SPLITTABILITY FOR FINITE PARTIALLY-ORDERED SETS

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[Received 22 July 1998. Read 7 October 1999. Published 30 December 1999.]

Abstract

Arhangel'skii and his co-workers have defined and developed notions of splittability or cleavability for topological spaces. We apply the analogous idea in partially-ordered sets (posets) and identify those posets that are splittable over a finite chain and those posets over which a given chain (finite or infinite) is splittable.

1. Introduction

A *partial order* on a non-empty set E is a binary relation on E (usually denoted by $\leq$) that is reflexive, transitive and antisymmetric. A mapping f from one partially-ordered set (poset) E to another E' is called *order-preserving* or *increasing* if $x \leq y$ in E implies $f(x) \leq f(y)$ in E'. The basic form of Arhangel'skii's definition [1] in topology is that a (topological) space X is splittable over a space X' if, to every subset A of X, there corresponds a continuous mapping from X into X' under which the images of A and of $X \backslash A$ are disjoint; by direct analogy, let us say that the poset E is *splittable* over the poset E' if, to every subset A of E, there corresponds an order-preserving mapping from E into E' under which the images of A and of $E \backslash A$ are disjoint. The fundamental problem is to determine convenient, necessary and sufficient conditions for this to happen and, as in topology, it appears to be a difficult problem. The purpose of this article is to solve it in the special cases where

(i) E is a chain or
(ii) E' is a finite chain.

(A poset is called a *chain* when, for every two of its elements x and y, either $x \leq y$ or $y \leq x$ is true.) The discussion falls naturally into two sections, according to whether it is the domain or the codomain that is taken to be a chain. We shall discuss the codomain case first.

2. Splittability over a chain

For each $n \geq 1$ let us denote by C_n the archetypal n-point chain $\{1, 2, 3, \ldots, n\}$ under its natural order. It is easy to see that C_{n+1} is not splittable over C_n since, choosing A to be the subset $\{1, 3, 5, \ldots\}$ of C_{n+1}, any order-preserving map f from C_{n+1} to C_n for which $f(A)$ was disjoint from $f(C_{n+1} \backslash A)$ would need to be one-to-one, which is clearly impossible. Also, if $C_n \oplus C_n$ denotes the direct sum of two copies of C_n (that is, the set $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ ordered by taking the natural orders within $\{1, 2, \ldots, n\}$ and $\{1', 2', \ldots, n'\}$) then $C_n \oplus C_n$ is not splittable over C_n: for, letting B denote the subset $\{1, 3, 5, \ldots, 2', 4', 6', \ldots\}$ of $C_n \oplus C_n$, an order-preserving map from $C_n \oplus C_n$ into C_n that 'kept B and its complement apart' would have to be one-to-one on *each* summand C_n, yielding again a contradiction. We readily deduce the following proposition.

Proposition 1. *If E is splittable over C_n then*
(i) *E contains no chain of $n+1$ (or more) elements, and*
(ii) *E contains no two disjoint n-element chains.*

The principal result of this section is that the converse of Proposition 1 is also valid: conditions (i) and (ii) are both necessary and sufficient for E to be splittable over C_n. For the demonstration, we shall regard n as fixed, and we shall call a poset E *restricted* if it satisfies conditions (i) and (ii).

Lemma 2. *Suppose that $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$ are two n-element chains in a restricted poset. Whenever $x_i = y_j$ we have $i = j$.*

Proof. If, for example, $i < j$, we obtain $\{y_1, y_2, \ldots, y_j, x_{i+1}, x_{i+2}, \ldots, x_n\}$ as a chain of more than n elements. ∎

Lemma 3. *Suppose that X, Y and Z are three n-element chains within a restricted poset. There is a point common to all three of them.*

Proof. Let us suppose otherwise. Enumerating the three chains as $X = \{x_1 < x_2 < \ldots < x_n\}$, $Y = \{y_1 < y_2 < \ldots < y_n\}$ and $Z = \{z_1 < z_2 < \ldots < z_n\}$, we note from Lemma 2 that $x_i = y_j$ or $x_i = z_j$ or $y_i = z_j$ are only possible when $i = j$. We describe how to construct, upwards from their least elements, two disjoint n-element chains S and T.

Without loss of generality, the first coincidence of x_i and y_i occurs 'before' the first coincidence of x_j and z_j: that is, we find the least value of i for which $x_i = y_i$, and we have that for each $j \leq i, x_j \neq z_j$. Let us define $s_k = x_k$ and $t_k = z_k$ for $1 \leq k \leq i$, and remark that the chains $\{s_1 < s_2 < \ldots < s_i\}$ and $\{t_1 < t_2 < \ldots < t_i\}$ are at present disjoint. Now seek the least value of $m > i$ for which $z_m \in \{x_m, y_m\}$. If $z_m = x_m$, define $s_k = y_k$ and $t_k = z_k$ for $i+1 \leq k \leq m$; on the other hand, if $z_m = y_m$, let $s_k = x_k$ and $t_k = z_k$ for $i+1 \leq k \leq m$. The chains $\{s_1 < s_2 < \ldots < s_m\}$ and $\{t_1 < t_2 < \ldots < t_m\}$ have remained disjoint. Since there are, by supposition, no three-way coincidences of the form $x_p = y_p = z_p$, this process of allowing the s- or t-sequence to 'change tracks to avoid the next threatened collision' will continue until $S = \{s_1 < s_2 < \ldots < s_n\}$ and $T = \{t_1 < t_2 < \ldots < t_n\}$ are completed as disjoint n-point chains, contrary to the 'restricted' nature of E. (A fuller account of this and of subsequent arguments can be found in [3].) ∎

The above is merely the first step in an induction process, and is included here only to make the following general step more accessible.

Lemma 4. *Suppose that $X^1, X^2, \ldots, X^m$ are finitely many n-point chains in a restricted poset, where $m \geq 2$. There is a point common to all m of them.*

PROOF. For $m = 2$ and $m = 3$ this is known from the definition or from Lemma 3. Proceeding inductively, we shall assume it to be valid for $m = k$ (where $k \geq 3$) and for every positive integer n, and seek an intersection for $k + 1$ chains $X^i = \{x_1^i < x_2^i < \ldots < x_n^i\}$ where $1 \leq i \leq k + 1$.

Supposing that there is no $(k + 1)$-fold coincidence of the form $x_p^1 = x_p^2 = \ldots = x_p^{k+1}$, we consider those values of p for which k-many of the terms x_p^i are equal, and we shall say that a *k-chain node occurs at p* for such an index p. Assuming that the labelling has been arranged so that the first k-chain node occurs at $p(0)$, and involves $X^1, X^2, \ldots, X^k$ but excludes X^{k+1}, we note that the subset $C = \{x_j^i : 2 \leq i \leq k + 1,\ 1 \leq j \leq p(0)\}$ consists of k chains that do not have a common point, but contains no chain of $p(0) + 1$ elements. By the inductive assumption, C contains two disjoint $p(0)$-element chains $s_1 < s_2 < \ldots < s_{p(0)}$ and $t_1 < t_2 < \ldots < t_{p(0)}$, and it is clear that $s_{p(0)} = x_{p_o}^1 (= x_{p(0)}^2 = \ldots = x_{p(0)}^k)$ and $t_{p(0)} = x_{p(0)}^{k+1}$ (or vice versa).

Now find the index $p(1)$ at which the next k-chain node occurs. By applying the same argument to the set of points x_j^i for $p(0) \leq j \leq p(1)$ and all values of *i excepting* an $i(0)$ chosen so that $X^{i(0)}$ is involved in the k-chain nodes at $p(0)$ and at $p(1)$, we find within this set two disjoint $(p(1) - p(0) + 1)$-element chains that have $s_{p(0)}$ and $t_{p(0)}$ as their least elements. Attaching these onto the chains that were previously identified gives two disjoint $p(1)$-element chains, and establishes a pattern that will generate disjoint n-element chains once again, leading to the desired contradiction. ■

Next, we show that we can remove the assumption of finiteness of the family of chains from Lemma 4.

Lemma 5. *Any family $\{X^\alpha : \alpha < \tau\}$ of n-element chains in a restricted poset must have a common point.*

PROOF. Suppose that there is no point common to every chain

$$X^\alpha = \{x_1^\alpha < x_2^\alpha < \ldots < x_n^\alpha\}.$$

For each i in the range from 1 to n we can therefore choose $\beta(i) < \tau$ such that $x_i^i \notin X^{\beta(i)}$. The chains

$$X^1, X^2, \ldots, X^n, X^{\beta(1)}, X^{\beta(2)}, \ldots, X^{\beta(n)}$$

must, by Lemma 4, share a common point q. Lemma 2 reminds us that there is therefore a single value of the index j such that

$$q = x_j^1 = x_j^2 = \ldots = x_j^n = x_j^{\beta(1)} = x_j^{\beta(2)} = \ldots = x_j^{\beta(n)}$$

and the observation

$$x_j^j = x_j^{\beta(j)} \in X^{\beta(j)}$$

reveals the contradiction. ■

Using Lemma 5, we shall now demonstrate how to 'split' any restricted poset over C_n. Recall first that, in a poset whose chains are uniformly finite, the *height* $ht(x)$ of a point x is defined as the largest number of elements in a chain having x as its maximum. This may be envisaged as stratifying the points in a natural way, where stratum 1 consists of the minimal elements and, thereafter, x is allocated to stratum k precisely when $k-1$ is the highest stratum in which there exists y with $y < x$. Now if E is restricted and, in addition, all of its maximal chains have the same length n, every maximal chain must include one point out of each stratum. In the notation of Lemma 5, then, x_i^α is the point in common between chain X^α and stratum i. Furthermore, to confirm that a mapping f defined on such a poset is order-preserving, it is enough to check that whenever x and y lie in *adjacent* strata and satisfy $x < y$, then $f(x) < f(y)$.

Lemma 6. *Let E be restricted. Given a subset A of E, we can construct an order-preserving mapping from E into C_n such that $f(A)$ is disjoint from $f(E\backslash A)$.*

Proof. We first consider the case in which E is a union of n-element chains. By Lemma 5 (and in the notation which it employed) there is a point x and an index $i(0)$ between 1 and n inclusive such that x is the $i(0)$th element $x_{i(0)}^\alpha$ of every chain X^α. There is no loss of generality in assuming that x belongs to A. We shall also suppose that $i(0)$ is odd—the argument for even values of $i(0)$ being similar.

The formula

$$f(x_i^\alpha) = \begin{cases} i & i \text{ odd}, \quad i \le i(0), \quad x_i^\alpha \in A \\ i+1 & i \text{ odd}, \quad i < i(0), \quad x_i^\alpha \notin A \\ i+1 & i \text{ even}, \quad i < i(0), \quad x_i^\alpha \in A \\ i & i \text{ even}, \quad i < i(0), \quad x_i^\alpha \notin A \\ i & i \text{ odd}, \quad i > i(0), \quad x_i^\alpha \in A \\ i-1 & i \text{ odd}, \quad i > i(0), \quad x_i^\alpha \notin A \\ i-1 & i \text{ even}, \quad i > i(0), \quad x_i^\alpha \in A \\ i & i \text{ even}, \quad i > i(0), \quad x_i^\alpha \notin A \end{cases}$$

will be found to satisfy the requirements. It maps each point of A to an *odd* member of C_n, namely its height *or* the height one step closer to the critical i_oth stratum, and it maps each point of the complement of A to an *even* member of C_n in a similar fashion. The order-preserving character of f can be confirmed by considering pairs of points in adjacent strata, as remarked above.

Now let us examine the general case, where some of the maximal chains in E have fewer than n elements. We partition E into (at most) n subsets $Z, S_1, S_2, \ldots, S_{n-1}$ where Z comprises all points that belong to an n-element chain in E and (for $1 \le k \le n-1$) S_k is the subset of stratum k that lies outside Z. The previous paragraph shows how to determine an order-preserving function f from Z into C_n that maps points of $Z \cap A$ to odd integers and points of $Z\backslash A$ to even integers. The corresponding task in the complement of Z is easier since, here, there are no n-element chains: whenever

$y \in S_i$, put

$$g(y) = \begin{cases} i & i \text{ odd}, \quad y \in A \\ i+1 & i \text{ odd}, \quad y \notin A \\ i+1 & i \text{ even}, \quad y \in A \\ i & i \text{ even}, \quad y \notin A \end{cases}$$

and we obtain another order-preserving map $g : \bigcup_1^{n-1} S_i \to C_n$ that takes points within A to odd integers and points outside A to even ones. Consider now $z \in Z$ and $s \in \bigcup_1^{n-1} S_i$. If $z < s$ we see that

$$f(z) \leq ht(z) + 1 \leq ht(s) \leq g(s).$$

Also, if $s < z$ then $ht(z)$ must exceed $ht(s)$ by 2 or more because, if not, the points from z upwards in a maximal (n-point) chain combined with a chain of $ht(s)$-many points having s as maximum would give an n-element chain incorporating s, contrary to its membership of $\bigcup_1^{n-1} S_i$. Therefore

$$g(s) \leq ht(s) + 1 \leq ht(z) - 1 \leq f(z).$$

It follows that the map $h : E \to C_n$ specified by

$$h(x) = \begin{cases} f(x) & \text{if } x \in Z \\ g(x) & \text{if } x \in \bigcup_1^{n-1} S_i \end{cases}$$

is order-preserving, and it gives A and $E \backslash A$ disjoint images. ■

Combining Proposition 1 and Lemma 6 we have the following theorem.

Theorem 7. *A poset is splittable over an n-point chain if and only if it contains neither an* $(n+1)$*-point chain nor a pair of disjoint n-point chains.*

Notice that whether the poset in Lemma 6 is finite or infinite is irrelevant, but that the finiteness of the codomain chain is essential since, for instance, the poset $(0, 1) \oplus (0, 1)$ is embeddable into $(0, 1)$ (where the interval $(0, 1)$ of real numbers is ordered in the natural way) and therefore *a fortiori* splittable over it.

3. Chains splittable over a poset

Since a one-to-one order-preserving map defined on a chain is necessarily an order-isomorphism, the argument in the first paragraph of section 2 already shows that the finite chain C_n is splittable over a poset E only in the extreme case where E contains an embedded copy of C_n. By consideration of order-dense subsets it is possible to dispense with the assumption of finiteness here. The full argument has already been published [4], so it is appropriate merely to state the conclusion.

Theorem 8. *A chain C is splittable over a poset E if and only if C is order-embeddable into E.*

This has a formal similarity to Arhangel'skii's classical result [2] that a compact Hausdorff space is splittable over the real line precisely when it is topologically embeddable into it, and raises the question of what other topological conclusions have analogues in ordered structures of one kind or another. Investigations are proceeding; see, for example, [5] and [6].

References

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