A FINITENESS CONDITION ON NORMAL CLOSURES OF CYCLIC SUBGROUPS

By Howard Smith
Department of Mathematics, Bucknell University, Lewisburg, Pennsylvania

(Communicated by M.L. Newell, M.R.I.A.)

[Received 14 November 1996. Read 24 May 1999. Published 30 December 1999.]

Abstract

The main result here is that a locally soluble-by-finite group G in which the normal closure of every element has bounded (Prüfer) rank has its commutator subgroup of bounded rank.

A group G has finite (Prüfer) rank if there is a positive integer r such that every finitely generated subgroup of G can be generated by r elements; the least such r is then the rank of G. We are concerned in this paper with groups G with the property that for some r the normal closure $\langle x\rangle^G$ of every element x of G has rank at most r. Now a group G is a BFC group if there exists an integer r such that every element of G has at most r conjugates, and a well-known result of B.H. Neumann [5] states that a group G is BFC if and only if G' is finite. Further, the order of G' is bounded in terms of the BFC number r; this is apparent from Neumann's original proof, and other authors have produced explicit (and improved) bounds for $|G'|$ (see e.g. [7]). For a group G, if G' has finite rank then certainly $\langle x\rangle^G$ has bounded rank for all $x \in G$ and, motivated by the BFC result, we might enquire to what extent the converse is true. The following is proved.

Theorem. *Let G be a locally (soluble-by-finite) group and suppose that $\langle x\rangle^G$ has rank at most r for every x in G, where r is some fixed positive integer. Then there is an integer s, depending only on r, such that G' has rank at most s.*

No attempt is made here to obtain anything like a 'best bound'. The bound that we obtain could indeed be made explicit, but it hardly seems worth it as it is likely to be quite an extravagant one. Before turning to the proof of the theorem, let us note a generalisation that may be obtained without difficulty. In attempting to extend the result so as to cover a larger class of groups, one might adopt the lazy approach and look for a class $\mathfrak{X}$ that contains all locally (soluble-by-finite) groups and has the property that finitely generated $\mathfrak{X}$-groups of finite rank are soluble-by-finite. A class with this property, and one that contains, for example, the class of residually finite groups and several of the more (and less) familiar classes of generalised soluble groups, is defined by Chernikov in [2]. His class $\mathfrak{X}$ is defined in terms of the class of periodic locally graded groups—a group G is locally graded if every non-trivial finitely generated subgroup of G has a non-trivial finite image. Groups of finite rank that belong to $\mathfrak{X}$ are almost locally soluble [2], and, together with our theorem, this implies the following, where $\acute{P}$, $\grave{P}$, L, R denote the usual closure operations (see [6, chap. 1]).

Corollary A. *Let $\mathfrak{X}$ be the $\langle \acute{P}, \grave{P}, L, R\rangle$-closure of the class of periodic locally graded groups and let $G \in \mathfrak{X}$. If $\langle x\rangle^G$ has rank at most r for all $x \in G$ then G' has rank s bounded in terms of r only.*

With the notation as above, it is easy to show that every $\mathfrak{X}$-group H is locally graded, but it is not at all easy to determine whether the converse is true, even in the case where H has finite rank. Accordingly, our method of proof does not apply in the locally graded case, let alone in the general case. Whereas the class of locally graded groups is sometimes accessible, on account of the fact that it excludes finitely generated infinite simple groups, it should be noted that these latter groups present no problem in the present context, as any simple group with our property has rank r. It may indeed be true that every group G in which the normal closure of every element has rank at most r has its derived subgroup of finite rank; this is left as an open question.

The theorem is proved in several stages. First we deal with the finite nilpotent case. The following result is a little stronger than we need here but suggests a possible starting point for the examination of a closely related problem.

Lemma 1. *Let G be a finite nilpotent group and suppose that $\langle x\rangle^G$ is r-generated for all $x \in G$, where r is some fixed positive integer. Then G' is generated by at most $\frac{1}{2}r^2(r+1)$ elements.*

Proof. Since a finite nilpotent group is the direct product of its Sylow subgroups we may assume that G is a p-group for some prime p. Further, if $G' = X\gamma_3(G)(G')^p$ for some subgroup X, then, since $(G')^p \leq \Phi(G')$, we have $G' = X\gamma_3(G)$ and hence $G' = X^G\gamma_3(G)$. It follows that $G' = X^G$, and so it suffices to prove that there exists such an X that is generated by at most $\frac{1}{2}r(r+1)$ elements. For this we may assume that $\gamma_3(G)(G')^p = 1$ and hence that G' is central and of exponent p. Now for $x \in G$, we have $\langle x\rangle^G$ abelian and r-generated, so $[x, G]$ has order p^r at most and x has at most p^r conjugates in G. But then every element of G has at most p^r conjugates in G and hence G' has order at most $p^{\frac{1}{2}r(r+1)}$, by [1]. The result follows. ∎

(I am grateful to E.I. Khoukhro for pointing out to me the above reduction to the case G' central and of exponent p.) If $\langle x\rangle^G$ has rank r for all $x \in G$, then the same bound $\frac{1}{2}r(r+1)$ is valid in the 'G' central' case and thus we have the following corollary as an immediate consequence.

Corollary 1. *Let G be a finite nilpotent group and suppose that $\langle x\rangle^G$ has rank at most r for all $x \in G$. Then G' has rank at most $\frac{1}{2}r^2(r+1)$.*

Lemma 2. *Let G be a finite soluble group with $\langle x\rangle^G$ of rank at most r for all $x \in G$. Then there is an integer d depending only on r such that $G^{(d)}$ has rank at most $\frac{1}{2}r^2(r+1)$.*

PROOF. Let F be the Fitting radical of G. If H is an arbitrary chief factor of G then H is abelian of prime exponent p and, being the normal closure of a single element, H has rank at most r. This $G/C_G(H)$ embeds as a soluble subgroup of $GL(r,p)$ and thus has r-bounded derived length c, say, by a result of Zassenhaus [6, theorem 3.23]. So $G^{(c)}$ centralises every chief factor of G and is therefore contained in F. By Corollary 1, F' has rank at most $\frac{1}{2}r^2(r+1)$ and therefore so does $G^{(c+1)}$. The lemma is proved. ■

We are now able to establish the result for finite groups.

Proposition 1. *Let G be a finite group and suppose that $\langle x\rangle^G$ has rank at most r for all $x \in G$, where r is some positive integer. Then G' has rank bounded in terms of r only.*

PROOF. Let $G^{(\omega)}$ denote the soluble residual of G; then $G^{(d)}/G^{(\omega)}$ has bounded rank for some $d = d(r)$, by Lemma 2. Further, $G^{(\omega)}$ is of course perfect and is therefore the normal closure of a single element. (This is well known and easy to prove by considering the intersection N of all maximal normal subgroups of G.) It follows that $G^{(\omega)}$ has rank at most r and, by factoring if necessary, we may therefore assume that $G^{(d)} = 1$. By induction on d we may suppose that G'' has rank bounded in terms of $d-1$ and r and hence of r only. Thus we may suppose that G is metabelian. Let K be the nilpotent residual of G. By Corollary 1 G'/K has bounded rank. If K/K^p has rank at most b for all primes p, then K too has rank at most b, and so we may assume that K has prime exponent p. Then G has a normal Sylow p-subgroup P and, by the Schur–Zassenhaus Theorem, we have $G = P]H$ for some p'-subgroup H which in this case is nilpotent. Again by Corollary 1, P' and H' have r-bounded rank; it follows that $(H')^G$ and P' are normal subgroups of bounded rank and, factoring once more, we may suppose that P and H are abelian. Now $K = [K,G] = [K,H]$ and so K is the nilpotent residual of KH. Since we need only bound the rank of K we shall assume (finally) that $G = K]H$, so that $G' = K$.

If k is an arbitrary element of K then $A =: \langle k\rangle^G$ has rank at most r. Thus $H/C_H(A)$ embeds in $GL(r,p)$ and is an abelian p'-group, and therefore has rank at most α for some α depending only on r (this follows easily from proposition II.1 of [4]). Also, $K/C_K(h)$ is r-generated for all $h \in H$ since $[h,K]$ has rank at most r. Now choose an element g of G with the maximum number of conjugates, write $g = hk$ where $h \in H$, $k \in K$ and note that $C_G(g) \supseteq C_H(k)C_K(h)$. By the above we have $K = C_K(h)L, H = C_H(k)M$, where $L = \langle k_1,\dots,k_r\rangle, M = \langle h_1,\dots,h_\alpha\rangle$ for some elements h_i, k_i. Since $C_K(h) \lhd G$ we have $G = HK = C_H(k)MC_K(h)L = C_H(k)C_K(h)ML$, and so we may choose a right transversal U for $C_G(g)$ in G with $U \subseteq ML$.

Now let $X = C_K(M), Y = C_H(L)$ and write $C = C_G(M) \cap C_G(L)$. Then $C \geq XY$ and, again by the above, we have each of $K/X, H/Y$ $r\alpha$-generated, and hence there exist elements $g_1,\dots,g_\beta$ of G such that $G = \langle C, g_1,\dots,g_\beta\rangle$, where $\beta = 2r\alpha$. Let $N = \langle g, g_1,\dots,g_\beta\rangle^G$; then $G = CN$. Now let $x \in C$; since the set of all conjugates of g in G is precisely $\{g^u : u \in U\}$, and since $(xg)^u = xg^u$ for each $u \in U$, we have by the choice of g that every conjugate in G of xg is of the form $(xg)^u$, for some $u \in U$. Thus, for $y \in C$, we have $[x, y] = x^{-1}x^y = g^u(g^y)^{-1}$ for some $u \in U$, and hence $[x, y] \in N$. Then $G' \leq C'N \leq N$, and the result follows easily. ■

We remark that the last stage of the above proof imitates the corresponding argument in [5]. It is evident that the property that we are discussing is a local one, in the sense that a group G has its derived subgroup of rank at most s if and only if every finitely generated subgroup of G has this property. In particular, we have the following consequence of Proposition 1.

Corollary 2. *Let G be a locally finite group and suppose that $\langle x\rangle^G$ has rank at most r for all $x \in G$. Then G' has rank bounded in terms of r only.*

One requirement for the proof of the locally soluble case is the following (easy) generalisation of a result of Roseblade (see [6, lemma 7.45]).

Lemma 3. *Let r be a positive integer and let G be a group that is the product of abelian normal subgroups of rank r. Then G is soluble of derived length bounded in terms of r only.*

Proof. Let F be an arbitrary finitely generated subgroup of G, so that $F \leq A_1 \dots A_k = B$, say, where each A_i is normal and abelian of rank at most r. Let T be the torsion subgroup of B; then B/T is torsion-free nilpotent and so $A_iT/T \leq Z_r(B/T)$ for each i [6, lemma 6.37], and B/T is nilpotent of class r (at most). It suffices to show that $F \cap T$ has bounded derived length. Since $F \cap T$ is finite (as B is nilpotent), we have $F \leq C_1 \dots C_k$, where each C_i is finite and characteristic in A_i. Thus $C_1 \dots C_k$ is a product of normal r-generator abelian subgroups and is therefore of bounded derived length by [6, lemma 7.45]. The result follows. ■

Proposition 2. *Let G be a locally soluble group and suppose that $\langle x\rangle^G$ has rank at most r for all $x \in G$. Then G' has finite rank bounded in terms of r only.*

Proof. Let $x \in G$. By [6, lemma 10.39] there is an integer c depending only on r such that $(\langle x\rangle^G)^{(c)}$ is periodic and hypercentral, and hence contained in the torsion subgroup T of the Hirsch–Plotkin radical H of G. By Corollary 2, T' has r-bounded rank and, factoring, we may therefore assume that $\langle x\rangle^G$ has derived length at most d for all $x \in G$, where $d = c + 1$. Lemma 3 and an easy induction on d now show that G has derived length bounded in terms of d and r and hence in terms of r only. Again by induction, G'' has $(d-1, r)$-bounded rank and hence r-bounded rank, so we may assume that G is metabelian. By the remark preceding Corollary 2, we may further assume that G is finitely generated and therefore of finite rank and hence a minimax group [6, theorem 10.38]. Let $A = G'$ and let n be an arbitrary positive integer. Then G/A^n is polycyclic and hence residually finite and so $\exists\, N \lhd G, G/N$ finite such that $A \cap N = A^n$. By Proposition 1, AN/N has r-bounded rank and hence A/A^n has r-bounded rank m, say. Now let T be the torsion subgroup of A; since G is residually finite [3] and minimax, T is finite and so $A^k = D$, say, is torsion-free for some $k > 0$. Further, there is a finitely generated subgroup B of D such that D/B is a divisible π-group for some finite (possibly empty) set π of primes. Choose a prime $q \notin \pi$; then $D = BD^q$ and hence $D/D^q \cong B/B \cap D^q = B/B^q$. Since $A/D^q = A/A^{kq}$ has rank at most m, so has B/B^q. It follows that B has rank at most m and therefore so has D. Finally, A/D also has rank at most m and so A has rank at most $2m$ and the proof is complete. ■

PROOF OF THE THEOREM. Let G be as stated. We may assume that G is finitely generated and hence that there is a normal soluble subgroup H of G such that G/H is finite. By Proposition 1 $G'H/H$ has r-bounded rank and so $G'H = FH$ for some subgroup F that is generated by boundedly many elements. Since F^G has bounded rank we may factor and thus assume that G is soluble. The result now follows by Proposition 2. ■

We have already remarked that, with $\mathfrak{X}$ as defined in Corollary A, a group of finite rank that belongs to $\mathfrak{X}$ is almost locally soluble. Suppose that $G \in \mathfrak{X}$ and that G' has finite rank; then G' has a characteristic locally soluble subgroup H of finite index, and G/H is soluble-by-finite. Now G is locally of finite rank, and it follows from [6, theorem 10.38] that G is almost locally soluble. The following generalisation of Chernikov's result [2] is therefore an easy consequence of Corollary A.

Corollary B. *Let G belong to the $\langle \acute{P}, \grave{P}, L, R\rangle$-closure of the class of periodic locally graded groups, and suppose that $\langle x\rangle^G$ has rank at most r for all $x \in G$, where r is some positive integer. Then G is almost locally soluble.*

REFERENCES

[1] I.M. Bride, Second nilpotent BFC-groups, *Journal of the Australian Mathematical Society* **11** (1970), 9–18.

[2] N.S. Chernikov, A theorem on groups of finite special rank, *Ukrainskii Matematicheskii Zhurnal* **42** (1990), 962–70; transl. *Ukrainian Mathematical Journal* **42** (1990), 855–61.

[3] P. Hall, On the finiteness of certain soluble groups, *Proceedings of the London Mathematical Society* **9** (1959), 595–622.

[4] C.R. Leedham-Green and W. Plesken, Some remarks on Sylow subgroups of general linear groups, *Mathematische Zeitschrift* **191** (1986), 529–35.

[5] B.H. Neumann, Groups covered by permutable subsets, *Journal of the London Mathematical Society* **29** (1954), 236–48.

[6] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin–Heidelberg–New York, 1972.

[7] J. Wiegold, Groups with boundedly finite classes of conjugate elements, *Proceedings of the Royal Society of London Series* A **238** (1957), 389–401.