

# EQUALITIES INVOLVING THE MODULUS OF AN OPERATOR

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[Received 9 January 1998. Read 1 October 1998. Published 30 September 1999.]

## ABSTRACT

Let  $E$  and  $F$  be Banach lattices and  $T : E \rightarrow F$  be a regular operator with a modulus  $|T|$ . Suppose also that  $j : F \rightarrow F''$  is the natural embedding. It is known that in general we need have neither  $|j \circ T| = j \circ |T|$  nor  $|T'| = |T|'$ . We show that these two equalities either hold or fail to hold together and obtain results that go some way towards describing exactly when these equalities do hold, thus obtaining a partial converse of a result of Synnatzschke. In the course of this we study weak\* sequential continuity of the lattice operations in  $E'$ .

## 1. Introduction

The space of regular operators from a Banach lattice  $E$  into a Banach lattice  $F$  need not form a lattice. Various ways of embedding this space into a lattice of operators have been suggested in the literature. Given that one obtains a lattice if the range space is Dedekind complete and that the dual of a Banach lattice is always Dedekind complete, people have looked at the operators from  $E$  into  $F''$  and those from  $F'$  into  $E'$ . In both cases there is a natural embedding of a regular operator from  $E$  into  $F$  into this new space. In the first case map  $T$  to  $JT$ , where  $J$  is the natural embedding of  $F$  into  $F''$ , whilst in the second map  $T$  to  $T'$ . Unfortunately neither of these constructions interacts very well with the order structure of the original space of operators. In particular if it should happen that  $|T|$  exists then we need not have  $J|T| = |JT|$  (see the example after lemma 2 in [2]) nor need we have  $|T'| = |T|'$  (see example 5.9 of [1], but be aware that the domain of the operator needs to be taken to be  $c$  rather than  $\ell_1$ ). Both of these counterexamples even have the range space being Dedekind complete, so there seems to be little prospect of a very general positive result here.

Synnatzschke proved that  $|T'| = |T|'$  provided  $F$  has an order continuous norm (theorem 5.11 of [1] yields this easily). It is also easy to verify that if  $E$  is atomic with an order continuous norm then  $|T'| = |T|'$ . In §2 we show that both possible equalities either hold or fail to hold simultaneously, whilst in §4 we come close to showing that the two cases where equality is known to hold are the only ones. The proofs there involve weak\* sequential continuity of the lattice operations in the dual of a Banach lattice, so we study this in §3 first.

We refer to [1] or [5] for any unexplained terms from Banach lattice theory.

## 2. Equivalence of the equalities

We start this section with its main result.

**Theorem 2.1.** Suppose that  $E$  and  $F$  are Banach lattices and  $T$  is a regular operator from  $E$  into  $F$  such that the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$ . Then the following assertions are equivalent:

- (1)  $J|T| = |JT|$ ,
- (2)  $|T|' = |T'|$ .

PROOF. Before starting the proof let us gather some simple facts about the operators involved. Each of these may be verified in a routine manner. Apart from using  $J$  to denote the natural embedding of  $F$  into  $F''$ , we also let  $i : E \rightarrow E''$  and  $j : F' \rightarrow F'''$  denote natural embeddings.

- ( $\alpha$ )  $|JT| \leq J|T|$ ,
- ( $\beta$ )  $|T'| \leq |T|'$ ,
- ( $\gamma$ )  $J'j$  is the identity on  $F'$  and hence  $(JS)'j = S'J'j = S'$  for all  $S \in \mathcal{L}(E, F)$ ,
- ( $\delta$ )  $JS = S''i$  for all  $S \in \mathcal{L}(E, F)$ ,
- ( $\epsilon$ )  $(U'i)'j = U$  for all  $U \in \mathcal{L}(F', E')$ ,
- ( $\eta$ )  $V = j'V''i = (V'j)'i$  for all  $V \in \mathcal{L}(E, F'')$ .

PROOF THAT (1)  $\Rightarrow$  (2). Given that  $T \in \mathcal{L}^r(E, F)$  has a modulus, we have

$$\begin{aligned}
 |JT|'j &= |T''i'j && [(\delta) \text{ with } S \text{ replaced by } T] \\
 &\leq (|T''i)'j \\
 &\leq (|T'|'i)'j && [(\beta) \text{ applied to } T'] \\
 &= |T'| && [(\epsilon) \text{ applied to } |T'|] \\
 &= |(JT)'j| && [(\gamma) \text{ with } S \text{ replaced by } T] \\
 &\leq |(JT)'j \\
 &\leq |JT|'j && [(\beta)]
 \end{aligned}$$

so that  $|JT|'j = |T'|$ . Applying the same reasoning to  $|T|$  rather than to  $T$  we see that

$$(J|T|)'j = |J|T||'j = ||T'| = |T'|.$$

Thus if  $J|T| = |JT|$  then we have

$$|T'| = |JT|'j = (J|T|)'j = |T'|.$$

PROOF THAT (2)  $\Rightarrow$  (1). Note that by ( $\gamma$ ) we have  $(JT)'j = T'$  so that  $|T'| \leq |(JT)'j \leq |JT|'j$  using ( $\beta$ ). Hence

$$\begin{aligned}
 |T'|'i &\leq (|JT|'j)'i \\
 &= j'|JT|''i \\
 &= |JT| && [(\eta) \text{ replacing } V \text{ by } |JT|] \\
 &= |T''i && [(\delta)] \\
 &\leq |T''i \\
 &\leq |T'|'i && [(\beta)]
 \end{aligned}$$

so that  $|T'|'i = |JT|$ . Again if we replace  $T$  by  $|T|$  then we have

$$|T''i = ||T'|'i = |J|T|| = J|T|.$$

Thus if  $|T'| = |T'|$  then

$$|JT| = |T'|'i = |T''i = J|T|. \quad \blacksquare$$

In what follows we shall say that the pair of Banach lattices  $(E, F)$  has the *invariant modulus property* if the two equivalent conditions in Theorem 2.1 hold for every regular operator from  $E$  into  $F$  that has a modulus. A direct consequence of this result and the formula, in theorem 1.16 (iii) of [1], for the modulus of an operator is the following result.

**Corollary 2.2.** *Let  $E$  and  $F$  be Banach lattices,  $T$  a regular operator from  $E$  into  $F$  such that the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$ . Then the following assertions are equivalent:*

- (1)  $|JT| = J|T|$ ,
- (2)  $|T'|f = \sup\{\sum_{i=1}^n |T'f_i| : n \in \mathbb{N}, 0 \leq f_i, \sum_{i=1}^n f_i = f\}$  for all  $f \in F'_+$ ,
- (3)  $f(|T|x) = \sup\{\sum_{i=1}^n f_i(Tx_i) : n \in \mathbb{N}, |x_i| \leq x, 0 \leq f_i, \sum_{i=1}^n f_i = f\}$  for all  $x \in E_+$  and all  $f \in F'_+$ .

**Theorem 2.3.** *Let  $E$  and  $F$  be Banach lattices,  $T$  a regular operator from  $E$  into  $F$  such that the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$ . If there exists an  $x \in E_+$  such that  $|T|x \notin I$ , where  $I$  is the closed ideal of  $F$  generated by  $T[0, x]$ , then  $|JT| \neq J|T|$ .*

PROOF. From the Hahn–Banach Theorem there exists a  $f \in F'$  such that  $f|_I = 0$  and  $f(|T|x) > 0$ . We may assume that  $f \geq 0$  as  $I$  is an ideal. Thus  $f(|T|x) > 0 = \sup\{\sum_{i=1}^n f_i(Tx_i) : |x_i| \leq x, 0 \leq f_i, \sum_{i=1}^n f_i = f\}$ . It follows from Corollary 2.2 that  $|JT| \neq J|T|$ .  $\blacksquare$

**Corollary 2.4.** *Suppose that  $E$  and  $F$  are Banach lattices and  $T$  is a regular operator from  $E$  into  $F$  such that the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$ . If  $|T|E$  is not contained in the closed ideal of  $F$  generated by  $TE$ , then  $|JT| \neq J|T|$ .*

### 3. Weak\* sequential continuity of lattice operations

In §4 we shall see that, under certain fairly weak assumptions, the equality in Theorem 2.1 is intimately related to the question of whether or not the lattice operations in  $E'$  are weak\* sequentially continuous. This is clearly the case if  $E$  is atomic with an order continuous norm and we have no other examples. Nor do we have a general proof of the equivalence of the two conditions. Nevertheless we do have a number of different hypotheses that guarantee the equivalence not only of these two conditions but also of a third, which is apparently rather weaker.

**Theorem 3.1.** *Let  $E$  be a Banach lattice that is (i) Dedekind  $\sigma$ -complete or (ii) separable or (iii) isomorphic to a space  $C_0(\Sigma)$  of continuous real-valued functions vanishing at infinity on a locally compact Hausdorff space (equipped with the pointwise order and the supremum norm). The following three conditions are equivalent:*

- (1)  $E$  is atomic with an order continuous norm,
- (2) the lattice operations in  $E'$  are weak\* sequentially continuous,
- (3) if  $(f_n)$  is a sequence in  $E'$  such that  $f_n \rightarrow f$  (weak\*) and  $|f_n| \rightarrow h$  (weak\*) then  $h = |f|$ .

PROOF. The proof that (1) $\Rightarrow$ (2) is routine, whilst the proof that (2) $\Rightarrow$ (3) is simple. It remains to show that (3) $\Rightarrow$ (1). We give different proofs in the three cases.

Suppose that  $E$  is Dedekind  $\sigma$ -complete. We show first, by contradiction, that  $E$  has an order continuous norm. By corollary 2.3 of [3] the lattice operations in an L-space are weakly sequentially continuous if and only if the L-space is atomic. In fact it is shown there that in this case there is a sequence in the L-space that converges weakly to 0 but all having the same (non-zero) modulus (essentially a sequence of Rademacher functions). Hence in  $\ell'_\infty = \mathcal{M}(\beta(\mathbb{N}))$ , the space of measures on the Stone-Ćech compactification of  $\mathbb{N}$ , there is a sequence  $(f_n)$  that converges weakly to 0, but with  $|f_n| = h \neq 0$  for all  $n \in \mathbb{N}$ . In [4], Grothendieck showed that a sequence of measures on a Stonean space (such as  $\beta(\mathbb{N})$ ) is weakly convergent if and only if it is weak\* convergent, so it follows immediately that (3) fails.

By corollary 2.4.3 of [5] there is a sublattice  $H$  of  $E$  that is isomorphic to  $\ell_\infty$ . As  $\ell_\infty$  is an injective Banach lattice there is a positive projection  $\mathcal{P}$  of  $E$  onto  $H$ . Using the result in the preceding paragraph we conclude that there is a sequence  $(f_n)$  in  $H'$  with  $f_n(x) \rightarrow 0$  for all  $x \in H$  but with  $|f_n| = h \neq 0$  for all  $n \in \mathbb{N}$ . The sequence  $(f_n \circ \mathcal{P})$  in  $E'$  is clearly weak\*-convergent to 0. On the other hand, it is easy to see that  $|f_n \circ \mathcal{P}| = |f_n| \circ \mathcal{P} = h \circ \mathcal{P} \neq 0$  so that (3) certainly fails.

Now we show that  $E$  is atomic. Without loss of generality we may assume that  $E$  has a weak order unit so that by theorem 2.7.8 of [5] there exists a probability measure space  $(\Omega, \Sigma, \mu)$  such that

$$L^\infty(\mu) \subset E \subset L^1(\mu) \quad L^\infty(\mu) \subset E' \subset L^1(\mu)$$

and  $x'(x) = \int_\Omega x'x d\mu$  for all  $x' \in E'$  and  $x \in E$ . If  $E$  is not atomic then  $\mu$  is not atomic so there exist  $r_n \in L^\infty(\mu) \subset E'$  with  $|r_n| = \chi_B$  for some  $B \in \Sigma$  with  $\mu(B) > 0$  and  $\int_\Omega r_n f d\mu \rightarrow 0$  as  $n \rightarrow \infty$  for all  $f \in L^1(\mu)$  (see proposition 2.1 of [3] for details). Therefore  $r_n \rightarrow 0$  (weak\*) and it follows from (3) that  $|r_n| = \chi_B \rightarrow 0$ , which is impossible. Thus  $E$  is atomic as claimed.

Now suppose that  $E$  is separable. Note that (3) is easily seen to be equivalent to the requirement that if  $f_n \rightarrow f$ ,  $g_n \rightarrow g$  and  $f_n \vee g_n \rightarrow h$  (weak\*) then  $h = f \vee g$  and also to the corresponding assertion with ' $\vee$ ' replaced by ' $\wedge$ '. Let us suppose that (1) fails, then there is  $x \in E_+$  such that the order interval  $[0, x]$  is not norm compact. Let  $E_x$  denote the ideal generated by  $x$  in  $E$ , normed by the order unit norm induced by  $x$ , and let  $I : E_x \rightarrow E$  denote the natural embedding, which is bounded and interval preserving but not compact. It follows from Schauder's theorem that  $I'$  also fails to be compact. Choose a sequence  $(g_n)$  in the unit ball of  $E'$  such

that the sequence  $(f_n) = (I'g_n)$  has no norm convergent subsequence. Without loss of generality, we may suppose that the  $g_n$  (and hence the  $f_n$ ) are positive. As  $E$  is separable the weak\* topology on the unit ball of  $E'$  is metrisable so we may assume, by extracting a subsequence, that  $g_n \rightarrow g$  (weak\*). By hypothesis, we may now assume that  $g_n \wedge g \rightarrow g$  (weak\*). As  $I$  is interval preserving,  $I'$  is a lattice homomorphism [1, theorem 7.7] so that

$$f_n \wedge I'g = I'g_n \wedge I'g \tag{3.1}$$

$$= I'(g_n \wedge g) \tag{3.2}$$

$$\rightarrow I'g \text{ (weak*)}. \tag{3.3}$$

As  $f_n \wedge I'g \leq I'g$  in the unital M-space  $E_x$ , we see that

$$\|f_n \wedge I'g - I'g\| = (I'g - f_n \wedge I'g)(x) \rightarrow 0.$$

Similarly we see that  $f_n \vee I'g \rightarrow I'g$  in norm and it follows easily that  $f_n \rightarrow I'g$  in norm, contradicting the choice of the sequence  $(f_n)$ .

Finally let us turn our attention to the case when  $E = C_0(\Sigma)$  for some locally compact Hausdorff space  $\Sigma$ . We need to show that every compact subset of  $\Sigma$  is finite. First we note that there can be no convergent sequence in  $\Sigma$ , for if  $\sigma_n \rightarrow \sigma_0$  then  $\delta_{\sigma_n} - \delta_{\sigma_0} \rightarrow 0$  (weak\*) but  $|\delta_{\sigma_n} - \delta_{\sigma_0}| = \delta_{\sigma_n} + \delta_{\sigma_0} \rightarrow 2\delta_{\sigma_0} \neq 0$  (weak\*), where  $\delta_\tau(f) = f(\tau)$ . We may thus apply theorem 1.6 of [7] to deduce that there exists a positive Borel measure  $\nu$  on  $B$  and Borel measurable functions  $\phi_n$  on  $B$  such that  $\nu(B) = 1$ ,  $|\phi_n| = 1$   $\nu$ -almost everywhere on  $B$ , and  $\int_B \phi_n \phi_m d\nu = 0$  for all  $n, m \in \mathbb{N}$  with  $n \neq m$ . Since  $\{\phi_n\}$  is an orthonormal sequence in  $L^2(\nu)$  it follows that  $\int_B h \phi_n d\nu \rightarrow 0$  as  $n \rightarrow \infty$  for all  $h \in C(B)$  and hence for all  $h \in C_0(\Sigma)$ . Thus if we define  $f_n(h) = \int_B h \phi_n d\nu$  and  $f_0(h) = \int_B h d\nu$  for all  $h \in C_0(\Sigma)$ , then  $f_n \rightarrow 0$  (weak\*) as  $n \rightarrow \infty$  and  $|f_n| = f_0$  for all  $n \in \mathbb{N}$ . By (3) this would certainly force  $f_0 = 0$  and hence  $\nu$  is the zero measure, contradicting its definition. ■

Whilst we always have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) we do not know if either of the reverse implications always holds. Weak\* sequential continuity of the lattice operations featured in [6] in a slightly more general context as a necessary and sufficient condition for the regular operators from  $E$  into  $c$  to form a lattice. In theorem 10.3 of [6], van Rooij noted the equivalence of this space of operators being a lattice with  $E$  being atomic with an order continuous norm, provided  $E$  was separable. Combining our Theorem 3.1 with other results from [6] we obtain two other cases where this equivalence is true. Although it may look a rather artificial condition, our condition (3) is actually precisely what is needed for our equalities involving the modulus to hold when the range space is taken to be  $c$ , providing more incentive to study the relationship between the three conditions in Theorem 3.1 in a general setting.

**Theorem 3.2.** *If  $E$  is a Banach lattice then the following conditions are equivalent:*

- (1) *the pair  $(E, c)$  has the invariant modulus property,*
- (2) *if  $(f_n)$  is a sequence in  $E'$  such that  $f_n \rightarrow f$  (weak\*) and  $|f_n| \rightarrow h$  (weak\*) then  $h = |f|$ .*

PROOF. It is routine to verify that bounded operators  $T$  from  $E$  into  $c$  correspond to weak\* convergent sequences  $(f_n)$  of bounded linear functionals on  $E$  by means of the formula  $Tx = (f_n(x))$  and that  $|T|$  exists if and only if  $(|f_n|)$  is weak\* convergent and that then  $|T|x = (|f_n|(x))$ . The natural embedding  $J$  of  $c$  into  $\ell_\infty$  may be regarded as mapping a sequence  $(t_n) \in c$  to  $(\lim_{n \rightarrow \infty} t_n, t_1, t_2, \dots) \in \ell_\infty$ . Consequently  $JTx = (f(x), f_1(x), f_2(x), \dots)$ , and hence  $|JT|x = (|f|(x), |f_1|(x), |f_2|(x), \dots)$ , whilst  $J|T|x = (\lim_{n \rightarrow \infty} |f_n|(x), |f_1|(x), |f_2|(x), \dots)$  and it is clear that the two are equal if and only if  $\lim_{n \rightarrow \infty} |f_n| = |f|$ . This suffices to prove the result. ■

The following result may be proved by the methods used in the separable case of the proof of Theorem 3.1 plus elementary arguments for the converse implication.

**Theorem 3.3.** *If  $E$  is a Banach lattice then the following conditions are equivalent:*

- (1)  $E$  is atomic with an order continuous norm,
- (2) the lattice operations in  $E'$  are weak\* continuous on bounded sets.

#### 4. The invariant modulus property

We should like to characterise precisely which pairs have the invariant modulus property, but are unable to do so. If we assume that  $F$  is Dedekind  $\sigma$ -complete then we have a result that gives both a necessary condition and a sufficient condition for this to happen, which coalesce under the hypotheses of Theorem 3.1.

**Theorem 4.1.** *Let  $E$  and  $F$  be Banach lattices with  $F$  Dedekind  $\sigma$ -complete and consider the following three conditions:*

- (1) at least one of following two conditions holds:
  - a)  $E$  is atomic with an order continuous norm,
  - b) the norm on  $F$  is order continuous;
- (2) the pair  $(E, F)$  has the invariant modulus property;
- (3) at least one of the following two conditions holds:
  - a) the lattice operations in  $E'$  are weak\* sequentially continuous,
  - b) the norm on  $F$  is order continuous.

We always have (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) and if  $E$  satisfies (i), (ii) or (iii) of Theorem 3.1 then (3)  $\Rightarrow$  (1).

PROOF. That (1) $\Rightarrow$ (2) is Synnatschke's result in case (b) and is routine in case (a). The cases when (3) $\Rightarrow$ (1) are a direct consequence of Theorem 3.1, so we need only prove that (2) $\Rightarrow$ (3).

Suppose that the lattice operations on  $E'$  are not  $\sigma(E', E)$ -sequentially continuous and the norm on  $F$  is not order continuous. Then there exist a disjoint sequence  $y_n \in [0, y]$  with  $\|y_n\| \geq 1$  for all  $n \in \mathbb{N}$  and  $y = \sup\{y_n : n \in \mathbb{N}\}$ , and  $f_n \in E'$  such that  $f_n \rightarrow 0$  (weak\*) but such that for some  $x_0 \in E_+$  we have  $|f_n|(x_0) \geq 1$  for all  $n \in \mathbb{N}$ .

Define  $T : E \rightarrow F$  by  $Tx = \sum_{n=1}^{\infty} f_n(x)y_n$ , then the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$  and is given by the order convergent series

$$|T|x = \sum_{n=1}^{\infty} |f_n|(x)y_n.$$

Thus  $|T|x_0 \geq y$ . Moreover we claim that  $|T|x_0 \notin \bar{I}$ , where  $I$  is the ideal generated by the image of  $T$ . Indeed, for each  $z \in I$  there exist  $m \in \mathbb{N}$  and  $x_i \in E$  with  $|z| \leq |Tx_1| + |Tx_2| + \dots + |Tx_m| = \sum_{n=1}^{\infty} \lambda_n y_n$ , where  $\lambda_n = \sum_{i=1}^m |f_n(x_i)| \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$\begin{aligned} \left\| |T|x_0 - z \right\| &\geq (|T|x_0 - z)^+ \\ &\geq (y - z)^+ \\ &\geq (y - \sum_{n=1}^{\infty} \lambda_n y_n)^+ \\ &\geq (y_k - \sum_{n=1}^{\infty} \lambda_n y_n)^+ \\ &\geq (1 - \lambda_k)y_k \end{aligned}$$

for all  $k \in \mathbb{N}$ . It follows that

$$\left\| |T|x_0 - z \right\| \geq \|(y - z)^+\| \geq (1 - \lambda_n)$$

for all sufficiently large  $n$ , which implies that  $\left\| |T|x_0 - z \right\| \geq 1$  for all  $z \in I$ , as  $\lambda_n \rightarrow 0$ . Thus  $|T|x_0 \notin \bar{I}$  and by Corollary 2.4 we have  $|JT| \neq J|T|$ , which contradicts (2). ■

If we allow the domain or range to vary then our results are rather more complete. To start with, if we take for  $E$  any space for which the lattice operations in  $E'$  are not weak\* sequentially continuous then we obtain immediately the following corollary.

**Corollary 4.2.** *For a Dedekind  $\sigma$ -complete Banach lattice  $F$  the following assertions are equivalent:*

- (1)  $F$  has an order continuous norm,
- (2) for every Banach lattice  $E$  the pair  $(E, F)$  has the invariant modulus property.

With only slightly more work we have the following theorem.

**Theorem 4.3.** *For Banach lattice  $E$  the following assertions are equivalent:*

- (1)  $E$  is atomic with an order continuous norm,
- (2) for each Banach lattice  $F$  the pair  $(E, F)$  has the invariant modulus property,
- (3) for each unital AM-space  $F$  the pair  $(E, F)$  has the invariant modulus property.

PROOF. Clearly (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3), so we need only prove that (3)  $\Rightarrow$  (1). We do this by proving that the lattice operations in  $E'$  are weak\* continuous on bounded sets and then appealing to Theorem 3.3.

Let  $(f_\gamma)_{\gamma \in \Gamma}$  be a bounded net in  $E'$  that converges weak\* to 0, but with  $(|f_\gamma|)_{\gamma \in \Gamma}$  not converging weak\* to 0. By Alaoglu's Theorem we may assume, by extracting a subnet if necessary, that  $(|f_\gamma|)_{\gamma \in \Gamma}$  converges to a non-zero limit  $g$ . Fix  $x_0 \in E_+$  such that  $g(x_0) \neq 0$ .

We take for  $F$  the unital AM-space of all bounded real-valued functions  $h$  on the directed set  $\Gamma$  such that  $h(\gamma)$  converges as  $\gamma \rightarrow \infty$ . Let also  $I$  denote the closed ideal consisting of all  $h \in F$  such that  $h(\gamma) \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Define  $T : E \rightarrow I \subset F$  by  $(Tx)(\gamma) = f_\gamma(x)$ . It is not difficult to verify that the modulus  $|T|$  of  $T$  exists in  $\mathcal{L}^r(E, F)$  and that  $(|T|x)(\gamma) = |f_\gamma|(x)$ . As  $\gamma \rightarrow \infty$  this tends to  $g(x)$ . As  $g(x_0) \neq 0$ , we see that  $|T|x_0 \notin I$ , so it follows from Corollary 2.4 above that  $|JT| \neq J|T|$ , which contradicts (3). ■

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