

MINIMAL WEAKLY SUBMAXIMAL TOPOLOGIES

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ABSTRACT

A new expansive but weak topological invariant is introduced and shown to have a distinguished pivotal role when logically compared with existing separation axioms in the literature. The structure of topologies, which minimally satisfy this property, is identified from both a topological and an order-theoretical point of view.

1. Introduction

Given an arbitrary infinite set X , we seek to identify those topologies for X that minimally satisfy the property 'weakly submaximal'. A topological space (X, \mathcal{T}) is said to be:

- T_1 if and only if for each $x \in X$, $\{x\}$ is \mathcal{T} -closed;
- T_{EF} if and only if each finite subset of X is either \mathcal{T} -open or \mathcal{T} -closed (see [5] and [11]);
- T_{ES} if and only if each singleton subset of X is either \mathcal{T} -open or \mathcal{T} -closed (see [5], [8], [10] and [11]);
- T_D if and only if for each $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is \mathcal{T} -closed (see [4] and [7]);
- T_0 if and only if for each $x \in X$, $\overline{\{x\}} \setminus \{x\}$ is a union of \mathcal{T} -closed sets;
- *door* if and only if each subset of X is either \mathcal{T} -open or \mathcal{T} -closed (see [6]);
- *submaximal* if and only if for each subset A of X , $\overline{A} \setminus A$ is \mathcal{T} -closed (equivalently, each \mathcal{T} -dense subset of X is \mathcal{T} -open, or, again, each subset of X is \mathcal{T} -locally closed) (see, in particular, [1] and [3]);
- *weakly submaximal* if and only if for each *finite* subset A of X , $\overline{A} \setminus A$ is \mathcal{T} -closed (equivalently, each finite subset of X is \mathcal{T} -locally closed);
- T_F if and only if each singleton subset of X is either \mathcal{T} -kernelled, as defined below, or \mathcal{T} -closed (see [2], [8], and [9]).

An obvious question with regard to weak submaximality can be anticipated. Let F be a finite subset of any topological space such that $|F| = n \geq 3$; say $F = \{x_1, x_2, \dots, x_n\}$ and write $F_i = F \setminus \{x_i\}$ for each i with $1 \leq i \leq n$. Then, with the understanding that $F_{n+1} = F_1$, the set-theoretic identity

$$\overline{F} \setminus F = \bigcap \{[(\overline{F_i} \setminus F_i) \cup (\overline{F_{i+1}} \setminus F_{i+1})] : 1 \leq i \leq n\}$$

(and the obvious inductive argument) confirms that it is equivalent to stipulate in the definition of weak submaximality that, for each *doubleton* (or two-point) subset A , $\overline{A} \setminus A$ is closed (that is, each doubleton subset of X is \mathcal{T} -locally closed).

Accordingly, it is immediate that the weak submaximality invariant is both expansive and hereditary. It is also routinely verifiable that door implies submaximality implies weak submaximality implies T_D , that submaximality implies T_{ES} implies T_F ,

that T_1 implies T_{EF} implies T_{ES} implies T_D , and that door implies T_{EF} implies weak submaximality implies T_F .

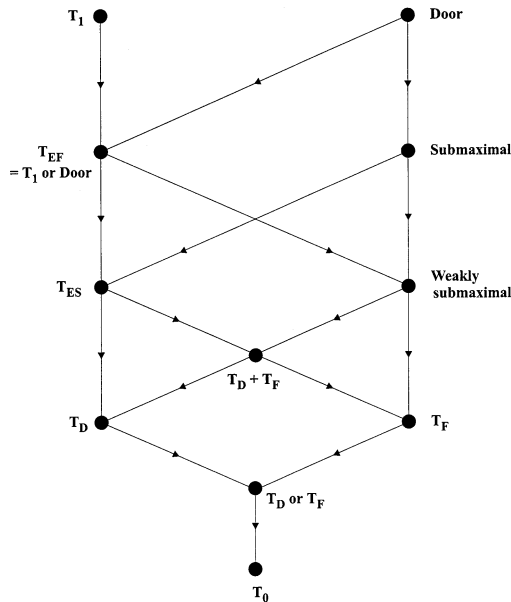
The property of weak submaximality is seen to occupy a special position in the logical hierarchy of expansive and hereditary topological invariant axioms which are weaker than either T_1 or door. In a sense, it bridges the ‘gap’ between submaximality and T_D on the one hand, and that between T_{EF} and T_F on the other. The diagram below and the later example serve to reveal the general relationships that prevail among these axioms.

By employing the well-developed contraction technique of intersecting a given topology with a suitably chosen principal ultratopology, we proceed by development of a purely topological approach to the question of minimality. We also indicate how a recognition of the underlying specialisation order structure of any topological space affords us new and valuable insight into the problem. As a consequence, we can offer alternative and comparative descriptions of previously established minimal structures in terms of the behaviour of the specialisation order and certain intrinsic topologies on the resulting partially-ordered set.

2. Main result

We begin with some definitions. Note that throughout this work, X shall denote an arbitrary infinite set in which, given subsets A and B , we abbreviate $A \cap (X \setminus B)$ by $A \setminus B$, whereas $|A|$ is the cardinality of A , and we write $|A| < \omega$ if A is finite.

Also, $LT(X)$ denotes the lattice of all topologies for X in which, given members \mathcal{T}_1 and \mathcal{T}_2 , $\mathcal{T}_1 \cap \mathcal{T}_2$ is the infimum of \mathcal{T}_1 and \mathcal{T}_2 , while \mathcal{T}_1 is said to be finer than \mathcal{T}_2 (equivalently, \mathcal{T}_2 is said to be coarser than \mathcal{T}_1) when $\mathcal{T}_2 \subseteq \mathcal{T}_1$ in $LT(X)$ (equivalently, $\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}_2$).



Of course, given a topological invariant P , a member \mathcal{T} of $LT(X)$ is *minimal P* if and only if \mathcal{T} is P and no strictly coarser member of $LT(X)$ is P .

Definition 1. Given $\mathcal{T} \in LT(X)$, $A \subseteq X$, and $x \in X$, the intersection of all \mathcal{T} -open subsets of X that contain A is called the \mathcal{T} -kernel of A and is denoted by \widehat{A} (assuming no danger of ambiguity). When $A = \widehat{A}$ we say that A is \mathcal{T} -kernelled. We often refer to $\widehat{\{x\}}$ as a *point-kernel*. As usual, \overline{A} denotes the \mathcal{T} -closure of A , and we similarly refer to $\overline{\{x\}}$ as being a *point-closure*. Of course, given $x, y \in X$, $x \in \overline{\{y\}}$ if and only if $y \in \widehat{\{x\}}$.

We adopt the notation of [2] and [10] by writing:

$$N_D(\mathcal{T}) = \{x \in X : \{x\} = \overline{\{x\}}\},$$

$$N_S(\mathcal{T}) = \{x \in X : \{x\} = \widehat{\{x\}}\}.$$

Note that the condition, $X = N_S(\mathcal{T}) \cup N_D(\mathcal{T})$, characterises a T_F -space (X, \mathcal{T}) , in which $\overline{\{y\}} \subseteq \{y\} \cup N_D(\mathcal{T})$ for each $y \in N_S(\mathcal{T})$. In addition, the equivalent conditions, $N_D(\mathcal{T}) = X, N_S(\mathcal{T}) = X$, give an alternative characterisation of a T_1 -space (X, \mathcal{T}) .

Definition 2. Given $x \in X$ and $Y \subseteq X$, we designate the following members of $LT(X)$:

$$\begin{aligned} \mathcal{J}(Y) &= \{G \subseteq X : Y \subseteq G\} \cup \{\phi\}, \\ \mathcal{J}(x) &= \{G \subseteq X : x \in G\} \cup \{\phi\}, \\ \mathcal{E}(Y) &= \{G \subseteq X : G \subseteq X \setminus Y\} \cup \{X\}, \\ \mathcal{E}(x) &= \{G \subseteq X : x \notin G\} \cup \{X\}, \\ \mathcal{D}(Y) &= \{G \subseteq X : \text{either } G \subseteq Y \text{ and } |Y \setminus G| < \omega \text{ or } Y \subseteq G \text{ and } |X \setminus G| < \omega\}, \\ \mathcal{M}(Y) &= \{G \subseteq X : \text{either } G \subseteq Y \text{ or } Y \subseteq G\}, \\ \mathcal{C} &= \{G \subseteq X : |X \setminus G| < \omega\} \cup \{\phi\}, \text{ the cofinite (or minimum } T_1) \text{ member of } LT(X), \\ \mathcal{R}(Y) &= \{G \subseteq X : \text{either } Y \subseteq G \text{ or } G \subseteq Y \text{ and } |Y \setminus G| < \omega\} \cup \{\phi\}. \end{aligned}$$

Clearly we have $\mathcal{D}(Y) \subseteq \mathcal{R}(Y) \subseteq \mathcal{M}(Y)$ in $LT(X)$. Furthermore, if $|Y| < \omega$ then $\mathcal{R}(Y) = \mathcal{M}(Y)$, while if $|X \setminus Y| < \omega$ then $\mathcal{D}(Y) = \mathcal{R}(Y)$.

Theorem 3. Let A be a non-empty proper subset of X .

- (i) A subset K of X is $\mathcal{R}(A)$ -closed if and only if either $X \setminus A \subseteq K$ and $|K \cap A| < \omega$ or $K \subseteq X \setminus A$.
- (ii) The space $(X, \mathcal{R}(A))$ is weakly submaximal.
- (iii) If $\mathcal{T} \subseteq \mathcal{R}(A)$ in $LT(X)$ and the space (X, \mathcal{T}) is weakly submaximal, then

$$\overline{\{a\}}^{\mathcal{T}} = \{a\} \cup (X \setminus A) \text{ for each } a \in A.$$

PROOF. (i) This is immediate by the above definition.

(ii) Let F be a finite subset of X . Either $F \subseteq X \setminus A$ in which case, by (i), F is $\mathcal{R}(A)$ -closed and so $\overline{F} \setminus F = \phi$, or $F \cap A \neq \phi$ in which case, by (i), $\overline{F} = F \cup (X \setminus A)$ and so $\overline{F} \setminus F = X \setminus (F \cup A) \subseteq X \setminus A$, whence by (i) it is $\mathcal{R}(A)$ -closed.

(iii) For each $a \in A$, $\{a\} \cup (X \setminus A)$ is the $\mathcal{R}(A)$ -closure of $\{a\}$ by (i), implying that $\{a\} \cup (X \setminus A) \subseteq \overline{\{a\}}^{\mathcal{T}}$.

On the other hand, suppose there exists $b \in A \cap (\overline{\{a\}}^{\mathcal{F}} - \{a\})$, and let $z \in X \setminus A$. Then, $X \setminus A \not\subseteq K = \overline{\{a, z\}}^{\mathcal{F}} \setminus \{a, z\}$ since $z \notin K$, and $K \not\subseteq X \setminus A$ since $b \in K$. Hence, K is not $\mathcal{R}(A)$ -closed by (i) and, therefore, not \mathcal{F} -closed. But this contradicts the weak submaximality of (X, \mathcal{F}) . We conclude that $A \cap (\overline{\{a\}}^{\mathcal{F}} \setminus \{a\}) = \emptyset$ and so $\{a\} \cup (X \setminus A) = \overline{\{a\}}^{\mathcal{F}}$. ■

Corollary 4. *Let A be a non-empty proper subset of X . The space $(X, \mathcal{R}(A))$ is minimal weakly submaximal.*

PROOF. Suppose $\mathcal{F} \subseteq \mathcal{R}(A)$ in $LT(X)$ and that the space (X, \mathcal{F}) is weakly submaximal. Observe that, for each $a \in A$ and each $x \in X \setminus A$, by Theorem 3 (iii) it is clear that

$$\overline{\{x\}}^{\mathcal{F}} \subseteq X \setminus A = \overline{\{a\}}^{\mathcal{F}} \setminus \{a\}, \text{ and so}$$

$$\overline{\{a, x\}}^{\mathcal{F}} = \overline{\{a\}}^{\mathcal{F}} \cup \overline{\{x\}}^{\mathcal{F}} = \overline{\{a\}}^{\mathcal{F}} = \{a\} \cup (X \setminus A).$$

Now let K be any $\mathcal{R}(A)$ -closed subset of X . By Theorem 3 (i) and (iii), either $K \subseteq X \setminus A$ in which case, for any $b \in A$,

$$\begin{aligned} K &= \cap \{X \setminus (A \cup \{x\}) : x \in X \setminus (K \cup A)\} \\ &= \cap \{[\{b\} \cup (X \setminus A)] \setminus \{b, x\} : x \in X \setminus (K \cup A)\} \\ &= \cap \{\overline{\{b, x\}}^{\mathcal{F}} \setminus \{b, x\} : x \in X \setminus (K \cup A)\} \end{aligned}$$

is \mathcal{F} -closed, or $X \setminus A \subseteq K$ and $|K \cap A| < \omega$ in which case

$$\begin{aligned} K &= \cup \{\{a\} \cup (X \setminus A) : a \in K \cap A\} \\ &= \cup \{\overline{\{a\}}^{\mathcal{F}} : a \in K \cap A\} \end{aligned}$$

is \mathcal{F} -closed.

That is, $\mathcal{F} = \mathcal{R}(A)$ in $LT(X)$, and the result follows by Theorem 3 (ii). ■

Example 5. Let $x \in X$ and let A be an infinite subset of X such that $X \setminus A$ is also infinite. Then, in $LT(X)$

- \mathcal{C} is T_1 and T_{EF} (indeed minimally so [5]) and weakly submaximal, but is not submaximal (since $\overline{A} \setminus A = X \setminus A$) and therefore not door.
- $(\mathcal{C} \cap \mathcal{I}(x)) \cup \{x\}$ is T_{ES} (indeed minimally so [5]) and T_D , but is neither T_{EF} nor weakly submaximal (since, for any $y \neq x$, $\overline{\{y, x\}} \setminus \{y, x\} = X \setminus \{y, x\}$ is not closed).
- $\mathcal{C} \cap \mathcal{I}(x) = \mathcal{D}(\{x\})$ is T_F (indeed minimally so [2]), but is neither T_D nor T_{ES} nor weakly submaximal.
- $\mathcal{C} \cap \mathcal{E}(x) = \mathcal{R}(X \setminus \{x\})$ is weakly submaximal (indeed minimally so by the above corollary) and T_D , but is not T_{ES} .
- $\mathcal{M}(A)$ is submaximal (indeed minimally so [3]), but is not T_{EF} and therefore not door.

- $\mathcal{I}(x) = \mathcal{M}(\{x\})$ is door, T_{EF} , and submaximal (indeed minimally so, [5], [6] and [2]), but is not T_1 .

Lemma 6 [10]. *Let $\mathcal{T} \in LT(X)$, $B \subseteq X$, $x \neq y$ in X , and let $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$ in $LT(X)$. The \mathcal{T}^* -closure of B is described by:*

$$\overline{B}^* = \begin{cases} \overline{B}, & \text{if } y \notin \overline{B} \\ \overline{B} \cup \overline{\{x\}}, & \text{if } y \in \overline{B}. \end{cases}$$

Lemma 7. *Let $\mathcal{T} \in LT(X)$, and let $x \neq y$ in X with $y \in N_S(\mathcal{T})$ and $x \in N_D(\mathcal{T})$. If (X, \mathcal{T}) is weakly submaximal then (X, \mathcal{T}^*) is weakly submaximal.*

PROOF. Let F be a finite subset of X . Observe first that $y \in \overline{F}$ implies that there exists $z \in F$ with $y \in \overline{\{z\}}$, which in turn implies that $z \in \widehat{\{y\}} = \{y\}$, and so $y \in F$. We conclude that $y \notin \overline{F} \setminus F$.

So, by Lemma 6,

$$\overline{F}^* = \begin{cases} \overline{F}, & \text{if } y \notin F \\ \overline{F} \cup \{x\}, & \text{if } y \in F, \end{cases}$$

whence

$$\overline{F}^* \setminus F = \begin{cases} \overline{F} \setminus F, & \text{if } y \notin F \\ (\overline{F} \setminus F) \cup \{x\}, & \text{if } y \in F \end{cases}$$

is \mathcal{T}^* -closed, since it is \mathcal{T} -closed and excludes y . ■

Corollary 8. *Let (X, \mathcal{T}) be a minimal weakly submaximal space.*

- (i) *If $y \in N_S(\mathcal{T})$ then $\overline{\{y\}} = \{y\} \cup N_D(\mathcal{T})$.*
- (ii) *$N_S(\mathcal{T}) \cap N_D(\mathcal{T}) = \emptyset$.*

PROOF. (i) Suppose $y \in N_S(\mathcal{T})$ and $x \in N_D(\mathcal{T})$ with $x \notin \overline{\{y\}}$, then, by Lemma 7, $\mathcal{T}^* = \mathcal{T} \cap (\mathcal{I}(y) \cup \mathcal{E}(x))$ is weakly submaximal, and strictly weaker than \mathcal{T} in $LT(X)$, since $\overline{\{y\}}^* = \overline{\{y\}} \cup \{x\}$ by Lemma 6.

The converse is immediate since (X, \mathcal{T}) is a T_F -space.

(ii) If $t \in N_S(\mathcal{T}) \cap N_D(\mathcal{T})$ then, by (i), $N_D(\mathcal{T}) = \{t\}$, whence $N_S(\mathcal{T}) = X$ and so $N_D(\mathcal{T}) = X$, an obvious contradiction. ■

Theorem 9. *(X, \mathcal{T}) is minimal weakly submaximal if and only if there exists a non-empty proper subset A of X such that $\mathcal{T} = \mathcal{R}(A)$ in $LT(X)$. (Moreover, the representation is canonical: $A = N_S(\mathcal{T})$ and $X \setminus A = N_D(\mathcal{T})$.)*

PROOF. Let (X, \mathcal{T}) be minimal weakly submaximal and let $A = N_S(\mathcal{T})$. Then $A \neq \emptyset$ and $A \neq X$ (otherwise, for any $x \in X$, $\mathcal{C} \cap \mathcal{E}(x) \subset \mathcal{C} \subseteq \mathcal{T}$ in $LT(X)$, which contradicts the minimality of \mathcal{T} , by the example above).

By the hypothesis and Corollary 8, $N_D(\mathcal{T}) = X \setminus A$, and so a repetition of the argument contained in the proof of Corollary 4 ensures that $\mathcal{R}(A) \subseteq \mathcal{T}$ in

$LT(X)$. The desired equality is then confirmed by Theorem 3(ii) and the minimality of \mathcal{F} .

The converse is immediate by Corollary 4. ■

3. Order

Some basic definitions, notation, and terminology need to be recorded.

Definition 10. Let (X, \leq) be a partially ordered set, let $x \in X$, and let $Y \subseteq X$.

- (i) x is *maximal (minimal)* if and only if $x \leq z (z \leq x)$ and $z \in X$ implies that $x = z$.
- (ii) x is said to be *ultramaximal* if and only if x is maximal and for any non-maximal $z \in X$, $z \leq x$.
- (iii)

$$\begin{aligned} \uparrow \{x\} &= \{z \in X : x \leq z\}, \\ \downarrow \{x\} &= \{z \in X : z \leq x\}. \end{aligned}$$
- (iv) Y is *increasing* if and only if $Y = \cup \{\uparrow \{y\} : y \in Y\}$ and Y is *decreasing* if and only if $Y = \cup \{\downarrow \{y\} : y \in Y\}$.
- (v) Y is a *chain* if and only if $y, z \in Y$ implies that either $y \leq z$ or $z \leq y$.
- (vi) If $n \in \omega$ and Y is a chain with $|Y| = n$ then Y is said to have *length* $n - 1$.

Definition 11. A partially ordered set in which every maximal member is ultramaximal and all maximal chains have unit length is called a *duplex*.

The essential characteristics of a duplex are that it contains both maximal and non-maximal members and that every non-maximal member is minimal.

Definition 12. Given a partially ordered set (X, \leq) , we designate the following intrinsic members of $LT(X)$:

- the *weak* topology, \mathcal{W} , whose closed sets are generated by the family $\{\phi, X, \downarrow \{x\} : x \in X\}$;
- the topology, denoted by \mathcal{M} , whose closed sets are generated by the family $\{\phi, X, \downarrow \{x\}, \downarrow \{x\} \setminus \{x\} : x \in X\}$;
- the topology, denoted by \mathcal{N} , whose closed sets are precisely those which are decreasing and contain only finitely many maximal members of X ;
- the topology, denoted by \mathcal{L} , which has as an (open) base the family $\mathcal{M} \cup \{\{x\} : x \text{ is ultramaximal}\}$;
- the *Scott* topology, \mathcal{S} , whose closed sets are precisely those which are decreasing and contain the suprema of up-directed subsets;
- the *Alexandroff* topology, \mathcal{A} , whose closed sets are precisely those which are decreasing.

It is fairly immediate that generally $\mathcal{W} \subseteq \mathcal{M} \subseteq \mathcal{L} \subseteq \mathcal{A}$, $\mathcal{W} \subseteq \mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{A}$ and $\mathcal{W} \subseteq \mathcal{S} \subseteq \mathcal{A}$ in $LT(X)$, and that, for each of these topologies, $\overline{\{x\}} = \downarrow \{x\}$.

The *specialisation order* for any topological space (X, \mathcal{F}) is defined by agreeing that $x \leq y$ in X when $x \in \overline{\{y\}}$. Indeed, given a partial order \leq and a topology \mathcal{F} for X , it is well known that \leq is the specialisation order for (X, \mathcal{F}) if and

only if $\mathcal{W} \subseteq \mathcal{T} \subseteq \mathcal{A}$ in $LT(X)$. That is, \mathcal{W} is the coarsest and \mathcal{A} is the finest of the topologies with a given specialisation order (acknowledging that, for all such topologies, we have $\overline{\{x\}} = \downarrow \{x\}$ and $\widehat{\{x\}} = \uparrow \{x\}$ for all $x \in X$).

In a duplex the ascending chain condition is evidently satisfied and so, for such a partially ordered set, the Scott and Alexandroff topologies coincide. Some further pertinent observations can be made.

Lemma 13. *Let (X, \leq) be a duplex in which A denotes the subset of maximal members. Then*

- (i) $\mathcal{D}(A) = \mathcal{W}$ in $LT(X)$;
- (ii) $\mathcal{E}(X/A) \cup (\mathcal{C} \cap \mathcal{J}(A)) = \mathcal{L}$ in $LT(X)$;
- (iii) if A is a singleton, $\mathcal{L} = \mathcal{M}$ in $LT(X)$, otherwise $\mathcal{W} = \mathcal{M}$ in $LT(X)$;
- (iv) $\mathcal{R}(A) = \mathcal{N}$ in $LT(X)$;
- (v) $\mathcal{M}(A) = \mathcal{S} = \mathcal{A}$ in $LT(X)$.

We now present an order-theoretic description of the previously established minimality result. For the sake of completeness and comparison we include also some corresponding characterisations for minimal T_F , minimal T_{ES} , minimal $(T_D$ and $T_F)$, and minimal submaximality.

Let $\mathcal{T} \in LT(X)$ with specialisation order \leq .

Theorem 14 [2]. *(X, \mathcal{T}) is minimal T_F if and only if (X, \leq) is a duplex and $\mathcal{T} = \mathcal{W}$.*

Theorem 15 [5]. *(X, \mathcal{T}) is minimal T_{ES} (and not T_1) if and only if (X, \leq) is a duplex and $\mathcal{T} = \mathcal{L}$.*

Theorem 16. *(X, \mathcal{T}) is minimal $(T_D$ and $T_F)$ if and only if (X, \leq) is a duplex and $\mathcal{T} = \mathcal{M}$.*

Theorem 17. *(X, \mathcal{T}) is minimal weakly submaximal if and only if (X, \leq) is a duplex and $\mathcal{T} = \mathcal{N}$.*

Theorem 18 [3]. *(X, \mathcal{T}) is minimal submaximal (and not T_1) if and only if (X, \leq) is a duplex and $\mathcal{T} = \mathcal{A} = \mathcal{S}$.*

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