

MONOIDS ARISING FROM POSITIVE MATRICES OVER COMMUTATIVE C^* -ALGEBRAS

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ABSTRACT

In this paper we study the structure of the Cuntz monoid $S(C_0(X))$ associated with a locally compact space X . It is shown that for a wide class of spaces, namely locally compact σ -compact spaces, the Riesz decomposition on the monoid forces the (covering) dimension of the space to be zero. It is possible then to diagonalise matrices over $C_0(X)$ in a unique way with respect to an equivalence relation. We give a representation of $S(C_0(X))$ as a monoid of lower semicontinuous functions over X , from which order-cancellation on $S(C_0(X))$ follows.

Introduction

In [3], Cuntz introduced an equivalence relation for positive elements of $A \otimes K$, where A is a C^* -algebra and K the algebra of compact operators on a separable infinite-dimensional Hilbert space. The set of all such equivalence classes was denoted $S(A)$ in [11]; it is an abelian monoid that can be endowed with a partial order, which turns out to extend the Murray–von Neumann subequivalence of projections to general positive elements. The study of the monoid $S(A)$ has proved to be useful in different instances. For example, it was used in [3] to prove the existence of dimension functions for stably finite simple unital C^* -algebras (see also [1]), and in [11] to study structural properties of C^* -algebras of the form $B \otimes D$, where D is unital, simple and stably finite and B is a *UHF*-algebra.

If A is a σ -unital C^* -algebra with real rank zero and stable rank one, the author established in [9] the exact relation between $S(A)$ and $V(A)$. The aim in this paper is to push forward that relation in case the C^* -algebra is commutative. This means that it is of the form $C_0(X)$ for some locally compact σ -compact zero-dimensional topological space X . It turns out then that interesting aspects of the algebraic structure of the monoid $S(C_0(X))$ are stored in decomposition properties of X .

In [8, corollary 5.8], it is shown that totally disconnected compact Hausdorff spaces with the property that for every n each normal element in $M_n(C(X))$ can be continuously diagonalised are precisely sub-Stonean spaces (see [7]). In order to achieve our goal, we also prove that it is possible to eliminate the sub-Stonean condition (for locally compact and σ -compact spaces), thus obtaining a weaker form of diagonalisation, namely with respect to the equivalence relation mentioned before.

Section 1 is devoted to summarising the necessary basic notions. In Section 2 we relate the zero-dimensionality of a σ -compact Hausdorff space to a decomposition property of σ -compact open sets. For σ -compact Hausdorff spaces with covering

dimension equal to zero, $S(C_0(X))$ can be represented as a monoid of lower semi-continuous functions over X , which is cancellative. This is done in the first part of Section 3. We prove further that the Riesz decomposition property on $S(C_0(X))$ is equivalent to X having covering dimension zero. From this, a special diagonalisation of matrices over $C_0(X)$ follows.

1. Preliminaries

Let A be a C^* -algebra and $a, b \in A$. Following [3] and [11], we write $a \lesssim b$ if there is a sequence $\{x_n\}$ in A such that $x_n = r_n b s_n$ for some $r_n, s_n \in A$ and $x_n \rightarrow a$ in norm. We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. Then \sim is an equivalence relation. Let $M_\infty(A) = \varinjlim M_n(A)$ with inclusions $M_n(A) \ni x \mapsto \text{diag}(x, 0) \in M_{n+1}(A)$. For $x, y \in M_\infty(A)_+$, say that $x \sim y$ if and only if $x \sim y$ in $M_n(A)$ for some n . By [11, proposition 2.1], the relation \lesssim , when restricted to projections, gives the usual Murray–von Neumann subequivalence. For the case of \sim , its restriction to projections gives the Murray–von Neumann equivalence when the algebra has stable rank one. Denote by $\langle x \rangle$ the \sim -equivalence class of x and define $\langle x \rangle + \langle y \rangle = \langle x' + y' \rangle$ where $x', y' \in M_\infty(A)_+$ satisfy $x' \sim x$, $y' \sim y$ and $x'y' = 0$, or by $\langle x \rangle + \langle y \rangle = \langle \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \rangle$. We sometimes write $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ as $x \oplus y$. Note that if $xy = 0$, then $x + y \sim x \oplus y$. Define $S(A)$ to be the set of all \sim -equivalence classes in $M_\infty(A)_+$, and define a partial ordering on it by $\langle x \rangle \leq \langle y \rangle$ if and only if $x \lesssim y$. Thus $(S(A), \leq)$ is a partially ordered abelian monoid.

Let X be a locally compact Hausdorff space, and let $f \in C_0(X)$. We denote by $\text{Coz}(f)$ the cozero set of f , that is, $\text{Coz}(f) = \{x \in X \mid |f(x)| > 0\}$.

Lemma 1.1. *Let A be a C^* -algebra and $x, y \in A_+$. Then $x \lesssim y$ if and only if there exists $\{r_j\}$ in A with $x = \lim_{j \rightarrow \infty} r_j y r_j^*$. If $g, h \in C(\text{Spec}(x))_+$ and $\text{Coz}(g) \subseteq \text{Coz}(h)$, then $g(x) \lesssim h(x)$. In particular, if A is commutative, it follows that $f \lesssim g$ for $f, g \in A$ if and only if $\text{Coz}(f) \subseteq \text{Coz}(g)$.*

PROOF. The first conclusion is [11, proposition 2.4], while the second is observed in [11, section 1]. The last assertion follows from the first two. ■

Let M be a monoid. All monoids in this paper will be abelian, written additively. We say that M is a *refinement monoid* provided that, whenever $x_1, x_2, y_1, y_2 \in M$ satisfy $x_1 + x_2 = y_1 + y_2$, then $\sum_{j=1}^2 z_{ij} = x_i$ and $\sum_{i=1}^2 z_{ij} = y_j$ for some elements $z_{ij} \in M$.

If (M, \leq) is preordered, then we say that M is a *Riesz monoid* provided that M satisfies the *Riesz Decomposition Property*, that is, whenever $x, y_1, y_2 \in M$ satisfy $x \leq y_1 + y_2$, then there exist $x_1, x_2 \in M$ such that $x = x_1 + x_2$ and $x_i \leq y_i$ for all i . Finally, M is said to satisfy the *Riesz Interpolation Property* provided that, whenever $x_1, x_2, y_1, y_2 \in M$ such that $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in M$ such that $x_i \leq z \leq y_j$ for $i, j = 1, 2$.

If M is the positive cone of a partially ordered abelian group, then, by [6, proposition 2.1], these three properties are all equivalent. However, for general orderings, there is no relation among them.

A non-zero element $u \in M$ is called an *order-unit* provided that, for any $x \in M$, there exists $n \in \mathbb{N}$ such that $x \leq nu$. If A is a unital C^* -algebra, then $\langle 1_A \rangle$ is trivially an order-unit for $S(A)$.

Let M be a partially ordered abelian monoid. An *interval* in M is a nonempty subset $I \subseteq M$ that is upward directed and order-hereditary (that is, if $x \leq y$ and $y \in I$, then $x \in I$). Denote by $\mathcal{A}(M)$ the set of all intervals in M . We can endow $\mathcal{A}(M)$ with a natural abelian monoid structure, namely, if $I, J \in \mathcal{A}(M)$,

$$I + J = \{z \in M \mid z \leq x + y \text{ for some } x \in I, y \in J\}.$$

If M is a Riesz monoid, then $I + J = \{x + y \mid x \in I, y \in J\}$.

We say that an interval I is *countably generated* if there exists a countable cofinal subset $X \subseteq I$ (that is, there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq I$ such that, for all $x \in I$, there exists $n \in \mathbb{N}$ such that $x \leq x_n$). Notice that in this context we can write $I = \{y \in M \mid y \leq x_n \text{ for some } n\}$. If given intervals I and J , we have that $I \subseteq J$, then we say that I is *bounded* by J .

2. Some general topology

The purpose of this section is to express the condition of zero-dimensionality in terms of decomposition of cozero sets of continuous functions. We will make use of the following topological property, similar to normality, which is a modification of the open reduction property introduced by Wehrung in [13, definition 3.2].

Definition 2.1. Let X be a topological space. We say that X satisfies the σ -compact open reduction property if, given open and σ -compact subsets U and V of X , there exist open and σ -compact subsets $U' \subseteq U$ and $V' \subseteq V$ such that $U' \cap V' = \emptyset$ and $U \cup V = U' \cup V'$.

Note that the last condition may be written as $U \cup V = U' \sqcup V'$, where \sqcup denotes disjoint union of sets.

Our goal is to show that, for locally compact σ -compact Hausdorff spaces, this property coincides precisely with having zero covering dimension. To prove the desired equivalence we need some preliminary notions. These can be found in [5].

Assume now that X is a completely regular space. We will consider two notions for dimension of X : the usual covering dimension, denoted by $\dim X$ (see [4, 1.6.7]), and the one considered in [5, chapter 16], which we will denote by $\dim_1 X$. The definition of $\dim_1 X$ is the same as $\dim X$, except that only covers consisting of cozero sets are considered. It is shown in [5, corollary 16.9] that $\dim X = \dim_1 X$ if X is a normal space. If $\dim_1 X = 0$ then X is called *strongly zero-dimensional* (see [12, 3.34]), while if $\dim X = 0$ we call X *zero-dimensional*. As noted in [12, 3.39], not all zero-dimensional spaces are strongly zero-dimensional, so that the concepts of $\dim X$ and $\dim_1 X$ are not identical in general, but they coalesce for Lindelöf spaces by [12, 3.35].

Lemma 2.2. *Let X be a completely regular space. Then $\dim_1 X = 0$ if and only if, whenever $f, g \in C(X)$ satisfy $X = \text{Coz}(f) \cup \text{Coz}(g)$, there exist clopen subsets $U \subseteq \text{Coz}(f)$ and $V \subseteq \text{Coz}(g)$ of X such that $U \cap V = \emptyset$ and $U \cup V = X$.*

PROOF. It is a rephrasing of the equivalence (a) \Leftrightarrow (b) in [5, theorem 16.17]. ■

Remark 2.3. If X is a locally compact Hausdorff space, then the σ -compact open subsets of X are precisely the cozero sets of functions in $C_0(X)_+$. In fact, this follows from an easy application of Urysohn's Lemma (for a proof, see [7]).

Proposition 2.4. *Let X be a locally compact Hausdorff space. If X is strongly zero-dimensional, then X satisfies the σ -compact open reduction property. If X is σ -compact, then the converse is also true.*

PROOF. First, we show that if $Y \subseteq X$ is a σ -compact open subset of X , then Y is Lindelöf.

Take an open cover $\{U_\alpha\}_\alpha$ of Y and write $Y = \cup_n K_n$, where K_n are compact subsets of X . Each K_n is covered by finitely many elements of $\{U_\alpha\}_\alpha$, say $K_n \subseteq \cup_{i=1}^{s(n)} U_{\alpha_{i,n}}$, whence $Y \subseteq \cup_n \cup_i U_{\alpha_{i,n}}$.

Now assume that X is strongly zero-dimensional. Let $Y \subseteq X$ be a σ -compact open subset of X . Clearly, Y has a basis of clopen sets. Therefore, since Y is Lindelöf, it follows after the implication (c) \Rightarrow (a) of [5, theorem 16.17] that Y is strongly zero-dimensional.

Let U and V be open and σ -compact subsets of X . By Remark 2.3, there exist $f, g \in C_0(X)_+$ such that $U = \text{Coz}(f)$ and $V = \text{Coz}(g)$. Let $Y = \text{Coz}(f) \cup \text{Coz}(g)$. Then Y is open and σ -compact, so by the previous observation Y is strongly zero-dimensional. Notice that $f|_Y$ and $g|_Y$ belong to $C(Y)$ (because $\text{Coz}_X(f), \text{Coz}_X(g) \subseteq U$). Then, there exist relatively clopen sets $U' \subseteq \text{Coz}(f)$ and $V' \subseteq \text{Coz}(g)$ of Y such that $U' \cap V' = \emptyset$ and $U' \cup V' = Y$. Now, if $Y = \cup_n K_n$ for some compact sets K_n , then $U' = U' \cap Y = \cup_n (K_n \cap U')$. Notice that U' is a relatively closed subset of Y , hence $U' = Y \cap T$, for a closed set T of X . Then $K_n \cap U' = K_n \cap Y \cap T = K_n \cap T$, because $K_n \subseteq Y$. Observe that $K_n \cap T$ is closed in K_n , a compact space, so $K_n \cap T$ is compact. Thus $K_n \cap U'$ is compact for all n , whence U' is σ -compact open. Similarly, V' is σ -compact open. Therefore X satisfies the σ -compact open reduction property. For the converse, assume that X is σ -compact and that satisfies the σ -compact open reduction property. Then X is Lindelöf and hence completely regular by [5, 3.15].

Suppose that there exist $f, g \in C(X)$ such that $X = \text{Coz}(f) \cup \text{Coz}(g)$. Then $\text{Coz}(f)$ and $\text{Coz}(g)$ are open and σ -compact subsets of X . Therefore there exist open and σ -compact sets $U' \subseteq \text{Coz}(f)$ and $V' \subseteq \text{Coz}(g)$ such that $U' \cap V' = \emptyset$ and $U' \cup V' = \text{Coz}(f) \cup \text{Coz}(g) = X$. In particular, U' and V' are also clopen sets, so that X is strongly zero-dimensional by Lemma 2.2. ■

Corollary 2.5. *Let X be a locally compact σ -compact Hausdorff space. Then $\dim X = 0$ if and only if X satisfies the σ -compact open reduction property.*

PROOF. Since X is σ -compact, it is Lindelöf and therefore normal (see, e.g. [5, exercise 3D]). By [5, corollary 16.9], we have that $\dim X = \dim_1 X$. Thus the equivalence follows from Proposition 2.4. ■

3. The structure of $S(C_0(X))$

In this section we analyse the structure of the monoid $S(C_0(X))$ for a locally compact σ -compact Hausdorff space X . By using techniques from [9] and [13], we first give an explicit order-isomorphism between $S(C_0(X))$ and a monoid of lower semicontinuous functions over X . Since the Riesz decomposition property on $S(C_0(X))$ is the key that leads to a diagonal representation of the elements in the monoid, we establish for which spaces this decomposition holds. It is possible then to diagonalise matrices over $C_0(X)$ in a unique way with respect to the equivalence relation \sim .

Let X be a compact space. From [10, proposition 1.7] and [2, proposition 1.1] we have that $\text{sr}(C(X)) = \lceil \dim X/2 \rceil + 1$ and that $\text{RR}(C(X)) = \dim X$. In case the space is locally compact, then by definition $\text{sr}(C_0(X)) = \text{sr}(C_0(\tilde{X})) = \text{sr}(C(\alpha X))$ and $\text{RR}(C_0(X)) = \text{RR}(C(\alpha X))$, where αX is the one-point compactification of X . It follows that, if X is a zero-dimensional σ -compact Hausdorff space, then the real rank of $C_0(X)$ is zero and the stable rank is one.

When dealing with arbitrary matrices over a ring of continuous functions it is often useful to evaluate them at some point of the space where the functions are defined. It is easy to see that the usual matrix rank is an invariant for the order-relation \lesssim (that is, if $\langle a \rangle \leq \langle b \rangle$ in $S(C_0(X))$, then we have $\text{rank}(a(x)) \leq \text{rank}(b(x))$ for all $x \in X$).

Let X be a locally compact space. Denote by $LSC_{\sigma,b}(X, \mathbb{Z}^+)$ the abelian monoid of bounded lower semicontinuous functions over X that are countable (pointwise) suprema of functions from $C_0(X, \mathbb{Z}^+)$. This monoid has a natural ordered structure given by the pointwise ordering. Notice that if X is moreover σ -compact and zero-dimensional, then there is an isomorphism $V(C_0(X)) \cong C_0(X, \mathbb{Z}^+)$. Indeed, this isomorphism can be induced by applying the rank function pointwise on a class $[p]$, for a projection $p \in M_\infty(C_0(X))$.

Let M be a monoid. Let $A_\sigma(M)$ be the monoid (under addition) of countably generated intervals in M , (partially) ordered under set inclusion. Fix $D \in A_\sigma(M)$, and let $A_{\sigma,D}(M)$ be the hereditary submonoid of $A_\sigma(M)$ generated by D . Note that the elements of $A_{\sigma,D}(M)$ are countably generated intervals in M bounded by nD for some n .

For a C^* -algebra A , denote $D(A) = \{[p] \in V(A) \mid p \text{ is a projection from } A\}$. In the case where the algebra is commutative and σ -unital and has real rank zero, that is, of the form $C_0(X)$ for some zero-dimensional locally compact σ -compact Hausdorff space X , we can write

$$D = D(C_0(X)) = \{\chi_U \mid U \text{ is a compact open subset of } X\}.$$

The next result is the σ -compact version of [13, lemma 3.6], and it can be established using a similar argument.

Proposition 3.1. *Let X be a locally compact σ -compact Hausdorff space with $\dim X = 0$. Define the following maps:*

$$\Phi : A_{\sigma,D}(C_0(X, \mathbb{Z}^+)) \rightarrow LSC_{\sigma,b}(X, \mathbb{Z}^+), \quad I \mapsto \sup_{g \in I} g$$

$$\Psi : LSC_{\sigma,b}(X, \mathbb{Z}^+) \rightarrow A_{\sigma,D}(C_0(X, \mathbb{Z}^+)), \quad f \mapsto \{g \in C_0(X, \mathbb{Z}^+) \mid g \leq f\}.$$

Then Φ and Ψ are (order-preserving) mutually inverse maps and Φ is an ordered-monoid isomorphism. ■

Definition 3.2. Let (M, \leq) be a preordered monoid. We say that M is *order-cancellative* if, whenever $x, y, z \in M$ satisfy $x + z \leq y + z$, then $x \leq y$.

Theorem 3.3. *Let X be a locally compact σ -compact Hausdorff space with $\dim X = 0$. Then $S(C_0(X))$ is order-isomorphic to $LSC_{\sigma,b}(X, \mathbb{Z}^+)$. In particular, $S(C_0(X))$ is order-cancellative.*

PROOF. By [9, theorem 2.8], $S(C_0(X))$ is order-isomorphic to $A_{\sigma,D}(V(C_0(X)))$, where $D = D(C_0(X))$. Recalling that $V(C_0(X))$ is isomorphic to $C_0(X, \mathbb{Z}^+)$, we can apply Proposition 3.1 to achieve the first conclusion. The last assertion is clear. ■

The isomorphism in Theorem 3.3 can also be described by applying the rank function pointwise. Indeed, since every element in $M_{\infty}(C_0(X))_+$ can be diagonalised with respect to \sim (see Lemma 3.6 below), we only have to check that, if $a \in C_0(X)_+$, then $\text{Coz}(a) = \cup_{p \leq a} \text{Coz}(p)$, where p runs over the set of projections of $C_0(X)$. But this is the case so long as X is zero-dimensional. In fact, as observed in [11], the monoid $S(A)$ can be thought of as a noncommutative analogue for matrix rank.

As shown in [9], for σ -unital C^* -algebras with real rank zero and stable rank one, the monoid $S(A)$ is completely determined by $V(A)$, hence it does not carry any additional information that is not encoded in $V(A)$. From this point of view, the next result may be rephrased by saying that, whenever $S(C_0(X))$ carries extra information, it is not a Riesz monoid.

Theorem 3.4. *Let X be a locally compact σ -compact Hausdorff space. Then the following conditions are equivalent:*

- (i) $\dim X = 0$;
- (ii) $RR(C_0(X)) = 0$;
- (iii) $S(C_0(X))$ is a Riesz monoid.

PROOF. (i) \Leftrightarrow (ii) is clear.

(ii) \Rightarrow (iii). Let $A = C_0(X)$. If $RR(A) = 0$, then $\dim X = 0$. Hence $\text{sr}(A) = 1$ and so $S(A)$ is a Riesz monoid by [9, theorem 2.13].

(iii) \Rightarrow (i). By Corollary 2.5, it suffices to show that X satisfies the σ -compact open reduction property. Let $A = C_0(X)$ and let $f, g \in A$. Since $\text{Coz}(f) = \text{Coz}(|f|)$ and $\text{Coz}(g) = \text{Coz}(|g|)$, we can assume with no loss of generality that $f, g \geq 0$.

Notice that $\text{Coz}(f + g) = \text{Coz}(f) \cup \text{Coz}(g)$. As X is σ -compact, the C^* -algebra $C_0(X)$ is σ -unital, hence there exists a sequence $\{f_n\}$ of functions that form an approximate unit. Now observe that

$$\begin{pmatrix} f_n & f_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} f_n & 0 \\ f_n & 0 \end{pmatrix} = \begin{pmatrix} f_n(f + g)f_n & 0 \\ 0 & 0 \end{pmatrix},$$

and this tends to $\begin{pmatrix} f+g & 0 \\ 0 & 0 \end{pmatrix}$ as $n \rightarrow \infty$. Therefore $f + g \lesssim f \oplus g$. By the Riesz decomposition property, there exist $F, G \in M_n(A)$ for some n such that $f + g \sim F \oplus G$ while $F \lesssim f$ and $G \lesssim g$. In particular, if $F = (f_{ij})$ for some $f_{ij} \in A$, then for each $k \in \mathbb{N}$ there exist functions $a_1^k, \dots, a_n^k \in A$ such that

$$\begin{pmatrix} a_1^k & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n^k & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{a_1^k} & \dots & \overline{a_n^k} \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \rightarrow F,$$

hence $f_{ij} = \lim_{k \rightarrow \infty} f a_i^k \overline{a_j^k}$ and so $\text{Coz}(f_{ij}) \subseteq \text{Coz}(f)$ for all i and j .

Similarly, if $G = (g_{ij})$, then $\text{Coz}(g_{ij}) \subseteq \text{Coz}(g)$ for all i and j . Hence, we have $\text{Coz}(\sum_{i,j} |f_{ij}|) = \cup_{i,j} \text{Coz}(|f_{i,j}|) \subseteq \text{Coz}(f)$, and $\text{Coz}(\sum_{k,l} |g_{k,l}|) \subseteq \text{Coz}(g)$.

Now, using the fact that the matrix rank is an invariant for the relation \lesssim , the condition $f + g \sim F \oplus G$ implies that

$$\left(\sum_{i,j} |f_{i,j}| \right) \left(\sum_{k,l} |g_{k,l}| \right) = 0,$$

as well as $\text{Coz}(\sum_{i,j} |f_{i,j}|) \sqcup \text{Coz}(\sum_{k,l} |g_{k,l}|) = \text{Coz}(f + g) = \text{Coz}(f) \cup \text{Coz}(g)$. Hence, letting $U = \text{Coz}(\sum_{i,j} |f_{i,j}|)$ and $V = \text{Coz}(\sum_{k,l} |g_{k,l}|)$, we get that $U \sqcup V = \text{Coz}(f) \cup \text{Coz}(g)$ and that $U \subseteq \text{Coz}(f)$ and $V \subseteq \text{Coz}(g)$, whence X satisfies the σ -compact open reduction property. ■

Corollary 3.5. *Let X be a locally compact σ -compact Hausdorff space. If $S(C_0(X))$ is a Riesz monoid, then it satisfies the refinement and interpolation properties.*

PROOF. Let $A = C_0(X)$. If $S(A)$ is a Riesz monoid, then $RR(A) = 0$ by Theorem 3.4, and so $\text{sr}(A) = 1$. Apply then [9, theorem 2.13]. ■

We will see now that it is possible to describe more precisely the structure of $S(C_0(X))$, diagonalising matrices in a special way. First we state a general fact concerning σ -unital C^* -algebras with real rank zero and stable rank one. It follows from [9, lemma 2.2] that, if A is a σ -unital C^* -algebra and $\{e_n\}$ is an increasing approximate unit, then the element $e = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \in A$ satisfies that the class $u := \langle e \rangle$ is an order-unit for $S(A)$.

Lemma 3.6. *Let A be a σ -unital C^* -algebra with real rank zero, stable rank one, and let $u = \langle e \rangle$ (for some $e \in A$) be an order-unit for $S(A)$. If $a \in M_\infty(A)_+$ and $\langle a \rangle \leq u$, then there exists $a' \in A$ such that $a \sim a'$. Furthermore, every element in $M_\infty(A)_+$ is \sim -equivalent to a diagonal matrix over A .*

PROOF. Let $a \in M_\infty(A)_+$ and assume that $\langle a \rangle \leq u$. Take an (increasing) approximate unit $\{p_n\}$ of projections for the hereditary algebra generated by a . Let $r = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n$.

Then $a \sim r$ by [9, lemma 2.2]. Observe that $e \lesssim 1$ (in \tilde{A}). Now, since $p_1 \lesssim a \lesssim e \lesssim 1$, we have that $p_1 \sim q_1 \leq 1$ for some projection q_1 and, in fact, $q_1 \in A$. Moreover, since $p_2 \lesssim 1$, there exists a projection $r_2 \in M_\infty(A)$ such that $(p_2 - p_1) \oplus p_1 \oplus r_2 \sim 1$. By cancellation on projections, it follows from the relation $(p_2 - p_1) \oplus p_1 \oplus r_2 \sim q_1 \oplus (1 - q_1)$ that $(p_2 - p_1) \oplus r_2 \sim (1 - q_1)$, whence there exists a projection $q'_2 \leq 1 - q_1$ such that $(p_2 - p_1) \sim q'_2$. Notice that $q'_2 \in A$, and define $q_2 = q_1 \oplus q'_2 \in A$. Observe that $p_2 = (p_2 - p_1) \oplus p_1 \sim q'_2 \oplus q_1 = q_2$.

By induction on n , we obtain an increasing sequence $\{q_n\}$ of projections in A such that $p_n \sim q_n$. Let $a' = \sum_{n=1}^{\infty} \frac{1}{2^n} q_n \in A$. By [9, proposition 2.3] $r \sim a'$ and so $a \sim r \sim a'$.

Now let $a \in M_\infty(A)_+$. There exists $n \in \mathbb{N}$ such that $\langle a \rangle \leq nu$. Using the Riesz decomposition property ([9, theorem 2.13]), there exist $a_1, \dots, a_n \in M_\infty(A)_+$ such that $\langle a \rangle = \langle a_1 \rangle + \dots + \langle a_n \rangle$ and $\langle a_i \rangle \leq u$ for all i . By the argument in the preceding paragraph, there exist $a'_i \in A_+$ such that $a_i \sim a'_i$, so $a \sim \text{diag}(a'_1, \dots, a'_n)$. ■

Proposition 3.7. *Let X be a zero-dimensional locally compact σ -compact Hausdorff space. Given continuous functions $f_1, \dots, f_n \in C_0(X)_+$, there exist $g_1, \dots, g_n \in C_0(X)_+$ such that $\text{Coz}(g_i) \subseteq \text{Coz}(g_{i-1})$ for $i = 2, \dots, n$ and $\text{Coz}(g_1) = \cup_{i=1}^n \text{Coz}(f_i)$, which satisfy*

$$f_1 \oplus \dots \oplus f_n \sim g_1 \oplus \dots \oplus g_n.$$

PROOF. We proceed by induction on n .

$n = 2$: Suppose we have $f, g \in C_0(X)_+$. Because of the zero-dimensionality of X and Corollary 2.5, we get $\text{Coz}(f) \cup \text{Coz}(g) = \text{Coz}(f') \sqcup \text{Coz}(g')$ for some $f', g' \in C_0(X)$ such that $\text{Coz}(f') \subseteq \text{Coz}(f)$ and $\text{Coz}(g') \subseteq \text{Coz}(g)$. Therefore $f \sim ff' \oplus fg'$ and $g \sim gf' \oplus gg'$, hence

$$\begin{aligned} f \oplus g &\sim ff' \oplus fg' \oplus gf' \oplus gg' \sim f' \oplus fg' \oplus gf' \oplus g' \\ &\sim f' \oplus g' \oplus fg' \oplus gf'. \end{aligned}$$

Let $g_1 = f' \oplus g'$ and $g_2 = fg' \oplus gf'$. Then $\text{Coz}(g_2) \subseteq \text{Coz}(g_1) = \text{Coz}(f) \cup \text{Coz}(g)$.

$n > 2$: Using the σ -compact open reduction property, write $\text{Coz}(f_1) \cup \dots \cup \text{Coz}(f_n) = \text{Coz}(f'_1) \sqcup \dots \sqcup \text{Coz}(f'_n)$ for some $f'_i \in C_0(X)_+$, with $\text{Coz}(f'_i) \subseteq \text{Coz}(f_i)$. Then, for $i = 1, \dots, n$ we have

$$f_i \sim f_i f'_1 \oplus \dots \oplus f_i f'_i \oplus \dots \oplus f_i f'_n \sim f_i f'_1 \oplus \dots \oplus f'_i \oplus \dots \oplus f_i f'_n.$$

Therefore

$$\begin{aligned} f_1 \oplus \dots \oplus f_n &\sim (f'_1 \oplus f_1 f'_2 \oplus \dots \oplus f_1 f'_n) \oplus \dots \oplus (f_n f'_1 \oplus \dots \oplus f_n f'_{n-1} \oplus f'_n) \\ &\sim (f'_1 \oplus \dots \oplus f'_n) \oplus (f_1(f'_2 \oplus \dots \oplus f'_n) \oplus f_2 f'_1) \oplus \dots \oplus (f_{n-1} f'_n \oplus f_n(f'_1 \oplus \dots \oplus f'_{n-1})). \end{aligned}$$

By induction hypothesis, there exist functions $G_2, \dots, G_n \in C_0(X)_+$ with $\text{Coz}(G_i) \subseteq \text{Coz}(G_{i-1})$ for $i = 3, \dots, n$ and

$$\text{Coz}(G_2) = \text{Coz}(f_1(f'_2 \oplus \dots \oplus f'_n) \oplus f_2 f'_1) \cup \dots \cup \text{Coz}(f_{n-1} f'_n \oplus f_n(f'_1 \oplus \dots \oplus f'_{n-1})).$$

Note that $\text{Coz}(G_2) \subseteq \text{Coz}(f'_1 \oplus \dots \oplus f'_n)$. Set $G_1 = f'_1 \oplus \dots \oplus f'_n \in C_0(X)_+$. Thus

$$f_1 \oplus \dots \oplus f_n \sim G_1 \oplus \dots \oplus G_n. \quad \blacksquare$$

We use now the previous results in order to diagonalise matrices over $C_0(X)_+$ in a certain unique way, as follows.

Theorem 3.8. *Let X be a locally compact σ -compact Hausdorff space with $\dim X = 0$. Then, for all $n \in \mathbb{N}$, every element in $M_n(C_0(X))_+$ is equivalent, with respect to \sim , to a diagonal matrix $\text{diag}(f_1, \dots, f_n)$, where $f_i \in C_0(X)_+$ for all i , and $\text{Coz}(f_n) \subseteq \dots \subseteq \text{Coz}(f_1)$. Further, if g_1, \dots, g_n are elements in $C_0(X)_+$ such that $\text{Coz}(g_n) \subseteq \dots \subseteq \text{Coz}(g_1)$, then $\text{diag}(f_1, \dots, f_n) \sim \text{diag}(g_1, \dots, g_n)$ if and only if $f_i \sim g_i$ for all i .*

PROOF. By Lemma 3.6 and Proposition 3.7, we get that, if $a \in M_n(C_0(X))_+$, then there exist functions $f_1, \dots, f_n \in C_0(X)_+$ such that $\text{Coz}(f_i) \subseteq \text{Coz}(f_{i-1})$, for $i = 2, \dots, n$, that satisfy $a \sim \text{diag}(f_1, \dots, f_n)$. The second assertion is easily checked. \blacksquare

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