A NOTE ON MORPHISMS AND FIXED POINT THEORY

By Donal O'Regan
Department of Mathematics, National University of Ireland, Galway

[Received 10 September 1998. Read 24 May 1999. Published 30 December 1999.]

Abstract

In this paper we use the idea of a morphism to present some new fixed point theory for admissible maps in the sense of Gorniewicz and Granas. These fixed point results will then be used to establish new coincidence theory, analytic alternatives and minimax inequalities.

1. Introduction

Let X and Y be metric spaces. A continuous single valued map $p : Y \to X$ is called a Vietoris map [2] if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a proper map, i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

In [6] and [7] we obtained a variety of fixed point results for strongly admissible maps. Recall a multifunction $\phi : X \to C(Y)$ (here $C(Y)$ denotes the family of non-empty, compact subsets of Y) is strongly admissible, and we write $\phi \in Ad(X, Y)$, if $\phi : X \to C(Y)$ is u.s.c. (upper semicontinuous), and if there exists a metric space Z and two continuous maps $p : Z \to X$ and $q : Z \to Y$ such that:

(i) p is a Vietoris map;

(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

[We remark that $\phi : X \to C(Y)$ u.s.c. is redundant in the above definition since (ii) and [4, p. 179] automatically guarantee it].

This paper is concerned with obtaining new fixed point theory and coincidence theory for maps that are admissible in the sense of Gorniewicz and Granas.

Definition 1.1 [3]. A multifunction $\phi : X \to C(Y)$ is *admissible* in the sense of Gorniewicz and Granas, and we write $\phi \in AGG(X, Y)$, if there exists a morphism $\psi : X \to Y$ such that the map determined by ψ is a selector of ϕ (i.e. $\psi(x) \subset \phi(x)$ for each $x \in X$).

The definition of a morphism in this paper is that given in [3]. For completeness we give the definition here. Let $D(X, Y)$ be the set of all pairs $X \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} Y$, where p is a Vietoris map and q is continuous. We will denote every such diagram by (p, q). Given two diagrams (p, q) and (p', q'), where $X \overset{p'}{\leftarrow} Z' \overset{q'}{\rightarrow} Y$, we write $(p, q) \sim (p', q')$ if there are maps $f : Z \to Z'$ and $g : Z' \to Z$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is called a *morphism* from X to Y. A multivalued map $\psi : X \to C(Y)$ is said to be *determined by a morphism* $(X \overset{p}{\leftarrow} Z \overset{q}{\rightarrow} Y)$ provided $\psi(x) = q\,(p^{-1}(x))$

for every $x \in X$; the morphism that determines ψ is also denoted by ψ. It is immediate [4, p. 179] that if $\psi : X \to Y$ is a morphism then the multivalued map determined by ψ is u.s.c.

For the remainder of this section we recall some theory which will be needed in Section 2.

Theorem 1.1 [6; 7]. *Let E be a Fréchet space, Q a closed, convex subset of E and $0 \in Q$. Let $H \in Ad(Q, E)$ be a compact map, and assume*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x, \lambda) \\ \text{with } x \in \lambda H(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, ...\} \\ \text{with } \{\lambda_j H(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{cases} \tag{1.1}$$

is satisfied. Then H has a fixed point in Q.

Remark 1.1. If E is a Hilbert space then $H \in Ad(Q, E)$ compact can be replaced [6] by the less restrictive assumption $H \in Ad(Q, E)$ condensing with $H(Q)$ bounded. If we consider condensing maps in Fréchet spaces which are not Hilbert then we need an extra technical assumption (see [5]).

Remark 1.2. In this paper we consider Q closed with $0 \in Q$. If $0 \in int\,(Q)$ then we could use a non-linear alternative of Leray–Schauder type [6; 7] instead of Theorem 1.1 if we wish. We will not mention this further since the analogue of the results presented in this paper are immediate if we wish to use such a Leray–Schauder alternative.

Let Z and W be subsets of Hausdorff topological vector spaces E_1 and E_2, respectively, and let F be a multifunction. We say [1], $F \in DKT(Z, W)$ if W is convex (i.e. a convex subset of E_2) and if there exists a map $B : Z \to W$ with $co\,(B(x)) \subseteq F(x)$ for all $x \in Z$, $B(x) \neq \emptyset$ for each $x \in Z$ and the fibres $B^{-1}(y) = \{z : y \in B(z)\}$ are open (in Z) for each $y \in W$. Finally we recall a result [1] concerning DKT maps (as above Z and W are subsets of Hausdorff topological vector spaces E_1 and E_2).

Theorem 1.2 [1]. *Let Z be paracompact and W be convex. If $F \in DKT(Z, W)$ then there exists a continuous selection (single valued) $f : Z \to W$ of F.*

2. Theory

We begin this section by presenting a fixed point result for admissible maps in the sense of Gorniewicz and Granas.

Theorem 2.1. *Let E be a Fréchet space, Q a closed, convex subset of E and $0 \in Q$. Let $\phi \in AGG(Q,E)$ be a compact map, and assume*

$$\begin{cases} \text{if } \{(x_j,\lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda\,\phi(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,....\} \\ \text{with } \{\lambda_j\,\phi(x_j)\} \subseteq Q \text{ for each } j \ge j_0 \end{cases} \tag{2.1}$$

is satisfied. Then ϕ has a fixed point in Q.

PROOF. Now since $\phi \in AGG(Q,E)$ there exists a morphism $\psi : Q \to E$ such that the map determined by ψ is a selection of ϕ (i.e. $\psi(x) \subset \phi(x)$ for each $x \in Q$). Notice $\psi \in Ad(Q,E)$ is a compact map (since ϕ is a compact map). Also if $\{(x_j,\lambda_j)\}_1^\infty$ is a sequence in $\partial Q \times [0,1]$ converging to (x,λ) with $x \in \lambda\,\psi(x)$ and $0 \le \lambda < 1$, then $x \in \lambda\,\phi(x)$. Now (2.1) implies that there exists $j_0 \in \{1,2,....\}$ with $\{\lambda_j\,\phi(x_j)\} \subseteq Q$ for each $j \ge j_0$. Consequently $\{\lambda_j\,\psi(x_j)\} \subseteq Q$ for each $j \ge j_0$. Thus (1.1) holds with $H = \psi$. Apply Theorem 1.1 to deduce that ψ (and hence ϕ) has a fixed point in Q. ■

Remark 2.1. If E is a Hilbert space then we may replace $\phi \in AGG(Q,E)$ compact with $\phi \in AGG(Q,E)$ a condensing map with $\phi(Q)$ bounded. The result follows from Theorem 1.1 and Remark 1.1 once one notices that $\psi \in Ad(Q,E)$ is a condensing map with $\psi(Q)$ bounded.

Next we obtain three coincidence theorems for AGG maps.

Theorem 2.2. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$. Let $G \in AGG(Q,X)$ and $F \in DKT(X,E)$. Define the map $\Theta : Q \to E$ by*

$$\Theta(x) = F \circ G(x) \quad \text{for } x \in Q.$$

Assume the following conditions are satisfied:

$$\Theta : Q \to E \text{ is a compact map} \tag{2.2}$$

and

$$\begin{cases} \text{if } \{(x_j,\lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0,1] \text{ converging to } (x,\lambda) \\ \text{with } x \in \lambda\,\Theta(x) \text{ and } 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1,2,....\} \\ \text{with } \{\lambda_j\,\Theta(x_j)\} \subseteq Q \text{ for each } j \ge j_0. \end{cases} \tag{2.3}$$

Then G and F^{-1} have a coincidence. That is, there exists $(x_0,y_0) \in Q \times X$ with $y_0 \in G(x_0) \cap F^{-1}(x_0)$ (i.e. there exists $x_0 \in Q$ with $G(x_0) \cap F^{-1}(x_0) \neq \emptyset$).

PROOF. Let $f : X \to E$ be a continuous selection (guaranteed from Theorem 1.2) of F. Let

$$\phi = f \circ G.$$

Now [2] implies $\phi \in AGG(Q, E)$. Also ϕ is a compact map and Θ satisfying (2.3) implies that ϕ satisfies (2.1). Thus Theorem 2.1 guarantees that there exists $x_0 \in Q$ with $x_0 \in f\, G(x_0)$. Thus $x_0 = f\, y_0$ for some $y_0 \in G(x_0)$. Also $x_0 = f(y_0) \in F(y_0)$ so $y_0 \in G(x_0) \cap F^{-1}(x_0)$. ■

Remark 2.2. If E is a Hilbert space then (2.2) can be replaced by the less restrictive assumption

$$\Theta : Q \to E \text{ is a bounded, condensing map} \tag{2.4}$$

in Theorem 2.2.

Theorem 2.3. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$. Let $G \in DKT(Q, X)$ and $F \in AGG(X, E)$. Define the map $\Theta : Q \to E$ by*

$$\Theta(x) = F \circ G(x) \quad \text{for} \quad x \in Q,$$

and assume (2.2) *and* (2.3) *hold. Then G and F^{-1} have a coincidence.*

Proof. Let $g : Q \to X$ be a continuous selection of G and let $\phi = F \circ g$. ■

Theorem 2.4. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$, $0 \in X$. Let $G \in AGG(Q, X)$ and $F \in AGG(X, E)$. Define the map $J : Q \times X \to E \times X$ by*

$$J(x, y) = F(y) \times G(x) \quad \text{for} \quad (x, y) \in Q \times X.$$

Assume the following conditions are satisfied:

$$J : Q \times X \to E \times X \text{ is a compact map} \tag{2.5}$$

and

$$\begin{cases} \text{if } \{(z_j, \lambda_j)\}_1^\infty \text{ with } z_j = (x_j, y_j) \text{ is a sequence in } \partial(Q \times X) \times [0, 1] \\ \text{converging to } (z, \lambda), \text{ with } z = (x, y), \text{ with } (x, y) \in \lambda J(x, y) \text{ and} \\ 0 \le \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, ...\} \text{ with} \\ \{\lambda_j J(x_j, y_j)\} \subseteq Q \times X \text{ for each } j \ge j_0. \end{cases} \tag{2.6}$$

Then G and F^{-1} have a coincidence. That is, there exists $(x_0, y_0) \in Q \times X$ with $y_0 \in G(x_0) \cap F^{-1}(x_0)$.

Proof. Notice that [2] implies $J \in AGG(Q \times X, E \times X)$. Apply Theorem 2.1 (with J instead of ϕ) to deduce that there exists $(x_0, y_0) \in Q \times X$ with $(x_0, y_0) \in J(x_0, y_0)$. ■

Remark 2.3. If E is a Hilbert space then (2.5) can be replaced by the less restrictive assumption

$$J : Q \times X \to E \times X \text{ is a bounded, condensing map} \tag{2.7}$$

in Theorem 2.4.

We will now use our coincidence theory to establish some new analytic alternatives and minimax inequalities.

Theorem 2.5. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$. Let $F \in AGG(X, E)$ and suppose $f, g : Q \times X \to \mathbf{R}$ are two functions which satisfy*

$$f(x, y) \leq g(x, y) \quad \text{for all} \quad (x, y) \in Q \times X. \tag{2.8}$$

Fix $\alpha \in \mathbf{R}$ and let

$$G(x) = \{y \in X : g(x, y) > \alpha\} \quad \text{for} \quad x \in Q$$

and

$$B(x) = \{y \in X : f(x, y) > \alpha\} \quad \text{for} \quad x \in Q.$$

[*Notice that $B : Q \to X$ is a selection of G.*] *Assume the following condition is satisfied:*

$$\begin{cases} \text{if } B(x) \neq \emptyset \text{ for every } x \in Q \text{ then } G \in DKT(Q, X) \text{ (and let } g : Q \to X \\ \text{be a continuous selection of } G) \text{ with } \Lambda = F \circ g : Q \to 2^E \text{ a compact} \\ \text{map, and if } \{(x_j, \lambda_j)\}_1^\infty \text{ is a sequence in } \partial Q \times [0, 1] \text{ converging to } (x, \lambda) \\ \text{with } x \in \lambda \Lambda(x) \text{ and } 0 \leq \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, ...\} \\ \text{with } \{\lambda_j \Lambda(x_j)\} \subseteq Q \text{ for each } j \geq j_0. \end{cases} \tag{2.9}$$

Then either

(A1) *there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in X$,*
or
(A2) *there exists $(x_0, y_0) \in Q \times X$ with $x_0 \in F(y_0)$ and $g(x_0, y_0) > \alpha$*
occurs.

Remark 2.4. If E is a Hilbert space then Λ compact in (2.9) can be replaced by Λ condensing and bounded.

Remark 2.5. We can generalise Theorem 2.5 if we replace condition (2.8) with any condition that guarantees that there exists a map $B : Q \to X$ with $co(B(x)) \subseteq G(x)$ for $x \in Q$. Of course (A1) has to be appropriately adjusted (i.e. (A1) would become: there exists $z_0 \in Q$ with $B(z_0) \neq \emptyset$).

PROOF. There are two cases to consider.

Case (i). $B(x) \neq \emptyset$ for every $x \in Q$.

Thus $G \in DKT(Q, X)$ and Theorem 2.3 guarantees that there exists $(x_0, y_0) \in Q \times X$ with $x_0 \in F(y_0)$ and $y_0 \in G(x_0)$ (i.e. $x_0 \in F(y_0)$ and $g(x_0, y_0) > \alpha$), so (A2) occurs.

Case (ii). $B(x) \neq \emptyset$ for every $x \in Q$ does not hold.

Then there exists $z_0 \in Q$ with $B(z_0) = \emptyset$, i.e. there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in X$, so (A1) occurs. ■

We now use Theorem 2.4 to obtain another analytic alternative.

Theorem 2.6. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$, $0 \in X$. Let $g : Q \times X \to \mathbf{R}$ and $s : E \times X \to \mathbf{R}$ be two functions which satisfy*

$$g(x, y) \leq s(x, y) \quad \text{for all} \quad (x, y) \in Q \times X. \tag{2.10}$$

Fix $\alpha \in \mathbf{R}$ and let

$$G(x) = \{y \in X : \ g(x, y) > \alpha\} \quad \text{for} \quad x \in Q$$

and

$$F(y) = \{x \in E : \ s(x, y) < \alpha\} \quad \text{for} \quad y \in X.$$

Assume the following condition is satisfied:

$$\begin{cases} \text{if } G(x) \neq \emptyset \text{ for every } x \in Q \text{ then } G \in AGG(Q, X) \text{ and if } F(y) \neq \emptyset \text{ for} \\ \text{every } y \in X \text{ then } F \in AGG(X, E) \text{ with } J : Q \times X \to 2^{E \times X} \text{ (defined by} \\ J(x, y) = F(y) \times G(x)) \text{ compact and if } \{(z_j, \lambda_j)\}_1^\infty, \text{ with } z_j = (x_j, y_j), \\ \text{is a sequence in } \partial(Q \times X) \times [0, 1] \text{ converging to } (z, \lambda), \text{ with } z = (x, y), \\ \text{with } (x, y) \in \lambda J(x, y) \text{ and } 0 \leq \lambda < 1, \text{ then there exists } j_0 \in \{1, 2, \ldots\} \\ \text{with } \{\lambda_j J(x_j, y_j)\} \subseteq Q \times X \text{ for each } j \geq j_0. \end{cases} \tag{2.11}$$

Then either

(A1) *there exists $z_0 \in Q$ with $g(z_0, y) \leq \alpha$ for all $y \in X$,*
or
(A2) *there exists $w_0 \in X$ with $s(x, w_0) \geq \alpha$ for all $x \in E$*
occurs.

PROOF. There are three cases to consider.

Case (i). $G(x) \neq \emptyset$ for every $x \in Q$ and $F(y) \neq \emptyset$ for every $y \in X$.

Then Theorem 2.4 guarantees that there exists $(x_0, y_0) \in Q \times X$ with $y_0 \in G(x_0)$ and $x_0 \in F(y_0)$, i.e. $s(x_0, y_0) < \alpha < g(x_0, y_0)$. This contradicts (2.10).

Case (ii). $G(x) \neq \emptyset$ for every $x \in Q$ does not hold.

Then there exists $z_0 \in Q$ with $G(z_0) = \emptyset$, i.e. (A1) occurs.

Case (iii). $F(y) \neq \emptyset$ for every $y \in X$ does not hold.

Then there exists $w_0 \in X$ with $F(w_0) = \emptyset$, i.e. (A2) occurs. ■

Remark 2.6. If E is a Hilbert space, then J compact in (2.11) can be replaced by J condensing and bounded.

Next we obtain some new minimax inequalities (see [6; 7; 8; 9]).

Theorem 2.7. *Let E be a Fréchet space with X and Q closed, convex subsets of E and $0 \in Q$, $0 \in X$. Let $g : Q \times X \to \mathbf{R}$ and $s : E \times X \to \mathbf{R}$ and suppose* (2.10) *holds.*

For each $\alpha \in \mathbf{R}$ *let*

$$G_\alpha(x) = \{y \in X : g(x,y) > \alpha\} \quad \textit{for} \quad x \in Q$$

and

$$F_\alpha(y) = \{x \in E : s(x,y) < \alpha\} \quad \textit{for} \quad y \in X.$$

Assume the following condition is satisfied:

$$\begin{cases} \textit{for each } \alpha \in \mathbf{R}, \textit{ if } G_\alpha(x) \neq \emptyset \textit{ for every } x \in Q \textit{ then } G_\alpha \in AGG(Q,X) \\ \textit{and if } F_\alpha(y) \neq \emptyset \textit{ for every } y \in X \textit{ then } F_\alpha \in AGG(X,E) \textit{ with} \\ J_\alpha : Q \times X \to 2^{E \times X} \textit{ (defined by } J_\alpha(x,y) = F_\alpha(y) \times G_\alpha(x)) \textit{ compact and} \\ \textit{if } \{(z_j,\lambda_j)\}_1^\infty, \textit{with } z_j = (x_j,y_j), \textit{ is a sequence in } \partial(Q \times X) \times [0,1] \\ \textit{converging to } (z,\lambda), \textit{ with } z = (x,y), \textit{ with } (x,y) \in \lambda\, J_\alpha(x,y) \\ \textit{and } 0 \le \lambda < 1, \textit{ then there exists } j_0 \in \{1,2,....\} \textit{ with} \\ \{\lambda_j\, J_\alpha(x_j,y_j)\} \subseteq Q \times X \textit{ for each } j \ge j_0. \end{cases} \tag{2.12}$$

Then

$$a_0 \equiv \inf_{x \in Q} \sup_{y \in X} g(x,y) \le \sup_{y \in X} \inf_{x \in E} s(x,y) \equiv b_0. \tag{2.13}$$

PROOF. Let $b_0 < \infty$ and $a_0 > -\infty$. Suppose $a_0 > b_0$. Then there exists $\alpha \in \mathbf{R}$ with

$$b_0 < \alpha < a_0. \tag{2.14}$$

Apply Theorem 2.6. If (A1) occurs then there exists $z_0 \in Q$ with $g(z_0,y) \le \alpha$ for all $y \in X$. Thus $\sup_{y \in X} g(z_0,y) \le \alpha$ and so $a_0 \le \alpha$. This contradicts (2.14). If (A2) occurs then there exists $w_0 \in X$ with $s(x,w_0) \ge \alpha$ for all $x \in E$. Thus $\inf_{x \in E} s(x,w_0) \ge \alpha$ and so $b_0 \ge \alpha$. This contradicts (2.14). ■

Theorem 2.8. *Let* E *be a Fréchet space with* X *and* Q *closed, convex subsets of* E *and* $0 \in Q$. *Let* $F \in AGG(X,E)$ *and suppose* $f, g : Q \times X \to \mathbf{R}$ *satisfy* (2.8). *For each* $\alpha \in \mathbf{R}$, *let*

$$G_\alpha(x) = \{y \in X : g(x,y) > \alpha\} \quad \textit{for} \quad x \in Q$$

and

$$B_\alpha(x) = \{y \in X : f(x,y) > \alpha\} \quad \textit{for} \quad x \in Q.$$

Assume the following condition is satisfied:

$$\begin{cases} \textit{for each } \alpha \in \mathbf{R}, \textit{ if } B_\alpha(x) \neq \emptyset \textit{ for every } x \in Q \textit{ then } G_\alpha \in DKT(Q,X) \\ \textit{(and let } g_\alpha : Q \to X \textit{ be a continuous selection of } G_\alpha) \textit{ with } \Lambda_\alpha = F \circ g_\alpha : \\ Q \to 2^E \textit{ a compact map and if } \{(x_j,\lambda_j)\}_1^\infty \textit{ is a sequence in } \partial Q \times [0,1] \\ \textit{converging to } (x,\lambda) \textit{ with } x \in \lambda\, \Lambda_\alpha(x) \textit{ and } 0 \le \lambda < 1, \textit{ then there exists} \\ j_0 \in \{1,2,....\} \textit{ with } \{\lambda_j\, \Lambda_\alpha(x_j)\} \subseteq Q \textit{ for each } j \ge j_0. \end{cases} \tag{2.15}$$

Then

$$\inf_{x \in Q} \sup_{y \in X} f(x,y) \le \sup\{g(x,y) : x \in Q,\ y \in X,\ x \in F(y)\}. \tag{2.16}$$

PROOF. Let

$$\alpha = \sup\{g(x, y) : x \in Q,\ y \in X,\ x \in F(y)\}.$$

The case $\alpha = \infty$ is trivial so from now on we assume $\alpha < \infty$. Apply Theorem 2.5. Notice that (A2) cannot occur (see the definition of α). Then there exists $z_0 \in Q$ with $f(z_0, y) \leq \alpha$ for all $y \in X$. Thus $\sup_{y \in X} f(z_0, y) \leq \alpha$ and so (2.16) follows. ∎

REFERENCES

[1] X.P. Ding, W.K. Kim and K.K. Tan, A selection theorem and applications, *Bulletin of the Australian Mathematical Society* **46** (1992), 205–12.

[2] L. Gorniewicz, Homological methods in fixed point theory of multivalued mappings, *Dissertationes Mathematicae* **129** (1976), 1–71.

[3] L. Gorniewicz and A. Granas, Some general theorems in coincidence theory I, *Journal de Mathématiques Pures et Appliquées* **60** (1981), 361–73.

[4] L. Gorniewicz and M. Slosarski, Topological essentiality and differential inclusions, *Bulletin of the Australian Mathematical Society* **45** (1992), 177–93.

[5] D. O'Regan, Some fixed point theorems for concentrative mappings between locally convex linear topological spaces, *Nonlinear Analysis* **27** (1996), 1437–46.

[6] D. O'Regan, Coincidences for admissible and $\Phi^\star$ maps, *Journal of Mathematical Analysis and Applications* **220** (1998), 322–33.

[7] D. O'Regan, Continuation methods based on essential and 0–epi maps, *Acta Applicandae Mathematicae* **54** (1998), 319–30.

[8] S. Park, Generalized Leray–Schauder principles for compact admissible multifunctions, *Toplogical Methods in Nonlinear Analysis* **5** (1995), 271–7.

[9] S. Park, J.S. Bae and H.K. Kang, Geometric properties, minimax inequalities and fixed point theorems on convex spaces, *Proceedings of the American Mathematical Society* **121** (1994), 429–39.