A NOTE ON THE ENVELOPING SEMIGROUP OF A FLOW

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Abstract

In her paper 'Enveloping semigroups for flows' (*Proceedings of the Royal Irish Academy* **95**A (1995), 179–91) A. Köhler constructed a surjective semigroup homomorphism Φ mapping the associated enveloping operator semigroup onto the enveloping Ellis semigroup of a flow. We show that Φ is not injective in general by giving a counterexample based on a flow on the unit interval.

Introduction

In [5] and [6] Angela Köhler has established various relations between the dynamical behaviour of a flow, its enveloping Ellis semigroup (see [3] or Definition 1.2 below) and a semigroup of operators on a Banach space associated with the given flow. In particular, she showed the existence of a continuous, surjective semigroup homomorphism Φ from the semigroup of operators mentioned above onto the enveloping Ellis semigroup. In several special cases she could prove the injectivity of Φ. The general case, however, was formulated as an open problem. In this note we show that Φ is not injective in general.

This result gives a partial answer to a problem formulated by J. S. Pym in section 5 of his article [8]. There he asks for relations between the enveloping semigroups of the flows (X, H) and $(P(X), H)$ (see Definition 1.1 below), where $P(X)$ is the space of probability measures on X with its usual weak* topology.

The counterexample, which will be discussed in the second part of this note, is in agreement with the original definition of a flow in Köhler's doctoral thesis [5] (compare Definition 1.1 below). However, it is not a monothetic flow like those considered in [6].

1. Preliminaries

The notation, the definitions and the results in this section are taken from [5] and are similar to those given in [6].

Definition 1.1. A *flow* is a pair (X, H), where X is a compact Hausdorff space and H is a semigroup of continuous maps from X onto itself.

Definition 1.2. Let (X, H) be a flow. Then the *enveloping Ellis semigroup* of (X, H) is the closure of H in the topology of pointwise convergence in X^X. It will be denoted by $\Sigma(X, H)$.

The enveloping Ellis semigroup is frequently used in topological dynamics and reflects various properties of a flow (X, H) (see, e.g., [1], [3], [4] and [7]). It was

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introduced in [3] by R. Ellis, who showed that it is a compact, right topological semigroup with respect to the topology of pointwise convergence.

Definition 1.3. Let (X, H) be a flow. The *associated operator semigroup* of (X, H) is the semigroup $\{T_\phi \mid \phi \in H\}$ of bounded, linear operators on the Banach space $C(X)$ of all continuous functions on X defined by $T_\phi f := f \circ \phi$.

More generally, we look at bounded semigroups of operators and introduce the concept of the enveloping operator semigroup.

Definition 1.4. The *enveloping operator semigroup* $\mathscr{E}(S)$ of a bounded semigroup $S \subset \mathscr{L}(E)$, E a Banach space, is the closure of the adjoint semigroup $S' := \{T' \mid T \in S\}$ in the weak* operator topology of $\mathscr{L}(E')$.

It follows from the Banach–Alaoglu Theorem that $\mathscr{E}(S)$ is compact.

Proposition 1.5. *Let E be a Banach space. For every bounded semigroup $S \subset \mathscr{L}(E)$ the enveloping operator semigroup $\mathscr{E}(S)$ is a compact, right topological semigroup.*

The following theorem was proved in [5]. We give a sketch of its proof since the construction of the homomorphism Φ, and not only its existence, is important in the next section (compare [6, theorem 1.2]).

Theorem 1.6. *Let (X, H) be a flow and S the associated operator semigroup on the space $C(X)$. Then there is a continuous, surjective semigroup homomorphism Φ from $\mathscr{E}(S)$ onto the enveloping Ellis semigroup $\Sigma(X, H)$ mapping the adjoints of the operators in S onto the elements of H, i.e. $\Phi(T'_\phi) = \phi$ for any $\phi \in H$.*

Sketch of proof. For any $R \in \mathscr{E}(S)$ there is a net $(T'_{\phi_\alpha})_\alpha \subseteq S'$ converging to R in the weak* operator topology. For $x \in X$, $f \in C(X)$ we have

$$\lim_\alpha f(\phi_\alpha(x)) = \lim_\alpha \langle f \circ \phi_\alpha, \delta_x \rangle = \lim_\alpha \langle f, T'_{\phi_\alpha} \delta_x \rangle = \langle f, R\delta_x \rangle,$$

so the net $(\phi_\alpha)_\alpha$ converges pointwise to a map $\psi_R \in \Sigma(X, H)$. We define $\Phi : \mathscr{E}(S) \to \Sigma(X, H) : R \mapsto \psi_R$. The map Φ is well defined since ψ_R is uniquely determined by the equations $f(\psi_R(x)) = \langle f, R\delta_x \rangle$ for all $f \in C(X)$, $x \in X$. It is easily seen that Φ is a homomorphism. The continuity and surjectivity of Φ are proved in a similar way. ■

2. The relation between $\mathscr{E}(S)$ and $\Sigma(X, H)$

In this section we prove that the homomorphism Φ in Theorem 1.6 is not injective in general by giving a counterexample based on a flow on the unit interval. This is of special interest as flows on the unit interval have many properties not shared by arbitrary flows. As an example, we mention the striking theorem of Šarkovskii on the periods of periodic orbits (see [9] or [2, theorem 1]).

On $X := [0,1]$ we define for any $k \in \mathbb{N}$ and any k-tuple $(x_1, \ldots, x_k) \in X^k$ with $0 < x_1 < \ldots < x_k < 1$ a continuous, piecewise linear function $\phi_{(x_1,\ldots,x_k)} : X \to X$ by

$$\phi_{(x_1,\ldots,x_k)}(t) := \frac{2}{x_{i+1} - x_i}(t - x_i)$$

for $t \in [x_i, \frac{1}{2}(x_i + x_{i+1})]$ and

$$\phi_{(x_1,\ldots,x_k)}(t) := \frac{2}{x_i - x_{i+1}}(t - x_{i+1})$$

for $t \in [\frac{1}{2}(x_i + x_{i+1}), x_{i+1}]$, where $i = 0, \ldots, k$, $x_0 := 0$ and $x_{k+1} := 1$. The graph of $\phi_{(x_1,\ldots,x_k)}$ consists of $k+1$ isosceles triangles of height 1, and the zeros of $\phi_{(x_1,\ldots,x_k)}$ are precisely $x_0, x_1, \ldots, x_k, x_{k+1}$. We set

$$I := \{(x_1, \ldots, x_k) \in X^k \mid k \in \mathbb{N}, 0 < x_1 < \ldots < x_k < 1\}.$$

Under the composition of maps the set

$$H_I := \{\phi_{(x_1,\ldots,x_k)} \mid (x_1, \ldots, x_k) \in I\}$$

becomes a semigroup. Let $\phi_0 : X \to X$ denote the map which is identically 0. Then $H := H_I \cup \{\phi_0\}$ is a semigroup of continuous selfmaps of X.

Proposition 2.1. *Let (X, H) be the flow defined above. Then the map Φ defined in Theorem 1.6 is not injective.*

PROOF. Assume Φ to be injective. Then Φ is even a homeomorphism from $\mathscr{E}(S)$ onto $\Sigma(X, H)$ since $\mathscr{E}(S)$ is compact. We introduce an order on I by defining

$$(x_1, \ldots, x_k) \leq (y_1, \ldots, y_l) \overset{\text{def}}{\Longleftrightarrow} \{x_1, \ldots, x_k\} \subseteq \{y_1, \ldots, y_l\}.$$

Relative to this order the set $H_I = (\phi_\alpha)_{\alpha \in I}$ becomes a net. Choose now $x \in (0,1)$ arbitrarily. We then have $\phi_{(y_1,\ldots,y_k)}(x) = 0$ for all $(y_1, \ldots, y_k) \in I$ with $(y_1, \ldots, y_k) \geq (x)$, i.e. $x \in \{y_1, \ldots, y_k\}$. So the net $(\phi_\alpha)_{\alpha \in I}$ converges pointwise to $\phi_0 \in H$. Since Φ^{-1} is continuous, the limit $\lim_\alpha \Phi^{-1}(\phi_\alpha) = \Phi^{-1}(\phi_0)$ exists in the weak* operator topology. In particular, we obtain

$$\lim_\alpha \langle \mathrm{id}, \Phi^{-1}(\phi_\alpha)\lambda \rangle = \langle \mathrm{id}, \Phi^{-1}(\phi_0)\lambda \rangle,$$

where $\mathrm{id} : X \to \mathbb{C} : x \mapsto x$, and $\lambda \in C(X)'$ denotes the Lebesgue measure. However, we have $\int_X \phi_\alpha d\lambda = \frac{1}{2}$ and therefore

$$\langle \mathrm{id}, \Phi^{-1}(\phi_\alpha)\lambda \rangle = \langle \mathrm{id}, T'_{\phi_\alpha}\lambda \rangle = \langle T_{\phi_\alpha}\mathrm{id}, \lambda \rangle = \langle \mathrm{id} \circ \phi_\alpha, \lambda \rangle = \tfrac{1}{2}$$

for every $\alpha \in I$. This yields a contradiction because $\langle \mathrm{id}, \Phi^{-1}(\phi_0)\lambda \rangle = \langle \mathrm{id} \circ \phi_0, \lambda \rangle = 0$. ■

Remark. Using the compactness of X^k for each $k \in \mathbb{N}$, it is easy to see that there exists a finite subset $\tilde{H}_k \subset \{\phi_{(x_1,x_2,\dots,x_k)} \in H\}$, such that for each $(x_1,\dots,x_k) \in X^k$ there is some $\phi \in \tilde{H}_k$ with $\phi(x_i) < \frac{1}{k}$ for $i = 1,\dots,k$. For each $(x_1,\dots,x_k) \in I$, we choose such a map in $\tilde{H}_k$ and denote it by $\psi_{(x_1,\dots,x_k)}$. By construction, the net $(\psi_\alpha)_{\alpha\in I}$ converges pointwise to $\phi_0 \in H$. Let $\tilde{H}$ denote the countable subsemigroup of H generated by the subset $\bigcup_{k\in\mathbb{N}} \tilde{H}_k \cup \{\phi_0\} \subseteq H$. Then the proof of Proposition 2.1 shows that the homomorphism Φ is not injective for the flow $(X, \tilde{H})$. Hence Φ need not be injective for flows originating from a countable semigroup of continuous maps.

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