# ALMOST REGULAR OPERATORS III

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Abstract

'Almost regular' operators admit a 'bounded approximate generalised inverse', and share properties analogous to regular operators which have generalised inverses. In particular, if the spaces are complete then almost regular operators are regular. We show here that $T : X \to Y$ is almost regular if and only if its completion $T^{\sim} : X^{\sim} \to Y^{\sim}$ is regular.

## Introduction

Recall ([4, definition 7.3.1]) that an element $a \in A$ in a normed linear algebra $A$ is *regular*, or 'relatively Fredholm', if and only if

$$a \in aAa, \tag{0.1}$$

so that it has a *generalised inverse* $b \in A$ for which

$$a = aba \in A. \tag{0.2}$$

This includes the possibility that $A$ is either *left invertible* ($ba = 1$) or *right invertible* ($ab = 1$). All this makes sense if $A$ is taken to be a 'normed linear category', in particular the category $BL$ of bounded linear operators between normed spaces: thus $T \in BL(X, Y)$ is regular if there is $S \in BL(Y, X)$ for which

$$T = TST \in BL(X, Y). \tag{0.3}$$

When $a \in A$ satisfies (0.2) then

$$ba = p = p^2 \quad \text{and} \quad ab = q = q^2 \tag{0.4}$$

are idempotents intimately connected to $a$:

$$\begin{aligned} p^{-1}(0) = a^{-1}(0) \equiv \{x \in A : ax = 0\} \quad \text{and} \\ q_{-1}(0) = a_{-1}(0) \equiv \{y \in A : ya = 0\}; \end{aligned} \tag{0.5}$$

in particular, when $A = BL$ and $a = T$ satisfies (0.3) then $P = ST$ and $Q = TS$ satisfy

$$P^{-1}(0) = T^{-}(0) \subseteq X \quad \text{and} \quad Q(Y) = \mathrm{cl}(TX) \subseteq Y. \tag{0.6}$$

In general it is necessary and sufficient for $T \in BL$ to be regular that both its null space and the closure of its range be complemented, and in addition $T$ is *proper* or 'strict', in the sense ([4, definition 3.2.7]) that the induced operator

$$\text{core}(T) : X/T^{-1}(0) \to \text{cl}(TX) \quad \text{is invertible.} \tag{0.7}$$

In particular, it is a consequence of (0.7) that

$$T(X) = \text{cl}(TX) \subseteq Y \tag{0.8}$$

$T$ has closed range; conversely, when the normed spaces $X$ and $Y$ are complete then the closed range condition (0.8) is, by the open mapping theorem, sufficient for $T$ to be proper in the sense of (0.7). If in particular $X$ and $Y$ are Hilbert spaces, so that all closed subspaces are complemented, then the closed range condition (0.8) is even sufficient for regularity (0.3).

If $A$ is a Banach algebra, that is, a normed algebra which is complete, then it is familiar that the group $A^{-1}$ of invertible elements forms an open set, as do the semigroups $A^{-1}_{left}$ and $A^{-1}_{right}$ of left and of right invertible elements. This topological property fails in general when the normed algebra $A$ is incomplete, but is passed on to the larger semigroups of 'almost' left or right invertible elements ([4, definition 3.7.1]). 'Almost invertibility' would be a more familiar idea were it not for the fact ([4, theorem 4.4.5]) that when the algebra $A$ is complete its invertible and almost invertible elements are the same: in other words, in the passage from complete to incomplete algebras the theory splits, the good properties being shared between two different concepts which coalesce when completeness is imposed. It is these considerations which motivate us to offer ([4, preface, p. vii]; [6, definition 3.1]) an 'approximate' version of regularity.

**Definition 1.** The element $a \in A$ is said to be approximately regular if there is $b = (b_n)$ in $A$ for which

$$\|a - ab_na\| \to 0 \quad \text{as} \quad n \to \infty, \tag{1.1}$$

and almost regular if there is $(b_n)$ in $A$ for which in addition

$$\sup_n \|b_n\| < \infty. \tag{1.2}$$

Clearly

$$(0.1) \Rightarrow (1.1)\,\&\,(1.2) \Rightarrow (1.1). \tag{1.3}$$

If left and right invertibility are special cases of regularity, then 'almost' left and right invertibility should be special cases of 'almost' regularity, and 'approximate' left or right invertibility special cases of 'approximate regularity': we call $a \in A$ *approximately left invertible* if

$$\|1 - b_na\| \to 0, \tag{1.4}$$

and *almost left invertible* if in addition (1.2) holds. What is interesting here is the implication ([3, theorem 3.7]; [4, theorem 3.7.2]) that

$$a \in A \text{ approximately left invertible} \Rightarrow a \in A \text{ almost left invertible,} \tag{1.5}$$

while if $A$ is complete then also

$$a \in A \text{ almost left invertible} \Rightarrow a \in A \text{ left invertible}. \tag{1.6}$$

Indeed, if (1.4) holds there must be $b_0' \in A$ for which

$$c = 1 - b_0' a \Rightarrow \|c\| < 1, \tag{1.7}$$

and then

$$b_n' = (1 + c + \cdots + c^n) b_0' \Rightarrow \|1 - b_n' a\| \to 0 \quad \text{and} \quad \|b_n'\| \le \frac{\|b_0'\|.}{(1 - \|c\|)}. \tag{1.8}$$

This gives (1.5); if $A$ is complete then we can sum the series in (1.8) and get ([4, theorem 4.4.5, 4.5.7]) an actual left inverse.

One–oneness and regularity together (cf. [6, theorem 3.3]) give left invertibility.

**Proposition 2.** *Necessary and sufficient for $a \in A$ to be almost left invertible is that*

$$a \in A \text{ is approximately regular and not a topological left zero divisor}; \tag{2.1}$$

*necessary and sufficient for $a \in A$ to be almost right invertible is that*

$$a \in A \text{ is approximately regular and not a topological right zero divisor}. \tag{2.2}$$

PROOF. If $a \in A$ is almost left invertible then it is certainly almost regular, and also ([3, theorem 3.2]; [4, theorem 3.7.3]) not a topological left zero divisor: with $(b_n')$ as in (1.7) and $\|b_n'\| \le k < k'$ take $c' = b_N'$ so that $\|1 - c'a\| \le \delta$ with $k = (1-\delta)k'$ and argue, for arbitrary $x \in A$,

$$\|x\| \le \|(1 - c'a)x\| + \|c'ax\| \le \delta\|x\| + k\|ax\|.$$

Conversely, if $a \in A$ is not a topological left zero divisor there is implication, for arbitrary $x = (x_n)$ in $A$,

$$ax_n \to 0 \Rightarrow x_n \to 0, \tag{2.3}$$

and if in addition (1.1) holds then take $x_n = 1 - b_n a$. This means that $a \in A$ is approximately left invertible, and hence by (1.5) almost left invertible. The argument for (2.2) is identical. ■

For example, if $A = BL$ then the condition that $a = T$ is not a topological left zero divisor is ([3, theorem 3.2]; [4, theorem 3.3.4]) that $T$ is bounded below, and the condition that $a = T$ is not a right topological zero divisor is ([3, theorem 3.2]; [4, theorem 3.4.4]) that $T$ is almost open.

Even when $A$ is complete the set of regular elements $a \in aAa$ need not be open: it is not clear what 'good' properties should be inherited by the 'almost regular' elements in the incomplete case. Atkinson's lemmas ([4, theorem 3.8.3]) have such an extension.

**Theorem 3.** *If $a \in A$ and $b \in A$ then*

$$1 - ba \text{ approximately regular} \Rightarrow 1 - ab \text{ approximately regular} \tag{3.1}$$

*and*

$$1 - ba \text{ almost regular} \Rightarrow 1 - ab \text{ almost regular}. \tag{3.2}$$

*If*

$$a - aa'_n a \to b \tag{3.3}$$

*then*

$$b \text{ almost regular} \Rightarrow a \text{ almost regular}. \tag{3.4}$$

Proof. Observe

$$1 - ab - (1 - ab)(1 + acb)(1 - ab) = a(1 - ba - (1 - ba)c(1 - ba))b. \tag{3.5}$$

Thus if $1 - ba$ is approximately regular, with

$$\|1 - ba - (1 - ba)c_n(1 - ba)\| \to 0,$$

then $1 - ab$ is approximately regular, with $1 + ac_n b$ in place of $c_n$. This gives both (3.1) and (3.2). If (3.3) holds, write $b_n = a - aa'_n a$ and suppose

$$\|b - bb'_n b\| \to 0:$$

then

$$\|a - a(a'_n + (1 - a'_n a)b'_n(1 - aa'_n))a\| = \|b_n - b_n b'_n b_n\| \to 0,$$

giving (3.4). ■

When $A = BL$ then almost regularity gives ([6, theorem 3.2]) something approaching the closed range property (0.7).

**Proposition 4.** *If $T \in BL(X, Y)$ then*

$$T \text{ almost regular} \Rightarrow T \text{ relatively almost open}. \tag{4.1}$$

*If in particular $X$ is complete then almost regular operators on $X$ have closed range.*

Proof. If $\|T - TS_nT\| \to 0$ then

$$y \in T(X) \Rightarrow \|y - Tx_n\| \to 0 \text{ with } x_n = S_n y. \tag{4.2}$$

If in particular the space $X$ is complete then the easy half ([4, theorem 4.4.4]) of the open mapping theorem says that the operator $T^\vee : X \to T(X)$ is open, forcing the subspace $T(X)$ to be complete and hence closed. ■

Proposition 4 does not extend to approximate regularity; Bermudez and Gonzalez ([1, remark 3]) have noticed the following.

*Example 5.* If $X = Y = l_2$ and $T = W : (x_n) \mapsto (\frac{1}{n}x_n)$ then $T$ is approximately regular but not relatively almost open.

For (1.1) take

$$S_n(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots) = (x_1, 2x_2, \ldots, nx_n, 0, 0, \ldots). \quad (5.1)$$

Since, however, $W$ is totally bounded but not of finite rank it is easy to show that it cannot be relatively almost open: if $T^\vee : X \to T(X)$ is almost open and not finite rank there is $y = (y_n)$ in $T(X)$ with $\|y_n\| = 1$ and $\|y_n - y_m\| \geq \frac{1}{2}$ if $m \neq n \in \mathbf{N}$, while if also $T$ is totally bounded there is $k > 0$ and sequences $x^{(n)} = (x_m^{(n)})$ in $X$ with $\|x_m^{(n)}\| \leq k$ and $\|y_n - Tx_m^{(n)}\| \leq \frac{1}{m}$.

Alternatively note that $X = l_2$ is complete and that $T = W$ is dense but not onto, and therefore does not have closed range.

The *enlargement* of an almost regular operator is regular (cf. [1, theorem 2]).

**Proposition 6.** *If $T \in BL(X, Y)$ then*

$$T \text{ almost regular} \Rightarrow \mathbf{Q}(T) \text{ regular}. \quad (6.1)$$

*If $a \in A$ then*

$$a \text{ almost regular} \Rightarrow \mathbf{q}(a) \text{ regular}. \quad (6.2)$$

PROOF. Here $\mathbf{Q} = l_\infty(\cdot)/c_0(\cdot)$ is the enlargement functor ([4, definition 1.9.2, 2.7.2]), and $\mathbf{q} : X \to \mathbf{Q}(X)$ the natural embedding. If $\|T - TS_nT\| \to 0$ holds with bounded $(S_n)$, define $\mathbf{S} : \mathbf{Q}(Y) \to \mathbf{Q}(X)$, following Bermudez and Gonzalez [1], by setting

$$\mathbf{S}((y_n) + c_0(Y)) = (S_ny_n) + c_0(X) \text{ for each } y = (y_n) \in l_\infty(Y) \quad (6.3)$$

and observe that

$$\mathbf{Q}(T)\mathbf{S}\mathbf{Q}(T) = \mathbf{Q}(T). \quad (6.4)$$

If $\|a - ab_na\| \to 0$ holds with bounded $(b_n)$, define $\mathbf{b} \in \mathbf{Q}(A)$ by setting

$$\mathbf{b} = (b_n) + c_0(A) \quad (6.5)$$

and observe that

$$\mathbf{q}(a)\mathbf{b}\mathbf{q}(a) = \mathbf{q}(a). \quad \blacksquare \quad (6.6)$$

When $A = BL$ and the spaces $X$ and $Y$ are complete then Bermudez and Gonzalez [1] have shown that (1.1) and (1.2) are together equivalent to (0.1).

**Theorem 7.** *If $T \in BL(X, Y)$ for Banach spaces $X$ and $Y$ then*

$$T \text{ almost regular} \Rightarrow T \text{ regular}. \tag{7.1}$$

Proof. We offer a variation of the argument of Bermudez and Gonzalez [1]. Recall the 'canonical factorisation' ([4, theorem 2.3.3])

$$T = JT^{\wedge}K \text{ with } T^{\wedge} = \text{core}(T), \tag{7.2}$$

where $J : \text{cl } TX \to Y$ and $K : X \to X/T^{-1}(0)$ are the natural mappings. Now if $T$ is almost regular, then so is $T^{\wedge} = \text{core}(T)$: if $\|T - TS_nT\| \to 0$ with $S_n : Y \to X$, then

$$\|T^{\wedge} - T^{\wedge}S_n^{\vee}T^{\wedge}\| \to 0 \text{ with } S_n^{\vee} = KS_nJ. \tag{7.3}$$

By Proposition 4 $T$ has closed range, and by Proposition 2 core$(T)$ is almost invertible, therefore invertible. At the same time the product $JT^{\wedge} : X/T^{-1}(0) \to Y$ is almost regular and bounded below, therefore by Proposition 2 left invertible, so that $T^{-1}(0) = P^{-1}(0)$ is complemented in $X$, and the product $T^{\wedge}K : X \to$ cl $TX$ is almost regular and onto, therefore by Proposition 2 right invertible, so that cl $TX = Q(Y)$ is complemented in Y. ■

We have not succeeded in extending Theorem 7 to more general $A$, in particular to $a \in A$ for a Banach algebra $A$.

Recall [5] the 'almost closure' of the range of an operator $T \in BL(X, Y)$:

$$\text{cl}^{\sim}(T, X) = \{\lim_n Tx_n : x \in l_\infty(X) \text{ and } Tx \in c_1(Y)\}: \tag{7.4}$$

evidently

$$T(X) \subseteq \text{cl}^{\sim}(T, X) \subseteq \text{cl}(TX). \tag{7.5}$$

If $\mathbf{Q} = l_\infty(\cdot)/c_0(\cdot)$ is again the enlargement functor then ([5, theorem 7]; cf. [2, proposition 17])

$$\text{cl}^{\sim}(T, X) = \{y \in Y : \mathbf{q}(y) \in \mathbf{Q}(T)\mathbf{Q}(X)\}. \tag{7.6}$$

If we recall that the *completion* of a normed space $X$,

$$X^{\sim} = c(X)/c_0(X) \subseteq \mathbf{Q}(X), \tag{7.7}$$

is a subspace of the enlargement, then we have ([4, theorem 4.5.3])

$$\mathbf{Q}(X) = \mathbf{Q}(X^{\sim}), \tag{7.8}$$

the enlargement of the completion is the same as the enlargement of the original: for the embedding $J : X \to X^{\sim}$ is bounded below and dense, therefore ([3, theorem 4.2]; [4, theorem 3.5.1]) almost open, so that its enlargement $\mathbf{Q}(J) : \mathbf{Q}(X) \to \mathbf{Q}(X^{\sim})$ is bounded below and open, therefore invertible. Now from (7.4) it follows that

$$y \in \text{cl}^{\sim}(T, X) \Leftrightarrow J(y) \in \text{cl}^{\sim}(T^{\sim}, X^{\sim}). \tag{7.9}$$

The connection between 'almost regularity' and the almost closure of the range resides in the following observation, with $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$ acting on the normed space $A$.

**Lemma 8.** *If $a \in A$ then*

$$a \text{ regular} \Leftrightarrow a \in (L_a R_a)(A); \tag{8.1}$$

$$a \text{ almost regular} \Leftrightarrow a \in \text{cl}^{\sim}(L_a R_a, A); \tag{8.2}$$

$$a \text{ approximately regular} \Leftrightarrow a \in \text{cl}(L_a R_a A). \tag{8.3}$$

PROOF. Clear. ■

From (8.2) and (7.6) we can reverse the implication in Proposition 6.

**Theorem 9.** *If $a \in A$ is a normed algebra element then*

$$a \in A \text{ almost regular} \Leftrightarrow \mathbf{q}(a) \in \mathbf{Q}(A) \text{ regular}. \tag{9.1}$$

*In particular,*

$$a \in A \text{ almost left invertible} \Leftrightarrow \mathbf{q}(a) \in \mathbf{Q}(A) \text{ left invertible} \tag{9.2}$$

*and*

$$a \in A \text{ almost right invertible} \Leftrightarrow \mathbf{q}(a) \in \mathbf{Q}(A) \text{ right invertible}. \tag{9.3}$$

PROOF. For (9.1) combine (7.6), with $T = L_a R_a$, with (8.1) and (8.2). For (9.2) replace (8.1) with the observation

$$a \text{ almost left invertible} \Leftrightarrow 1 \in \text{cl}^{\sim}(\text{R}_\text{a}, \text{A}), \tag{9.4}$$

and for (9.3) with

$$a \text{ almost right invertible} \Leftrightarrow 1 \in \text{cl}^{\sim}(\text{L}_\text{a}, \text{A}). \quad ■ \tag{9.5}$$

From (1.6) it is clear that, for a normed algebra $A$,

$$a \in A \text{ almost left invertible} \Rightarrow \mathbf{q}(\text{a}) \in \text{A}^{\sim} \text{ left invertible} \tag{9.6}$$

and

$$a \in A \text{ almost right invertible} \Rightarrow \mathbf{q}(\text{a}) \in \text{A}^{\sim} \text{ right invertible}. \tag{9.7}$$

We are now in possession of the converse.

**Theorem 10.** *If $a \in A$ for a normed algebra $A$ then*

$$a \in A \text{ almost left invertible} \Leftrightarrow \mathbf{q}(a) \in A^{\sim} \text{ left invertible} \tag{10.1}$$

*and*

$$a \in A \text{ almost right invertible} \Leftrightarrow \mathbf{q}(a) \in A^{\sim} \text{ right invertible}. \tag{10.2}$$

*If $T \in BL(X, Y)$ is a bounded linear operator between normed spaces then*

$$T \in BL(X, Y) \text{ almost regular} \Leftrightarrow T^{\sim} \in BL(X^{\sim}, Y^{\sim}) \text{ regular}. \tag{10.3}$$

Proof. (9.1) applies equally to $T$ and to $T^{\sim}$, and Theorem 7 applies to $T^{\sim}$, giving (10.3). Also (9.2) and (9.3) apply equally to $a \in A$ and $\mathbf{q}(a) \in A^{\sim}$. ■

Two candidates for a 'decomposably almost regular' operator coincide, as follows.

**Theorem 11.** *If $a \in A$ then the following two conditions are equivalent:*

$$\|a - aa'_n a\| \to 0 \text{ with } \|a'_n c_n - 1\| \to 0 \text{ and } \sup_n(\|a'_n\| + \|c_n\|) < \infty; \tag{11.1}$$

$$\|a - c_n p_n\| \to 0 \text{ with } \|c'_n c_n - 1\| + \|p_n^2 - p_n\| \to 0 \text{ and } \sup_n(\|c_n\| + \|c'_n\| + \|p_n\|) < \infty. \tag{11.2}$$

Proof. If (11.2) holds, argue

$$a - ac'_n a = (1 - ac'_n)(a - c_n p_n) + a(1 - c'_n c_n)p_n - (a - c_n p_n)p_n + c_n(p_n - p_n^2),$$

which tends to 0 and gives (11.1) with $a'_n = c'_n$; conversely, if (11.1) holds then we can satisfy (11.2) by taking $c'_n = a'_n$ and $p_n = a'_n a$. ■

We have not related, when $A = BL$ and $a = T$, the conditions of Theorem 11 to the spatial condition

$$T^{-1}(0) \cong Y/\mathrm{cl}(TX). \tag{11.3}$$

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