CONFORMAL TRANSFORMATIONS OF DISSIPATIVE OPERATORS

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Abstract

In the first part of this paper we give an elementary proof of the Kreiss Theorem [6] and we slightly improve the Leveque–Trefethen–Spijker Theorem [8; 12] for the case of matrices. In the second part we use the previous results to prove that if T is a quasi-dissipative matrix, and if φ is a conformal transformation of the negative half-plane onto the unit disk, then $\varphi(T)$ has bounded powers.

In applied mathematics, it is very important to know when the powers of a matrix are bounded for a given norm, especially when solving a differential equation or a partial differential equation by discretisation. For more details the reader is referred to [10] and the recent textbook [5].

In the first section, we give an elementary proof of an old result of Kreiss [6], that is, $p(T) = \sup_{k\geq 0} \|T^k\|$ is finite if and only if the spectrum of T is in the closed unit disk and if T satisfies the Kreiss condition

$$\|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda| - 1}, \quad \text{for} \quad |\lambda| > 1.$$

Let $r(T)$ be the lower bound of all $C > 0$ that satisfy the Kreiss condition. A conjecture was made by Leveque and Trefethen [8], and was subsequently proved by Spijker [12], that

$$p(T) \leq enr(T), \text{ if } T \in M_n(\mathbb{C}).$$

This is not the best inequality when the eigenvalues of T are of modulus 1 because then $p(T) \leq mr_1(T)$, where m denotes the number of distinct eigenvalues, and $r_1(T) \leq r(T)$ is the function defined after the proof of Theorem 1.1. Consequently, we give a slightly improved version of the above inequality.

In the second section, we use the previous results to show that if $H \in M_n(\mathbb{C})$ is a Hermitian matrix for a given norm, then its Cayley transformation $U = (1 + iH)(1 - iH)^{-1}$ has all its powers U^k which are bounded ($k \in \mathbb{Z}$). We generalise this result by showing that if T is a quasi-dissipative operator and if φ is a conformal transformation of the half-plane $\Pi = \{z : \Re(z) \leq 0\}$ onto the unit disk $\Delta = \{z : |z| \leq 1\}$, then $\varphi(T)$ satisfies the Kreiss condition for some effective constant C.

1. Power bounded operators

If T is in $B(X)$ and satisfies $\sup_k ||T^k|| \le C < \infty$, then by Gelfand's formula [1, theorem 3.2.8(iii)] the spectrum of T is in the unit disk and we have

$$\|(\lambda - T)^{-1}\| = \left\| \frac{1}{\lambda} + \frac{T}{\lambda^2} + \frac{T^2}{\lambda^3} + \dots \right\| \le \frac{C}{|\lambda| - 1},$$

for $|\lambda| > 1$. It is easy to see that if the spectrum is included in the unit disk then the powers of T are not necessarily bounded. To see this, take $T = \begin{pmatrix} 1, & a \\ 0, & 1 \end{pmatrix} \in M_2(\mathbb{C})$, with $a \ne 0$. Then $T^k = (I + N)^k = I + kN$, where $N^2 = 0$.

A natural question to ask is: do the two previous conditions imply the power-boundedness of T? Unfortunately, the answer is no for infinite-dimensional spaces. We can only affirm that if $Sp(T) \subset \Delta$ and $||(\lambda - T)^{-1}|| \le \frac{C}{|\lambda|-1}$ for $|\lambda| > 1$, then $\|T^k\| = O(k)$, for $k \ge 0$. If, moreover, $C = 1$, then $\|T^k\| = O(\sqrt{k})$. To see this, it is sufficient to apply Cauchy's formula to T and to the circle Γ of radius $1 + \frac{1}{k}$, which gives $T^k = \frac{1}{2\pi i} \int_\Gamma \mu^k (\mu - T)^{-1} d\mu$. Consequently,

$$\|T^k\| \le \frac{1}{2\pi} \left(1 + \frac{1}{k}\right)^k \frac{C}{1 + \frac{1}{k} - 1} 2\pi \left(1 + \frac{1}{k}\right) = Ck \left(1 + \frac{1}{k}\right)^{k+1,} \sim Cek.$$

If $C = 1$, then we have $e^{zT} = \lim_{k\to\infty} \left(\frac{k}{z}\right)^k \left(\frac{k}{z} - T\right)^{-k}$, and therefore by the resolvent condition satisfied for $k > |z|$ we have

$$\|e^{zT}\| \le \lim_{k\to\infty} \left|\frac{k}{z}\right|^k \left\| \left(\frac{k}{z} - T\right)^{-1} \right\|^k \le \lim_{k\to\infty} \left|\frac{k}{z}\right|^k \frac{1}{\left(\frac{k}{|z|} - 1\right)^k} = e^{|z|}.$$

Hence by Cauchy's formula for the kth derivative, applied to e^{zT} and to the circle Γ' having centre at 0 and radius k, we obtain $T^k = \frac{k!}{2\pi i} \int_{\Gamma'} \frac{e^{zT}}{z^{k+1}} dz$. Thus, by Sterling's formula, $\|T^k\| \le k! \frac{e^k}{k^k} \sim \sqrt{2\pi k}$.

Throughout this paper $T \in B(X)$ is said to satisfy the Kreiss condition, with the constant $C > 0$ if $Sp(T) \subset \Delta$ and $\left\|(\lambda - T)^{-1}\right\| \le \frac{C}{|\lambda|-1}$ for $|\lambda| > 1$. We note that if this condition is locally satisfied for $1 < |\lambda| < \rho$, then it is also satisfied for $|\lambda| > 1$ with a new larger constant. To see this we apply Holomorphic Functional Calculus to T, to the function $h(z) = \frac{1}{\lambda - z}$ and to the circle $\Gamma = \{z: |z| = \sqrt{\rho}\}$. Then we have

$$\begin{aligned}\|(\lambda - T)^{-1}\| &\le \frac{1}{2\pi} \int_\Gamma |\lambda - z|^{-1} \frac{C}{|z| - 1} |dz| \\ &\le \frac{C\sqrt{\rho}}{(\sqrt{\rho} - 1)(|\lambda| - \sqrt{\rho})} \le C \frac{\sqrt{\rho} + 1}{\sqrt{\rho} - 1} \frac{1}{|\lambda| - 1},\end{aligned}$$

for $|\lambda| > \rho$.

It is known that in the finite-dimensional case, the Kreiss condition implies that all the powers of T are bounded. We shall give a more detailed outline later. For now we give an apparently unknown elementary proof of this result.

Let $T \in M_n(\mathbb{C})$, then $(\lambda - T)^{-1}$ is a matrix having coefficients $r_{ij}(\lambda)$ that are rational functions with numerators that are polynomials of degree inferior or equal to $n-1$ and denominators that are the characteristic polynomials of T. Applying the decomposition theorem for rational functions in simple elements to each r_{ij}, it is easy to see that there exist matrices $A_{i,j} \in M_n(\mathbb{C})$ such that

$$(\lambda - T)^{-1} = \sum_{i=1}^{\ell} \frac{A_{i,1}}{\lambda - \alpha_i} + \frac{A_{i,2}}{(\lambda - \alpha_i)^2} + \ldots + \frac{A_{i,n_i}}{(\lambda - \alpha_i)^{n_i}} \tag{1}$$

where $\alpha_1 \ldots \alpha_\ell$ denote the roots of the characteristic polynomial and $n_1, \ldots n_\ell$ their multiplicities $(n_1 + \ldots n_\ell = n)$.

Theorem 1.1. *Let $T \in M_n(\mathbb{C})$. The following properties are equivalent:*

(i) $\sup_{k \geq 0} \|T^k\| < \infty$,

(ii) *T satisfies the Kreiss condition,*

(iii) *$Sp(T) \subset \Delta$ and all the eigenvalues of modulus 1 are simple poles of the resolvent.*

PROOF. (i) implies (ii) results from the remark given at the beginning. Let us show that (ii) implies (iii). Suppose for example that α_1 has modulus 1. Let λ converge to α_1, with $|\lambda| > 1$ and λ colinear to α_1. For $n \geq 2$, $\lim(\lambda - \alpha_1)^n \ \|(\lambda - T)^{-1}\| = \lim(|\lambda| - 1)^n \|(\lambda - T)^{-1}\| = 0$, by the Kreiss condition. Thus by successively applying this to $n = n_1, n_1 - 1, \ldots, 2$ we conclude from formula (1) that we have $A_{1,2} = A_{1,3} = \ldots = A_{1,n_1} = 0$. We now prove that (iii) implies (i). Let $\alpha_1, \ldots \alpha_m$ be the eigenvalues of T with modulus 1, with multiplicities $n_1, \ldots n_m$. Choose $r > 0$ such that the disks $\overline{B}(\alpha_i, r)$ are disjoint, and $1 - 2r$ is greater than the moduli of the other eigenvalues. By applying Holomorphic Functional Calculus [1, theorem 3.3.3] to T and to the functions f_0, f_1 defined by

$$f_0(\lambda) = \begin{cases} \lambda, & \text{on } B(0, 1-2r) \\ 0, & \text{on } \bigcup_{i=1}^{m} B(\alpha_i, r) \end{cases} \quad \text{and} \quad f_1(\lambda) = \begin{cases} 0, & \text{on } B(0, 1-2r) \\ \lambda, & \text{on } \bigcup_{i=1}^{m} B(\alpha_i, r) \end{cases},$$

we define $T_0 = f_0(T)$ and $T_1 = f_1(T)$. Then $T = T_0 + T_1$, $Sp(T_1) = \{\alpha_1 \ldots \alpha_m\} \cup \{0\}$ and $Sp(T_0) = \big(Sp(T) \setminus \{\alpha_1, \ldots, \alpha_m\}\big) \cup \{0\}$. In particular, the spectral radius satisfies $\rho(T_0) \leq 1 - 2r < 1$. Hence, by Gelfand's formula [1, theorem 3.2.8 (iii)], $\lim_{k \to \infty} ||T_0^k|| = 0$. Also $T_0 T_1 = T_1 T_0 = 0$, and hence $T^k = T_0^k + T_1^k$. Since $\frac{1}{\lambda}(\lambda - T)^{-1} = (\lambda - T_0)^{-1}(\lambda - T_1)^{-1}$, and $(\lambda - T_0)^{-1}$ has no poles on the unit circle, hypothesis (iii) implies that $(\lambda - T_1)^{-1}$ has only simple poles on the unit circle. In other words we have

$$(\lambda - T_1)^{-1} = \frac{A_1}{\lambda - \alpha_1} + \ldots + \frac{A_m}{\lambda - \alpha_m} + \frac{B}{\lambda}, \tag{2}$$

for some $A_1, \dots A_m, B \in M_n(\mathbb{C})$. This formula implies that

$$A_i = \frac{1}{2\pi i} \int_{\Gamma_i} (\lambda - T_1)^{-1} d\lambda,$$

where Γ_i is the boundary of $\overline{B}(\alpha_i, r)$, which proves that A_i is the Riesz projection associated to T and α_i (see [1, pp 106–7]). By developing in series all the terms in (2), for $|\lambda| > 1$, we get

$$\frac{1}{\lambda} + \frac{T_1}{\lambda^2} + \dots = \frac{A_1}{\lambda}\left(1 + \frac{\alpha_1}{\lambda} + \dots\right) + \dots + \frac{A_m}{\lambda}\left(1 + \frac{\alpha_m}{\lambda} + \dots\right) + \frac{B}{\lambda}.$$

Consequently $T_1 = \alpha_1 A_1 + \dots \alpha_m A_m$. So, because the Riesz projections are orthogonal, we get

$$T_1^k = \alpha_1^k A_1 + \dots \alpha_m^k A_m. \tag{3}$$

Therefore $\|T_1^k\| \leq \|A_1\| + \dots + \|A_m\|$. This proves that $\|T^k\|$ is bounded. ■

Let $T \in M_n(\mathbb{C})$ be such that the spectral radius satisfies $\rho(T) \leq 1$. Let us introduce the following three functions:

$$p(T) = \sup_{k \geq 0} ||T^k||;$$
$$r(T) = \sup_{|\lambda|>1} (|\lambda| - 1)||(\lambda - T)^{-1}||;$$
$$r_1(T) = \overline{\lim}_{\substack{|\lambda|\to 1 \\ |\lambda|>1}} (|\lambda| - 1)||(\lambda - T)^{-1}||.$$

We have $r_1(T) \leq r(T) \leq p(T)$. Theorem 1.1 implies that if $r(T)$ is finite then $p(T)$ is finite. Since $\lim\limits_{|\lambda|\to\infty} (|\lambda| - 1)||(\lambda - T)^{-1}|| = 1$, then by the Maximum Theorem for continuous functions we conclude that if $r_1(T)$ is finite then $r(T)$ is also finite. If we take $\mathbb{C}^2$ with the Euclidean norm, then for

$$T = \begin{pmatrix} 0, & a \\ 0, & 0 \end{pmatrix},$$

we have $r_1(T) = 0$ and

$$r(T) = \sup_{|\lambda|>1} (|\lambda| - 1)\left\| \frac{1}{\lambda}\left(1 + \frac{T}{\lambda}\right) \right\| \geq \frac{1}{4}||2 + T|| > \frac{|a|}{4},$$

which can be as large as needed.

Many authors have been interested in knowing whether it is possible to estimate $p(T)$ in terms of $r(T)$. According to Tadmor [13] the original proof of Kreiss [6] gives a very bad estimate $p(T) \leq Cr(T)^n$ for an effective constant C. Subsequently, Morton, Strang and Miller obtained better estimates of the type $p(T) \leq C6^n(n+4)^{5n} r(T)$, $p(T) \leq Cn^n r(T)$, and $p(T) \leq Ce^{9n^2} r(T)$. Strang noticed that in the paper of Laptev [7] the estimate $p(T) \leq \frac{32e}{\pi} n^2 r(T)$ is implicitly contained, an estimate which was later improved by Tadmor [13] to $p(T) \leq \frac{32}{\pi} nr(T)$ by using a Cauchy integral. Using Bernstein's inequality for the derivative of a rational function, Leveque and Trefethen [8] were able to show that $p(T) \leq 2enr(T)$, and they also conjectured that $p(T) \leq enr(T)$. The former result was slightly improved by Smith [11], who obtained $p(T) \leq \left(1 + \frac{2}{\pi}\right) enr(T)$. Spijker [12] finally solved the conjecture and his proof is simple.

The example given in [8, pp 585–6] shows that in some way this is the best possible inequality, because we can find a series of matrices $T_k \in M_n(\mathbb{C})$ such that $T_k^n = 0$ and $\lim_{k\to\infty} \frac{p(T_k)}{nr(T_k)} = e$. This inequality, which is the best possible for most matrices, could be very bad for some extreme cases, for example matrices which have all their eigenvalues on the unit circle.

Theorem 1.2. *Let $T \in M_n(\mathbb{C})$ have all its eigenvalues on the unit circle. Then $p(T) \leq mr_1(T)$, where $m \leq n$ is the number of distinct eigenvalues of T.*

PROOF. Suppose, without any restriction, that $r_1(T)$ is finite, because otherwise $p(T)$ is also infinite. By using the same argument as in the proof of Theorem 1.1, we conclude that

$$(\lambda - T)^{-1} = \frac{A_1}{\lambda - \alpha_1} + \ldots + \frac{A_m}{\lambda - \alpha_m},$$

where the A_i are projections. If we take λ converging to α_i, with $|\lambda| > 1$ and λ colinear to α_i, then $(\lambda - \alpha_i)(\lambda - T)^{-1}$ converges to A_i. Thus we obtain

$$||A_i|| = \lim_{\substack{|\lambda|>1 \\ |\lambda|\to 1}} ||(\lambda - \alpha_i)(\lambda - T)^{-1}|| \leq \overline{\lim}_{\substack{|\lambda|>1 \\ |\lambda|\to 1}} (|\lambda| - 1)||(\lambda - T)^{-1}|| = r_1(T).$$

Hence, by (3), we obtain $||T^k|| \leq mr_1(T)$. ■

Example. For $M_2(\mathbb{C})$ we give an example of a sequence of matrices T_n having their two eigenvalues of modulus 1, such that $\lim_{n\to\infty} \frac{p(T_n)}{r_1(T_n)} = 2$. Take $M_2(\mathbb{C})$ with the Euclidean norm $\|\cdot\|_2$, a sequence (θ_n) converging to 0, with $0 \leq \theta_n < \frac{\pi}{2}$ and

$$P_n = \begin{pmatrix} 1, & \cot\theta_n \\ 0, & 0 \end{pmatrix}.$$

The P_n are projections and $||P_n||_2 = ||1 - P_n||_2 = \frac{1}{\sin\theta_n}$. Let $T_n = 1 - 2P_n$. Then we have $T_n^2 = 1$, and hence

$$T_n^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ T_n, & \text{if } k \text{ is odd.} \end{cases}$$

Also

$$p(T_n) = ||T_n||_2 = ||T_n T_n^*||^{1/2} = \rho(T_n T_n^*)^{1/2} = \frac{1 + \cos\theta_n}{\sin\theta_n}.$$

But since $T_n = 1 - P_n - P_n$, where P_n and $1 - P_n$ are orthogonal projections, we have

$r_1(T_n) = \mathrm{Max}(||P_n||, ||1 - P_n||) = \frac{1}{\sin\theta_n}$ (see the proof of Theorem 2.2). Also

$$\lim_{n\to\infty} \frac{p(T_n)}{r_1(T_n)} = \lim_{n\to\infty} (1 + \cos\theta_n) = 2.$$

The proof of the fundamental inequality $p(T) \leq enr(T)$ depends on an idea of Leveque and Trefethen [8, p. 587] and on the lemma proven by Spijker [12], which affirms the following property. Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function such that the polynomials P, Q with complex coefficients are of degree at most n, and R does not have poles on the unit circle. Then

$$\int_{|z|=1} |R'(z)||dz| \leq 2\pi n \max_{|z|=1} |R(z)|.$$

We now give a slight improvement to the fundamental inequality.

Theorem 1.3. *Let $T \in M_n(\mathbb{C})$ be such that $\rho(T) \leq 1$. Denote by m the number of distinct points of the spectrum of T on the unit circle. Then we have*

$$p(T) \leq mr_1(T) + e(n-m)r(T)\|1 - A_1\| \dots \|1 - A_m\|, \tag{4}$$

where $A_1, \dots A_m$ are the Riesz projections associated to the m eigenvalues on the unit circle.

Proof. Suppose $r(T)$ is finite, otherwise there is nothing to prove. By using the notation in the proof of Theorem 1.1 we can write

$$(\lambda - T)^{-1} = Q(\lambda) + \frac{A_1}{\lambda - \alpha_1} + \dots + \frac{A_m}{\lambda - \alpha_m},$$

where the rational function Q has at most $n-m$ poles inside the open unit disk. The same argument as in the proof of Theorem 1.2 shows that $||A_1||, \dots, ||A_m|| \leq r_1(T)$. Since $T^k = T_0^k + T_1^k = T_0^k + \alpha_1^k A_1 + \dots + \alpha_m^k A_m$, we obtain $p(T) \leq p(T_0) + mr_1(T)$. To estimate $p(T_0)$ we use the identity

$$\frac{1}{\lambda} + (\lambda - T)^{-1} = (\lambda - T_0)^{-1} + (\lambda - T_1)^{-1}, \quad \text{for} \quad \lambda \notin Sp(T). \tag{5}$$

Let us denote by $P = 1 - (A_1 + \dots + A_m) = (1 - A_1) \dots (1 - A_m)$ the Riesz projection associated to T_0; therefore $PT_1 = T_1 P = 0$, $PT_0 = T_0 P = T_0$ and of course P commutes with T. Therefore

$$P(\lambda - T)^{-1} = P(\lambda - T_0)^{-1}. \tag{6}$$

Let us consider the closed subalgebra $A = PM_n(\mathbb{C})P$, which has P as identity and which contains T_0. The resolvent of T_0 relative to the subalgebra A having identity P is therefore $R(\lambda) = P(\lambda - T_0)^{-1}$ because $(\lambda P - T_0)P(\lambda - T_0)^{-1} = P(\lambda - T_0)^{-1}(\lambda P - T_0) = P$. The poles of $R(\lambda)$ are exactly all the poles of the resolvent of T, which have modulus less than 1. Hence their number is at most $n - m$. Since $P(\lambda - T_0)^{-1}$

converges to zero at infinity, then $R(\lambda)$ necessarily has at most $n-m-1$ zeros. In addition, $||R(\lambda)|| \le ||P||r(T)$ for $|\lambda| > 1$, from (6) and $||P|| \le ||1-A_1||\dots||1-A_m||$. Using the same argument as Leveque and Trefethen [8] and Spijker [12], and applying it not to $M_n(\mathbb{C})$ but to the subalgebra $A = PM_n(\mathbb{C})P$ having the same norm and the identity P, we obtain $p(T_0) \le e(n-m)||P||r(T)$. Hence we get the result. ■

Remarks. The beginning of the argument shows that there are only two possibilities. Either all the A_i are zero, in which case $m = 0$, and hence $r_1(T) = 0$, or one of the A_i is not zero, in which case $r_1(T) \ge ||A_i|| \ge 1$, because the norm of the projection is greater than or equal to 1. In practice it is difficult to estimate $||P||$. It is easy to see that $||P|| \le 1 + mr_1(T)$. This gives a satisfactory formula for $k = 0$ or $k = m$. But for $1 \le k \le m-1$, this bound does not give anything useful. For the case where $||1-A_1|| = \dots = ||1-A_m|| = 1$, which occurs when the projections are Hermitian, (4) is better than the fundamental inequality. It would be interesting to find a good estimate for $||P||$.

2. Conformal transformation of quasi-dissipative operators

If H is a self-adjoint operator on a Hilbert space, its Cayley transform $U = (1+iH)\,(1-iH)^{-1}$ gives a unitary operator, such that, for any $n \in \mathbb{Z}$, the norm of U^n is always equal to 1. Is there a similar result if we take a Hermitian operator on a Banach space? More generally, if we take a quasi-dissipative operator T on a Banach space and if φ is a conformal transformation of the half-plane $\{z : \Re(z) \le 0\}$ onto the unit disk, can we conclude that the powers of $\varphi(T)$ are bounded? We shall see that, in the case of matrices, the answer is yes. We suspect that the result is not true in the infinite-dimensional case, but we were not able to provide a counter-example.

If $T \in B(X)$, where X is a Banach space, the numerical range of T is defined by

$$V(T) = \{f(Tx) : x \in X,\, f \in X^*,\, ||x|| = ||f|| = f(x) = 1\}.$$

Then T is a Hermitian operator on this Banach space if and only if $V(T) \subset \mathbb{R}$. If X is a Hilbert space, with the standard norm, then this concept coincides with that of the self-adjoint operator. But, in general, the two notions are different. The only things we can say are that $Sp(T) \subset \mathbb{R}$ and $||e^{itT}|| = 1$, for all $t \in \mathbb{R}$ (see [2, lemma 5.2, p. 46]). Unfortunately, the set of Hermitian elements does not have interesting algebraic properties. In general, T Hermitian does not imply that T^2 is Hermitian. For more details, see [2; 3; 4].

We say that $T \in B(X)$ is dissipative if $V(T) \subset \{z : \Re(z) \le 0\}$. By the Lumer–Phillips theorem [2, p. 30], T is dissipative, if and only if $||e^{tT}|| \le 1$, for all $t \ge 0$. If H is Hermitian then iH, $-iH$ and $-H^2$ are dissipative. Below are more examples of dissipative matrices. We say that T is quasi-dissipative if there exists a constant $C > 0$ such that $||e^{tT}|| \le C$, for all $t \ge 0$. In this case, there exists an equivalent norm on $B(X)$ for which T becomes dissipative (see [2, lemma 2.7, p. 21]).

If we take the nilpotent matrix

$$T = \begin{pmatrix} 0, & a \\ 0, & 0 \end{pmatrix} \in M_2(\mathbb{C})$$

with $a > 0$, then $Sp(T) \subset \{z\colon \Re(z) \leq 0\}$ and T is not quasi-dissipative because $e^{tT} = 1 + tT$, and hence $\lim_{t\to\infty} ||e^{tT}|| = \infty$. By the conformal transformation $\varphi_0(z) = \frac{1+z}{1-z}$, $U = \varphi_0(T) = 1 + 2T$ and $U^n = 1 + 2nT$. In this case the powers of U are not bounded. In other words, the fact that $Sp(T) \subset \{z\colon \Re(z) \leq 0\}$ is not sufficient to conclude that the powers of U are bounded. We need more, for example the hypothesis of quasi-dissipativity. Let us now look at two small interesting examples.

Example 1. Take

$$T = \begin{pmatrix} -a, & c \\ 0, & -b \end{pmatrix},$$

with $0 \leq a < b$ and $0 < c \leq b - a$, then

$$e^{tT} = \begin{pmatrix} e^{-at}, & \frac{c}{b-a}(e^{-at} - e^{-bt}) \\ 0, & e^{-bt} \end{pmatrix}.$$

If we take in $M_2(\mathbb{C})$ the norm $||\ \ ||_1$ defined by

$$\left\| \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix} \right\|_1 = \max\left[|\alpha| + |\gamma|, |\beta| + |\delta|\right],$$

then

$$||e^{tT}||_1 = \begin{cases} e^{-at}, & \text{for } t \geq 0, \\ \frac{c}{b-a}(e^{-bt} - e^{-at}) + e^{-bt}, & \text{for } t < 0. \end{cases}$$

In particular, T is dissipative and $||e^{tT}||$ increases rapidly on the negative half-line $\mathbb{R}_-$. If we take $a = 0, c = b$ and $U = (1 + T)(1 - T)^{-1}$, then

$$U^n = \begin{pmatrix} 1, & 1 - (\frac{1-b}{1+b})^n \\ 0, & (\frac{1-b}{1+b})^n \end{pmatrix}.$$

Therefore, if we take $b > 1$, then

$$||U^n|| = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 1 + 2(\frac{b-1}{b+1})^n > 1, & \text{if } n \text{ is odd.} \end{cases}$$

Example 2. Let P be a projection in $B(X)$. Then we have $e^{-tP} = 1 - P(1 - e^{-t}) = 1 - P + Pe^{-t}$. Hence $||e^{-tP}|| \leq ||1 - P|| + ||P||$ for $t \geq 0$ implies that P is quasi-dissipative. For this case we have $U = \frac{1-P}{1+P} = 1 - P$, hence $U^k = U$, for any $k \geq 1$.

Lemma 2.1. *If $T \in B(X)$ is quasi-dissipative with constant $C > 0$, then we have* $||(\lambda - T)^{-1}|| \leq \frac{C}{\Re(\lambda)}$, *for* $\Re(\lambda) > 0$.

PROOF. By the Laplace transform, we have $(\lambda - T)^{-1} = \int_0^\infty e^{-\lambda t} e^{tT} dt$. Therefore

$$||(\lambda - T)^{-1}|| \leq \int_0^\infty |e^{-\lambda t}| ||e^{tT}|| dt \leq C \int_0^\infty e^{-t\Re(\lambda)} dt = \frac{C}{\Re(\lambda)}. \quad \blacksquare$$

According to theorem 2 in [8] the converse is true in finite dimension.

By the Riesz theory, we know that all compact self-adjoint operators T on a Hilbert space can be written in the form $T = \sum \lambda_n P_n$, where λ_n are the eigenvalues of T and the orthogonal projections P_n are self-adjoint and of finite rank. This theorem was generalised for compact Hermitian operators on Banach spaces (see [3, theorem 1, p. 82] and [4, theorem 11, p. 33]), except that in general, even in finite dimensions, the P_n are not Hermitian (see the remark in [3, p. 84] and Crabb's example [2, p. 58]). The proof of this theorem is difficult because it is based on a fixed-point theorem (Schauder–Tychonoff or Markov–Kakutani). We now give an elementary proof of this result for matrices.

Theorem 2.2. (Spectral Theorem for Hermitian Matrices.) *Let $H \in M_n(\mathbb{C})$ be a Hermitian matrix for some norm. Then*

$$H = \alpha_1 P_1 + \ldots + \alpha_m P_m,$$

where $\alpha_1 \ldots \alpha_m$ are the distinct eigenvalues of H and $P_1, \ldots, P_m$ are the corresponding Riesz projections. These projections satisfy $P_1 + \ldots + P_m = 1$, $P_i P_j = 0$ for $i \neq j$ and $||P_i|| = 1$ for $i = 1, \ldots, m$.

PROOF. Since H is Hermitian, $T = iH$ is dissipative, then by the previous lemma we have

$$||(\lambda - T)^{-1}|| \leq \frac{1}{\Re(\lambda)}, \quad \text{for} \quad \Re(\lambda) \geq 0.$$

By the argument used in the proof of (ii) implies (iii) of Theorem 1.1, and by taking λ converging to $i\alpha_r (r = 1, \ldots m)$, with $\Re(\lambda) > 0$ and $\Im(\lambda) = \alpha_r$, we prove that the poles $i\alpha_1 \ldots i\alpha_m$ are simple. By formula (1), we have $(\lambda - T)^{-1} = \frac{A_1}{\lambda - i\alpha_1} + \ldots + \frac{A_m}{\lambda - i\alpha_m}$, for all $\lambda \neq i\alpha_1, \ldots, i\alpha_m$. Changing λ to $i\lambda$ we obtain

$$(\lambda - H)^{-1} = \frac{A_1}{\lambda - \alpha_1} + \ldots + \frac{A_m}{\lambda - \alpha_m}.$$

The same argument as the one used in the proof of Theorem 1.1 shows that $A_i = P_i$, the Riesz projection associated to H and α_i. Hence we have $P_1 + \ldots + P_m = 1$ and $P_i P_j = 0$ for $i \neq j$. By developing the series for $|\lambda| > ||H||$, we obtain

$$H = \alpha_1 P_1 + \ldots + \alpha_m P_m.$$

It is left to verify that $||P_i|| = 1$. The properties of the projections imply that

$$(\lambda - iH)^{-1} = \frac{P_1}{\lambda - i\alpha_1} + \ldots + \frac{P_m}{\lambda - i\alpha_m}.$$

Therefore, by making λ converge to $i\alpha_i$, with $\Re(\lambda) > 0$ and $\Im(\lambda) = \alpha_i$, we obtain

$$||P_i|| = \lim_{\lambda} |\lambda - i\alpha_i| ||(\lambda - iH)^{-1}|| = \Re(\lambda)||(\lambda - iH)^{-1}|| \leq 1,$$

which yields the result because $\|P_i\|$ is always greater than 1. ■

Corollary 2.3. *Let $H \in M_n(\mathbb{C})$ be a Hermitian matrix for some norm. Let $U = (1+iH)(1 - iH)^{-1}$ be its Cayley transform. Then $||U^k|| \leq n$, for any $k \in \mathbb{Z}$.*

Proof. By the previous arguments we have

$$U^k = \left(\frac{1 + i\alpha_1}{1 - i\alpha_1}\right)^k P_1 + \ldots + \left(\frac{1 + i\alpha_n}{1 - i\alpha_n}\right)^k P_n,$$

but

$$\left|\frac{1 + i\alpha_r}{1 - i\alpha_r}\right| = 1, \quad \text{for} \quad r = 1, \ldots, n.$$

Hence $||U^k|| \leq ||P_1|| + \ldots + ||P_n|| = n.$ ■

Let us now investigate the more general problem of the conformal transformation of a quasi-dissipative matrix. Let Π be the half-plane $\{z \colon \Re(z) \leq 0\}$ and Δ be the unit disk $\{z \colon |z| \leq 1\}$. The transformation

$$\varphi_0(z) = \frac{1 + z}{1 - z}$$

is a conformal transformation of Π onto Δ. Knowing that all the conformal transformations of the unit disk Δ are obtained by composition of the transformations

$$\psi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}, \quad |\alpha| < 1,$$

and the rotations $R_\theta \colon z \longrightarrow e^{i\theta} z$, we can conclude that all conformal transformations of Π onto Δ are of the form $R_\theta \circ \psi_\alpha \circ \varphi_0$.

Theorem 2.4. *Let T be a quasi-dissipative operator, with constant $C > 0$, and φ be a conformal transformation of Π onto Δ. Then $\varphi(T)$ satisfies the Kreiss condition outside Δ with a constant less than or equal to*

$$C_\varphi = C\frac{||1 - T|| + r||1 + T||}{1 - r},$$

where $r = |\varphi(-1)|$.

PROOF. By Holomorphic Functional Calculus it is clear that the spectrum of $\varphi(T)$ is contained in Δ.

(a) First suppose that $\varphi = \varphi_0$. Let $|\mu| > 1$, and then $\mu = \frac{1+\lambda}{1-\lambda}$ with $\lambda = \frac{\mu-1}{\mu+1}$ satisfies $\Re(\lambda) > 0$. We have $\mu - \varphi_0(T) = \frac{2(\lambda-T)}{1-\lambda}(1-T)^{-1}$, and consequently $(\mu - \varphi_0(T))^{-1} = \frac{1-\lambda}{2}(1-T)(\lambda-T)^{-1} = \frac{1}{\mu+1}(1-T)\left(\frac{\mu-1}{\mu+1} - T\right)^{-1}$; hence, by Lemma 2.1, we have

$$||(\mu - \varphi_0(T))^{-1}|| \leq \frac{||1-T||}{|1+\mu|}\frac{C}{\Re(\frac{\mu-1}{\mu+1})}.$$

Let $\mu = a + ib$, then

$$\frac{\mu-1}{\mu+1} = \frac{a-1+ib}{a+1+ib} = \frac{(a-1+ib)(a+1-ib)}{(a+1)^2+b^2};$$

hence

$$Re\frac{\mu-1}{\mu+1} = \frac{a^2+b^2-1}{(a+1)^2+b^2} = \frac{|\mu|^2-1}{|\mu+1|^2}. \tag{7}$$

Finally, we obtain

$$||(\mu - \varphi_0(T))^{-1}|| \leq C||1-T||\frac{|\mu+1|}{|\mu|^2-1} \leq C||1-T||\frac{1+|\mu|}{|\mu|^2-1} = C\frac{||1-T||}{|\mu|-1}.$$

(b) Since the Kreiss condition depends only on $|\mu|$, it suffices to assume, without loss of generality, that $\varphi = \psi_\alpha \circ \varphi_0$, with $0 < |\alpha| < 1$. We set $A = \varphi_0(T)$, so $\varphi(T) = \psi_\alpha(A)$. Let $|\mu| > 1$ with $\mu \neq -\frac{1}{\overline{\alpha}}$, then $\mu = \psi_\alpha(T)$, where $t = \frac{\mu+\alpha}{1+\overline{\alpha}\mu}$, with $|t| > 1$. Thus we obtain

$$\mu - \varphi(T) = \frac{(1-|\alpha|^2)}{1-\overline{\alpha}t}(t-A)(1-\overline{\alpha}A)^{-1},$$

hence

$$\begin{aligned}
(\mu - \varphi(T))^{-1} &= \frac{1-\overline{\alpha}t}{1-|\alpha|^2}(1-\overline{\alpha}A)(t-A)^{-1}\\
&= \frac{1-\overline{\alpha}t}{1-|\alpha|^2}(1-\overline{\alpha}(1+T)(1-T)^{-1})(t-(1+T)(1-T)^{-1})^{-1}\\
&= \frac{1-\overline{\alpha}t}{1-|\alpha|^2}(1-\overline{\alpha}-(1+\overline{\alpha})T)(t-1-(1+t)T)^{-1}\\
&= \frac{1-\overline{\alpha}t}{1-|\alpha|^2}(1-\overline{\alpha}-(1+\overline{\alpha})T)\frac{1}{1+t}\left(\frac{t-1}{t+1} - T\right)^{-1}.
\end{aligned}$$

Thus, for $|\mu| > 1$, $\mu \neq -\frac{1}{\overline{\alpha}}$, we have

$$\|(\mu - \varphi(T)^{-1}\| \leq \frac{\|1-\overline{\alpha}-(1+\overline{\alpha})T\|}{1-|\alpha|^2}\frac{|1-\overline{\alpha}t|}{|1+t|}\frac{C}{\Re\left(\frac{t-1}{t+1}\right)}.$$

Therefore, by formula (7), we have

$$\begin{aligned}\|(\mu-\varphi(T)^{-1}\| &\leq \frac{C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|}{1-|\alpha|^2}|1-\overline{\alpha}t|\frac{|t+1|}{|t|^2-1}\\ &= C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|\frac{|t+1|}{|1+\overline{\alpha}\mu|(|t|^2-1)}\\ &= C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|\frac{|1+\alpha+\mu(1+\overline{\alpha})|}{|\mu+\alpha|^2-|1+\overline{\alpha}\mu|^2}\\ &\leq C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|\frac{|\mu+\alpha|+|(1+\mu\overline{\alpha})|}{|\mu+\alpha|^2-|1+\overline{\alpha}\mu|^2}\\ &= C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|\frac{1}{|\mu+\alpha|-|1+\overline{\alpha}\mu|}.\end{aligned}$$

Since $|\mu+\alpha|-|1+\overline{\alpha}\mu| = |\mu+\alpha|-|\alpha||\mu+\frac{1}{\overline{\alpha}}|$ and α and $\frac{1}{\overline{\alpha}}$ are colinear, we can assume, without loss of generality, that α is real positive in order to find a majorant for $\frac{1}{|\mu+\alpha|-|1+\overline{\alpha}\mu|}$, which depends only on $|\mu|$. Suppose $\alpha > 0$, and let $\mu = a+ib$, then

$$\frac{1}{|\mu+\alpha|^2-|\alpha\mu+1|^2} = \frac{1}{(1-\alpha^2)(|\mu|^2-1)},$$

which implies that

$$\frac{1}{|\mu+\alpha|-|\alpha\mu+1|} = \frac{|\mu+\alpha|+|\alpha\mu+1|}{|\mu+\alpha|^2-|\alpha\mu+1|^2} \leq \frac{(1+\alpha)(1+|\mu|)}{(1-\alpha^2)(|\mu|^2-1)} = \frac{1}{(1-\alpha)(|\mu|-1)}.$$

This proves that for $|\mu| > 1$, and $\mu \neq -\frac{1}{\overline{\alpha}}$, we have

$$\begin{aligned}\|(\mu-\varphi(T))^{-1}\| &\leq \frac{C\|1-\overline{\alpha}-(1+\overline{\alpha})T\|}{1-|\alpha|}\frac{1}{|\mu|-1}\\ &\leq \frac{C\|1-T\|+|\alpha|\|1+T\|}{1-|\alpha|}\frac{1}{|\mu|-1}.\end{aligned}$$

By continuity we obtain the same inequality at $-\frac{1}{\overline{\alpha}}$. Hence the result. ■

Remark. In Theorem 2.4 the constant C_φ tends to $+\infty$ when $|\alpha| \longrightarrow 1$. Unfortunately, it is not possible to improve this. We can see this by taking $\alpha_n = 1-\frac{1}{n}$ and $\varphi_n = \psi_n$. If $|\mu| > 1$ is fixed, then we have

$$\lim_{n\to\infty} \|(\mu-\varphi_n(T))^{-1}\|(|\mu|-1) = +\infty.$$

Corollary 2.5. *Let $T \in M_n(\mathbb{C})$ be a quasi-dissipative matrix for some norm, and let φ be a conformal transformation of Π onto Δ. Then $\|\varphi(T)^k\|$ is bounded.*

Proof. This follows from the previous theorem and from results in the first section. ■

Many results of this paper can be extended to some operators in infinite dimension, for example Riesz operators (see [14, theorem 7]) or meromorphic operators (see [9]).

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