REGULARITY PROPERTIES OF (LM)-SPACES OF TYPE (M_0)

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ABSTRACT

Denote countable inductive limits of metrisable spaces (resp. Fréchet spaces) by (LM)-spaces (resp. (LF)-spaces). We investigate regularity properties of (LM)-spaces of type (M_0). Although (LF)-spaces of type (M_0) need not be regular, we prove that they are always β-regular. Furthermore, we give a number of equivalent conditions for (LM)-spaces of type (M_0) to be regular.

1. Introduction

Let $(E_n)_{n\in N}$ be an inductive sequence of locally convex spaces, i.e. E_n is continuously included into E_{n+1} for all $n \in N$; and let $E := \operatorname{ind} E_n$ be its inductive limit, i.e. $E = \bigcup_{n=1}^{\infty} E_n$ is endowed with the finest locally convex topology such that the injections E_n into E are continuous. If every step E_n is metrisable (resp. a Fréchet space), then $E = \operatorname{ind} E_n$ is called an (LM)-space (resp. (LF)-space) (see [1], [9], [12] or [18]). Here an (LM)-space is always assumed to be Hausdorff. In the sense of Palamodov [17], an inductive sequence $(E_n)_{n\in N}$ or its inductive limit $E = \operatorname{ind} E_n$ is called acyclic (resp. weakly acyclic) if the natural map

$$j\colon \bigoplus_{n\in N} E_n \to \bigoplus_{n\in N} E_n,\ (x_n)_{n\in N} \mapsto (x_n - x_{n-1})_{n\in N}\ (x_0 := 0)$$

is a topological isomorphism (resp. a weak topological isomorphism) onto its range. The notions of acyclicity and weak acyclicity are closely linked to the concepts of limit subspaces and well-located subspaces, which play an important role in the study of partial differential operators and convolution operators (see [9], [11]). Using methods of homological algebra, Retakh [23] proved that an (LF)-space $E = \operatorname{ind} E_n$ is acyclic (resp. weakly acyclic) if and only if it satisfies condition (M) (resp. condition (M_0)), i.e. there exists an increasing sequence $(U_n)_{n\in N}$ of absolutely convex 0-neighbourhoods U_n in E_n with the following property: for each n there is $m = m(n) \geq n$ such that for all $k \geq m$ the (weak) topology of E_k coincides on U_n with the (weak) topology of E_m, or equivalently E and E_m induce the same (weak) topology on U_n. We also say an (LM)-space is of type (M) or (M_0) if it satisfies condition (M) or (M_0). Besides acyclicity and weak acyclicity, many other regularity conditions for inductive limits have been defined for different purposes. For details, refer to [1], [2], [9], [18] and [24]. In the following, we always denote the topology of $E = \operatorname{ind} E_n$ by t and the topology of E_n by t_n.

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Recall that an inductive sequence $(E_n, t_n)_{n\in N}$ or its inductive limit $(E, t) := \operatorname{ind}(E_n, t_n)$ is said to be:

(a) α-regular if, for each bounded set B in (E, t), there exists $n = n(B) \in N$ such that B is contained in E_n;

(b) β-regular if, for each bounded set B in (E, t) that is contained in some E_n, there exists $m \in N$ such that B is bounded in (E_m, t_m);

(c) regular if, for each bounded set B in (E, t), there exists $n = n(B) \in N$ such that B is contained and bounded in (E_n, t_n);

(d) sequentially retractive if, for each null sequence (x_k) in (E, t), there exists $n = n(x_k) \in N$ such that (x_k) is contained and a null sequence in (E_n, t_n).

Obviously an inductive limit is regular if and only if it is both α-regular and β-regular. As is well known, Dieudonné–Schwartz Theorem [5] points out that a strict inductive limit where each E_n is closed in E_{n+1} is regular. The extensions of Dieudonné–Schwartz Theorem to general inductive limits have been considered. See, for example, [3], [8], [13], [14], [15], [20] and [21]. Sequentially retractive inductive limits were introduced and studied thoroughly by Floret (see [7]). He proved that sequential retractivity implies regularity. Vogt [25] gave the first purely functional analytic proof of Retakh's theorem. He proved that an acyclic (LF)-space (i.e. an (LF)-space of type (M)) is sequentially retractive and certainly regular. On the other hand, Fernandez [6] proved that every sequentially retractive (LF)-space $E = \operatorname{ind} E_n$ satisfies condition (Q), which is defined like (M) without the assumption that $U_n \subset U_{n+1}$ for every $n \in N$. Wengenroth [26] proved that condition (Q) already implies (M). Thus he established the following important relationship: for (LF)-spaces, type (M) $\Leftrightarrow$ type (Q) $\Leftrightarrow$ acyclicity $\Leftrightarrow$ sequential retractivity $\Rightarrow$ regularity. However, a weak acyclic (LF)-space (i.e. an (LF)-space of type (M_0)) need not be regular (see [25]). Moreover, Diaz and Dierolf [4] found that there even exists an (LF)-space of type (M_0) that is not quasi-regular (call an inductive limit $E = \operatorname{ind} E_n$ quasi-regular if, for each bounded set B in E, there is $n \in N$ and a bounded set A in E_n such that $B \subset \overline{A}^E$). Hence it is natural to ask under which condition an (LF)-space of type (M_0) is regular. In the present paper, we shall prove that every (LM)-space of type (M_0) is automatically β-regular. Thus we find that, for those inductive limits, α-regularity $\Leftrightarrow$ regularity (see [22]). We shall point out that for (LM)-spaces of type (M_0), many regularity conditions are indeed equivalent. Particularly, we prove that for an (LM)-space $E = \operatorname{ind} E_n$ of type (M_0), regularity is equivalent to the following very weak property: for each $n \in N$, there is $m = m(n) \geq n$ such that, for every weak Cauchy sequence in E_n, its E_k-closure is contained in E_m for all $k \geq n$.

2. Main results

Theorem 1. *Let* $(E, t) = \operatorname{ind}(E_n, t_n)$ *be an* (*LM*)*-space of type* (M_0), *then* (E, t) *is* β*-regular.*

Proof. Since $(E, t) = \operatorname{ind}(E_n, t_n)$ satisfies (M_0), there exists an increasing sequence $(U_n)_{n\in N}$ of absolutely convex 0-neighbourhoods U_n in (E_n, t_n) with the following property: for each $n \in N$, there is $m = m(n) \geq n$ such that $\sigma(E, E')|U_n = \sigma(E_m, E'_m)|U_n$. Let B be a bounded set in (E, t) and contained in some E_n. Then there is $p \in N$ such that $B \subset p\overline{U}_p^E$ (see [18, proposition 8.5.20]). Without loss of generality, we may take n large enough that $B \subset E_n$ and $B \subset n\overline{U}_n^E$ both hold. We have already proved that $\overline{U}_n^E = \bigcup_{k=n}^{\infty} \overline{U}_n^{E_k}$ (see [22]), hence $B \subset n(\bigcup_{k=n}^{\infty} \overline{U}_n^{E_k})$, or $\frac{1}{n}B \subset \bigcup_{k=n}^{\infty} \overline{U}_n^{E_k}$. Take any $x \in \frac{1}{n}B$, then there exists $k \geq n$ such that $x \in \overline{U}_n^{E_k}$. Thus there exists a sequence $(x_i) \subset U_n$ such that $x_i \xrightarrow{i} x$ in (E_k, t_k) and in $(E_k, \sigma(E_k, E_k'))$. On the other hand, $x \in \frac{1}{n}B \subset E_n$, hence there is $\lambda \geq 1$ such that $x \in \lambda U_n$. Now we have

$$\frac{1}{\lambda}x \in U_n, \ (\frac{1}{\lambda}x_i)_{i \in N} \subset U_n, \text{ and } \frac{1}{\lambda}x_i \xrightarrow{i} \frac{1}{\lambda}x \text{ in } (E_k, \sigma(E_k, E_k')).$$

Remarking that $\sigma(E, E')|U_n = \sigma(E_m, E_m')|U_n$, we conclude that

$$\frac{1}{\lambda}x_i \xrightarrow{i} \frac{1}{\lambda}x \text{ and } x_i \xrightarrow{i} x \text{ in } (E_m, \sigma(E_m, E_m')), \text{ hence } x \in \overline{U}_n^{\sigma(E_m, E_m')} = \overline{U}_n^{E_m}.$$

This means that $\frac{1}{n}B \subset \overline{U}_n^{E_m}$. Take any sequence (b_i) in $\frac{1}{n}B$, then $b_i \in \overline{U}_n^{E_m} \subset U_n + V_m^{(i)}$ for every $i \in N$. Here $V_m^{(1)} \supset V_m^{(2)} \supset \cdots$ is a base of absolutely convex 0-neighbourhoods in (E_m, t_m). Clearly, every b_i can be written in the form $b_i = y_i + z_i$, where $y_i \in U_n$ and $z_i \in V_m^{(i)}$. Thus $\frac{1}{i}b_i = \frac{1}{i}y_i + \frac{1}{i}z_i$ for every $i \in N$. Obviously, $\frac{1}{i}z_i \xrightarrow{i} 0$ in (E_m, t_m) and $\frac{1}{i}b_i \xrightarrow{i} 0$ in (E, t). Hence $\frac{1}{i}y_i = \frac{1}{i}b_i - \frac{1}{i}z_i \xrightarrow{i} 0$ in (E, t) and in $(E, \sigma(E, E'))$. Since $(\frac{1}{i}y_i)_{i \in N} \subset U_n$ and $\sigma(E, E')|U_n = \sigma(E_m, E_m')|U_n$, we have $\frac{1}{i}y_i \xrightarrow{i} 0$ in $(E_m, \sigma(E_m, E_m'))$. Combining this with the fact that $\frac{1}{i}z_i \xrightarrow{i} 0$ in (E_m, t_m) and in $(E_m, \sigma(E_m, E_m'))$, we conclude that $\frac{1}{i}b_i = \frac{1}{i}y_i + \frac{1}{i}z_i \xrightarrow{i} 0$ in $(E_m, \sigma(E_m, E_m'))$. From this, we know that $\frac{1}{n}B$ is bounded in $(E_m, \sigma(E_m, E_m'))$ and hence bounded in (E_m, t_m). Therefore B is bounded in (E_m, t_m) for some $m \geq n$. Thus we have shown that (E, t) is β-regular. ■

As an immediate consequence we have α-regular $\Leftrightarrow$ regularity for (LM)-spaces of type (M_0) (see [22]). In fact, we can prove that even some very weak conditions are equivalent to regularity for those spaces. Theorem 2 gives a number of equivalent descriptions of regularity for (LM)-spaces of type (M_0), which enrich the related result in [22].

Theorem 2. *Let* $(E, t) = \operatorname{ind}(E_n, t_n)$ *be an* (LM)*-space of type* (M_0). *Then the following conditions are equivalent:*

(a) (E, t) *is weakly boundedly retractive, i.e. for each bounded set* B *in* (E, t), *there is* $n = n(B) \in N$ *such that* B *is contained and bounded in* (E_n, t_n) *and* $\sigma(E, E')|B = \sigma(E_n, E_n')|B$;

(b) (E, t) *is weakly compactly regular, i.e. for each weakly compact set* K *in* (E, t), *there is* $n = n(K) \in N$ *such that* K *is contained and weakly compact in* (E_n, t_n);

(c) (E, t) *is weakly sequentially retractive, i.e. for each weak null sequence* (x_k) *in* (E, t), *there is* $n = n(x_k) \in N$ *such that* (x_k) *is contained and a weak null sequence in* (E_n, t_n);

(d) (E, t) *is regular*;

(e) (E,t) *is* α*-regular;*

(f) *for each* $n \in N$*, there is* $m = m(n) \geq n$ *and a* 0*-neighbourhood* V_n *in* (E_n, t_n) *such that* $\overline{V}_n^E \subset E_m$*;*

(g) (E,t) *admits a defining spectrum* $(F_n, s_n)_{n\in N}$ *of metrisable locally convex spaces, i.e.* $(E_n, t_n)_{n\in N}$ *has an equivalent inductive sequence* $(F_n, s_n)_{n\in N}$ *(see* [16] *or* [26]*) such that every* (F_n, s_n) *has a base of* 0*-neighbourhoods whose members are closed in* (E,t)*;*

(h) *for each* $n \in N$*, there is* $m = m(n) \geq n$ *such that for every weak Cauchy sequence* $(z_j)_{j\in N}$ *in* (E_n, t_n)*,* $\overline{\{z_j : j \in N\}}^E \subset E_m$*;*

(i) *for each* $n \in N$*, there is* $m = m(n) \geq n$ *such that for every weak Cauchy sequence* $(z_j)_{j\in N}$ *in* (E_n, t_n)*,* $\overline{\{z_j : j \in N\}}^{E_k} \subset E_m$ *for all* $k \geq n$*.*

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are obvious. The implications (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (f) were shown respectively in [22] and [19]. Thus (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f).

(g) $\Rightarrow$ (f). For each $n \in N$, there is $m = m(n) \geq n$ such that $E_n \subset F_m$ and the inclusion $i\colon (E_n, t_n) \to (F_m, s_m)$ is continuous. Take any 0-neighbourhood U_m in (F_m, s_m) such that U_m is closed in (E,t), then there is a 0-neighbourhood V_n in (E_n, t_n) such that $V_n \subset U_m$. Thus $\overline{V}_n^E \subset \overline{U}_m^E = U_m \subset E_m$.

(f) $\Rightarrow$ (g). Since $(E,t) = \operatorname{ind}(E_n, t_n)$ satisfies (M_0), for each $n \in N$, there is $m = m(n) \geq n$ and an absolutely convex 0-neighbourhood U_n in (E_n, t_n) such that $\sigma(E,E')|U_n = \sigma(E_m, E_m')|U_n$. Here we need not use the assumption that $U_n \subset U_{n+1}$ for every $n \in N$. By (f), we may assume without loss of generality that, for each $n \in N$, there is an absolutely convex 0-neighbourhood V_n in (E_n, t_n) such that $\overline{V}_n^E \subset E_{n+1}$ and $\sigma(E,E')|V_n = \sigma(E_{n+1}, E_{n+1}')|V_n$. For each $n \in N$, let $V_n \supset V_n^{(1)} \supset V_n^{(2)} \supset \cdots$ be a base of absolutely convex 0-neighbourhoods in (E_n, t_n). Put $F_n := span[\overline{V}_n^E]$, then $E_n \subset F_n \subset E_{n+1}$. Let F_n be equipped with the metrisable locally convex topology s_n that admits

$$\{(\frac{1}{2^k}\overline{V_n}^E) \bigcap \overline{(V_{n+1}^{(k)} \bigcap \frac{1}{2^k} V_{n+2})}^E \ : k = 1, 2, 3, \cdots\}$$

as a 0-neighbourhood base. Obviously $E_n \subset F_n$ and the inclusion $i\colon (E_n, t_n) \to (F_n, s_n)$ is continuous. On the other hand, $F_n \subset E_{n+1} \subset E_{n+3}$, we will prove that the inclusion $i\colon (F_n, s_n) \to (E_{n+3}, t_{n+3})$ is continuous. For any

$$x \in \overline{V_{n+1}^{(k)} \bigcap \frac{1}{2^k} V_{n+2}}^E \subset \overline{V_{n+1}^{(k)}}^E \subset \overline{V}_{n+1}^E \subset E_{n+2},$$

there is $\lambda \geq 1$ such that $\frac{1}{\lambda}x \in V_{n+2}$ and there is a net $(x_\delta) \subset V_{n+1}^{(k)} \bigcap \frac{1}{2^k} V_{n+2}$ such that $x_\delta \xrightarrow{\delta} x$ in (E,t) and in $(E, \sigma(E,E'))$. Clearly, $\frac{1}{\lambda}x_\delta \xrightarrow{\delta} \frac{1}{\lambda}x$ in $(E, \sigma(E,E'))$. Since

$$\frac{1}{\lambda}x \in V_{n+2}, \ (\frac{1}{\lambda}x_\delta) \subset V_{n+1}^{(k)} \bigcap \frac{1}{2^k} V_{n+2} \subset V_{n+2}, \ \sigma(E,E')|V_{n+2} = \sigma(E_{n+3}, E_{n+3}')|V_{n+2},$$

we have

$$\frac{1}{\lambda}x_\delta \xrightarrow{\delta} \frac{1}{\lambda}x \text{ in } (E_{n+3},\sigma(E_{n+3},E'_{n+3})) \text{ and } x_\delta \xrightarrow{\delta} x \text{ in } (E_{n+3},\sigma(E_{n+3},E'_{n+3})).$$

Thus

$$x \in \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{\sigma(E_{n+3},E'_{n+3})} = \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E_{n+3}}.$$

This means that

$$\overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E} = \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E_{n+3}},$$

and

$$\frac{1}{2^k}\overline{V}^{E}_{n} \bigcap \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E} = \frac{1}{2^k}\overline{V}^{E}_{n} \bigcap \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E_{n+3}}.$$

Since the inclusion $i:(E_{n+1},t_{n+1}) \to (E_{n+3},t_{n+3})$ is continuous, for any closed 0-neighbourhood W_{n+3} in (E_{n+3},t_{n+3}), there is $k \in N$ such that $V^{(k)}_{n+1} \subset W_{n+3}$. Thus $\overline{V^{(k)}_{n+1}}^{E_{n+3}} \subset \overline{W}^{E_{n+3}}_{n+3} = W_{n+3}$. Hence

$$\frac{1}{2^k}\overline{V}^{E}_{n} \bigcap \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E} = \frac{1}{2^k}\overline{V}^{E}_{n} \bigcap \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E_{n+3}} \subset \overline{V^{(k)}_{n+1}}^{E_{n+3}} \subset W_{n+3}.$$

That is to say, the inclusion $i:(F_n,s_n) \to (E_{n+3},t_{n+3})$ is continuous. Therefore, (E,t) has a defining spectrum of metrisable locally convex spaces $(F_n,s_n)_{n\in N}$, where each step (F_n,s_n) has a 0-neighbourhood base

$$\{\frac{1}{2^k}\overline{V}^{E}_{n} \bigcap \overline{V^{(k)}_{n+1} \bigcap \frac{1}{2^k} V_{n+2}}^{E} : k = 1,\ 2,\ 3, \cdots\},$$

whose members are closed in (E,t).

(f) ⇒ (h) ⇒ (i). They are obvious.

(i) ⇒ (a). Since $(E,t) = \operatorname{ind}(E_n,t_n)$ satisfies (M_0), there exists an increasing sequence $(U_n)_{n\in N}$ of absolutely convex 0-neighbourhoods U_n in (E_n,t_n) with the property that, for each $n \in N$, there is $m = m(n) \geq n$ such that $\sigma(E,E')|U_n = \sigma(E_m,E'_m)|U_n$. Let B be any bounded set in (E,t), then there is $n \in N$ such that $B \subset n\overline{U}^{E}_{n}$ (see [18, proposition 8.5.20]). For the above n, there is $m = m(n) \geq n$ such that $\sigma(E_m,E'_m)|U_n = \sigma(E,E')|U_n$. By [22, lemma 1], $\overline{U}^{E}_{n} = \bigcup\limits_{k=n}^{\infty} \overline{U}^{E_k}_{n} = \bigcup\limits_{k=m}^{\infty} \overline{U}^{E_k}_{n}$. Thus $B \subset n(\bigcup\limits_{k=m}^{\infty} \overline{U}^{E_k}_{n})$, or $\frac{1}{n}B \subset \bigcup\limits_{k=m}^{\infty} \overline{U}^{E_k}_{n}$. For the above m, by condition (i), there is $p = p(m) \geq m$ such that for every weak Cauchy sequence (z_j) in (E_m,t_m), $\overline{\{z_j : j \in N\}}^{E_k} \subset E_p$ for all $k \geq m$. Now we claim that $\frac{1}{n}B \subset E_p$. In fact, for any $x \in \frac{1}{n}B \subset \bigcup\limits_{k=m}^{\infty} \overline{U}^{E_k}_{n}$, there is $k \geq m$ such that $x \in \overline{U}^{E_k}_{n}$. Hence there exists a sequence $(x_j)_{j\in N} \subset U_n$ such that $x_j \xrightarrow{j} x$ in (E_k,t_k) and in $(E_k,\sigma(E_k,E'_k))$. Since $\sigma(E,E')|U_n = \sigma(E_m,E'_m)|U_n$,

$(x_j)_{j\in N}$ is also a Cauchy sequence in $(E_m, \sigma(E_m, E'_m))$. Thus $x \in \overline{\{x_j : j \in N\}}^{E_k} \subset E_p$, namely $\frac{1}{n}B \subset E_p$, and $B \subset E_p$. This means that (E, t) is α-regular. By Theorem 1, we know that (E, t) is regular. Hence for any bounded set B in (E, t), there is $n \in N$ such that B is contained and bounded in (E_n, t_n). Clearly, there is $\lambda > 0$ such that $\lambda B \subset U_n$. For the above n, by (M_0) there is $m = m(n) \geq n$ such that $\sigma(E, E')|U_n = \sigma(E_m, E'_m)|U_n$. From this, we know that $\sigma(E, E')|\lambda B = \sigma(E_m, E'_m)|\lambda B$ and $\sigma(E, E')|B = \sigma(E_m, E'_m)|B$, namely (E, t) is weakly boundedly retractive. This completes the proof of Theorem 2. ■

Remark 1. In the proof of (f) ⇔ (g), we may only assume that $(E, t) = \mathrm{ind}\,(E_n, t_n)$ satisfies condition (Q_0), which is defined like condition (M_0) without requirement $(U_n)_{n\in N}$ being increasing (see [10]). Recently the author was told that Wengenroth has already proved that $(Q_0) \Leftrightarrow (M_0)$, (see [28, theorem 3]).

Remark 2. After submitting this paper, the author was informed by the referee that Theorem 1 was obtained independently by Wengenroth [26], and that [26] had been partially published in [27].

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