |y(t)| &\leq |h(t) + \int_0^t k(t,s)\, \tau_\epsilon(y(s))\, f(s,y(s))\, ds| \\ &\leq |h|_0 + \int_0^t \alpha(s)\, g(|y(s)|)\, ds \end{aligned}$$

and again we have $|y(x)| \leq M_0$ for all $x \in [0,T]$. ■

Remark 1. The theory presented in this paper could be extended easily to the more general case of functional Volterra equations, i.e. to problems of the form

$$y(t) = V\, y(t) \quad \text{on } [0,T]. \tag{2.26}$$

Here V is an abstract Volterra equation and we also assume that there exists $x_0 \in \mathbf{R}^n$

with $V\,x(0) = x_0$ for all $x \in C([0,T],\mathbf{R}^n)$. We leave the details to the reader; again one must consider an adjusted problem

$$y(t) = V_\epsilon\, y(t) \quad \text{on} \quad [0,T]. \tag{2.27}$$

For example we can write (2.16) in the form (2.26) with

$$V\,y(t) = S \circ \mathscr{F}\, y(t),$$

where $S : L^q([0,T],\mathbf{R}^n) \to C([0,T],\mathbf{R}^n)$ is given by

$$S\,x(t) = h(t) + \int_0^t k(t,s)\,x(s)\,ds$$

and $\mathscr{F} : C([0,T],\mathbf{R}^n) \to L^q([0,T],\mathbf{R}^n)$ is given by

$$\mathscr{F}\,y\,(t) = f(t,y(t)).$$

Notice in this case $V_\epsilon\, y(t) = S \circ \mathscr{F}_\epsilon\, y(t)$ where $\mathscr{F}_\epsilon : C([0,T],\mathbf{R}^n) \to L^q([0,T],\mathbf{R}^n)$ is given by

$$\mathscr{F}_\epsilon\, y\,(t) = \tau_\epsilon(y(t))\,f(t,y(t)).$$

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