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THE FRÉCHET–URYSOHN PROPERTY, (*LM*)-SPACES AND THE STRONGEST LOCALLY CONVEX TOPOLOGY

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Abstract

We study the Fréchet–Urysohn property for inductive limits of sequences of metrisable locally convex spaces ((*LM*)-spaces). We carefully prove that an (*LM*)-space E is metrisable $\Leftrightarrow$ E is Fréchet–Urysohn $\Leftrightarrow$ E has property C_3 (in the sense of Webb) $\Leftrightarrow$ E has the Nyikos double sequence property $(**)$ $\Leftrightarrow$ E is b-Baire-like, and give related simple examples that disprove three published results concerning φ, an $\aleph_0$-dimensional space bearing the strongest locally convex topology.

1. Definitions and preliminary results

Our main purpose is to fortify a correct version of [12, theorem] while disproving the original and two related observations in [18]. We begin with definitions and some relevant results of independent interest and will end by determining which is valid of two three-space analogues to Michael's theorem [14] on products of Fréchet–Urysohn spaces.

Throughout, let 'tvs (lcs)' stand for 'Hausdorff (locally convex) topological vector space'. Recall that a topological space X is *sequential* [8] if every sequentially closed subset of X is closed; X is *Fréchet–Urysohn* if every point in the closure of a subset A of X is a limit of a sequence of points of A. Webb [25] used the notations C_1

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and C_2 for the Fréchet–Urysohn and sequential properties, respectively, and defined a tvs E to have property C_3 if the sequential closure of each set in E is sequentially closed. Clearly, metrisable $\Rightarrow C_1 \Leftrightarrow [C_2 \wedge C_3]$. These are distinct, intensely studied properties (cf. [2]; [6]; [7]; [9]; [10]; [11]; [17]; [24]); e.g., C_1 is vital in differential calculus (cf. [26], [4]).

We will (re)prove that, for (LM)-spaces, metrisable $\Leftrightarrow C_1 \Leftrightarrow C_3$ [12]. An lcs (E, τ) is an *(LM)-space* if it is the inductive limit of a sequence of metrisable lcs's, i.e. if there is an increasing sequence $\{(E_n, \tau_n)\}_n$ of metrisable lcs's such that $E = \bigcup_n E_n$, each (E_n, τ_n) is continuously included in (E_{n+1}, τ_{n+1}), and τ is the finest locally convex topology on E for which each (E_n, τ_n) is continuously included in (E, τ).

Nyikos [17, theorem 4] (cf. [5]) denoted by (**) the following double sequence property in slightly altered form.

(**) *For every double sequence $\{x_{k,n}\}$ in E such that each $\lim_k x_{k,n} = 0$, there are sequences $\{k_p\}_p$ and $\{n_p\}_p$ with $\lim_p n_p = \infty$ such that $\lim_p x_{k_p,n_p} = 0$.*

Averbukh and Smolyanov [3] (cf. [26, (1), p. 140]) proved that $C_1 \Rightarrow$ (**) for E a tvs. The Nyikos proof [17, theorem 4] actually shows that $C_3 \Rightarrow$ (**) for E a topological group. We offer the following.

Proposition 1.1. *A* tvs *E has property C_3 if and only if it has property* $(**)$.

Proof. Assume that $E \neq \{0\}$ has property C_3. Given $\{x_{k,n}\}$ as in the definition, select a non-zero $x \notin \{nx_{k,n} : k, n \in \mathbb{N}\}$ and put

$$A = \{n^{-1}x - x_{k,n} : k, n \in \mathbb{N}\}.$$

Each $n^{-1}x = \lim_k (n^{-1}x - x_{k,n}) \in A^s$ and $0 = \lim_n n^{-1}x \in A^{ss}$, where A^s denotes the sequential closure of the set A. By hypothesis $A^{ss} = A^s$, so there exist $\{k_p\}_p$ and $\{n_p\}_p$ in $\mathbb{N}$ such that $\lim_p (n_p^{-1}x - x_{k_p,n_p}) = 0$.

If $\{n_p\}_p$ is unbounded then, passing to a subsequence if necessary, we may assume that $\{n_p\}_p$ is increasing, which implies that

$$\lim_p x_{k_p,n_p} = \lim_p [n_p^{-1}x - (n_p^{-1}x - x_{k_p,n_p})] = 0 - 0 = 0,$$

as required, and the proof would be complete. By way of contradiction, assume that $\{n_p\}_p$ is bounded, so that for some subsequence $\{p_r\}_r$ of $\mathbb{N}$ and for some positive integer N we have $n_{p_r} = N$ for $r = 1, 2, \ldots$. Therefore

$$\lim_r (N^{-1}x - x_{k_{p_r},N}) = 0.$$

Now $\{k_{p_r}\}_r$ is either bounded or unbounded, and we may assume, passing to a subsequence if necessary, that $\{k_{p_r}\}_r$ either has constant value K or is strictly increasing. The first case implies that $N^{-1}x - x_{K,N} = 0$, i.e. that $x = Nx_{K,N}$, contradicting the way x was chosen. In the other case, $0 = \lim_r (N^{-1}x - x_{k_{p_r},N}) = N^{-1}x$, i.e. $x = 0$, again contradicting our choice of x.

Conversely, let $A \subset E$ and $x \in A^{ss}$ be given. There exists a sequence $\{y_n\}_n$ in A^s that converges to x, and for each n there exists a sequence $\{z_{k,n}\}_k$ in A that converges to y_n. Defining $x_{k,n} = y_n - z_{k,n}$ and applying property (**) yields sequences $\{k_p\}_p$ and $\{n_p\}_p$ with the latter strictly increasing such that $x_{k_p,n_p} \to 0$. Therefore $z_{k_p,n_p} = y_{n_p} - (y_{n_p} - z_{k_p,n_p}) \to x - 0 = x$, so $x \in A^s$, proving that $A^{ss} = A^s$. ■

Recall that an increasing sequence of convex balanced sets is *absorbing* (*bornivorous*) in the space E if each point (each bounded set) of E is absorbed by some member of the sequence. If each member is closed, the absorbing sequence is *closed*. An lcs E is *Baire-like* [21] (*b-Baire-like* [20]) if some member of any closed absorbing (bornivorous) sequence is necessarily a neighbourhood of zero. For an lcs it is clear that Baire ⇒ (b-)Baire-like ⇒ (quasi-)barrelled. Ruess [20] observed that every metrisable lcs is b-Baire-like, and Yamamuro [26, (4), p. 141] proved that every Fréchet–Urysohn lcs is bornological. We generalise both results in the following.

Proposition 1.2. *Every Fréchet–Urysohn* lcs *E is bornological and b-Baire-like. In fact, if $\{A_n\}_n$ is a bornivorous sequence in a locally convex Fréchet–Urysohn space E, then some A_n is a neighbourhood of zero.*

PROOF. Suppose that no A_n is a 0-neighbourhood. Then for each 0-neighbourhood U and each n there exists $x_{U,n} \in U \setminus nA_n$. Thus 0 is in the closure of $B_n = \{x_{U,n}\}_U$, so there is a sequence $\{U_n(k)\}_k$ of 0-neighbourhoods such that $\lim_k x_{Un(k),n} = 0$ for each n. Property (**) provides sequences $\{k_p\}_p$ and $\{n_p\}_p$ such that $\lim_p n_p = \infty$ and $\lim_p x_{U_{n_p}(k_p),n_p} = 0$. But the bounded sequence $\{x_{U_{n_p}(k_p),n_p}\}_p$ is not a subset of $n_pA_{n_p}$ for each p, which contradicts the fact that $\{A_n\}_n$ is bornivorous. ■

Here the C_1 hypothesis cannot be relaxed to C_3, as ℓ^1 with its weak topology shows. However, we have the following.

Proposition 1.3. *Every bornological* (*or even quasi-barrelled*) *space E with property C_3 is b-Baire-like.*

PROOF. If E is not b-Baire-like, then there exists a closed bornivorous sequence $\{B_n\}_n$ in E such that no B_n is a 0-neighbourhood in E. Since E is quasi-barrelled, no individual B_n is bornivorous. Hence for each $n \in \mathbb{N}$ there exists a bounded set S_n such that $S_n \nsubseteq knB_n$ for every $k \in \mathbb{N}$. For each $k, n \in \mathbb{N}$ choose $x_{k,n} \in S_n \setminus knB_n$ and put $y_{k,n} = k^{-1}x_{k,n}$. Then for each $k, n \in \mathbb{N}$ we have $y_{k,n} \notin nB_n$ and $\lim_k y_{k,n} = 0$. By Proposition 1.1 we obtain two sequences $\{k_p\}_p$ and $\{n_p\}_p$ in $\mathbb{N}$ with $\lim_p n_p = \infty$ such that $\lim_p y_{k_p,n_p} = 0$. But this convergent sequence is not bounded because $y_{k_p,n_p} \notin n_pB_{n_p}$ and $\{n_pB_{n_p}\}_p$ is a fundamental sequence of bounded sets, a contradiction. ■

Left open is the question of whether every bornological space with property C_3 is necessarily Fréchet–Urysohn. The answer is positive for (LM)-spaces, as we prove next.

2. A theorem and three corrections

Theorem 2.1. *For an (LM)-space E, the inductive limit of a sequence $\{(E_n, \tau_n)\}_n$ of metrisable spaces with each (E_n, τ_n) continuously contained in (E_{n+1}, τ_{n+1}), the six statements below are equivalent:*

(1) *E is metrisable;*
(2) *E is Fréchet–Urysohn;*
(3) *E has property C_3;*
(4) *E has property $(**)$;*
(5) *E is b-Baire-like;*
(6) *if $\{U_n\}_n$ is an increasing sequence of balanced convex sets with each U_n a neighbourhood of 0 in (E_n, τ_n), then $\overline{U}_m$ is a neighbourhood of 0 in E for some $m \in \mathbb{N}$.*

Proof. As the inductive limit of metrisable spaces, E is always bornological. The definitions and results of the previous section yield (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

[(5) $\Rightarrow$ (6)]: $\{\overline{U}_n\}_n$, by Makarov's theorem [18, 8.5.20], is bornivorous.

[(6) $\Rightarrow$ (1)]: We use a technique of Narayanaswami and Saxon [15]. For each $n \in \mathbb{N}$, let $\{U_{k,n}\}_k$ be a fixed (countable) base of neighbourhoods of 0 for the nth metrisable step (E_n, τ_n). If S is a sequence with $S(n) \in \{U_{k,n} : k \in \mathbb{N}\}$ for each n, the closed absolutely convex hull $\overline{\mathrm{acx}}\,(\bigcup_i S(i))$ of $\bigcup_i S(i)$ is a neighbourhood of 0 in E, and the totality of such sets forms a base of 0-neighbourhoods in E. Given S, (6) implies the existence of some $m(S) \in \mathbb{N}$ such that $V_S = \overline{\mathrm{acx}}\,(\bigcup_{i \le m(S)} S(i))$ is a neighbourhood of 0 in E, and these sets also constitute a base of 0-neighbourhoods in E. Furthermore, this latter base is countable, since each V_S is determined by a finite subset of $\{U_{k,n} : k, n \in \mathbb{N}\}$. Therefore E is metrisable. ■

The simplest example of an (LM)-space having property C_2 but not C_3 (equivalently, C_2 but not C_1) is the space φ whose nth defining step is n-dimensional, an example already known to Yoshinaga [27], Webb [25] and Nyikos [17]. Any lcs containing φ, then, cannot have property C_1. The essence of [13] is that *an (LM)- or (DF)-space has property C_2 but not C_1 if and only if it is a Montel (DF)-space containing φ*. Moreover, a barrelled space is Baire-like if it is metrisable [1] or, more generally, does not contain φ [21]. Converses hold in Grothendieck's class of (LF)-spaces: *an (LF)-space E is metrisable $\Leftrightarrow$ E is Baire-like $\Leftrightarrow$ E does not contain φ* [15].

But these latter equivalences fail for (LM)-spaces. While every Baire-like (LM)-space is metrisable, no $\aleph_0$-dimensional metrisable space is Baire-like. And, even though no metrisable space contains φ, not containing φ fails to guarantee the metrisability of an (LM)-space.

Example 2.1. For each n let D_n be the space ℓ^1 with the topology induced by the Banach space ℓ^n, and let D be the (LM)-space that is the inductive limit of the D_n. Now D cannot be metrisable, for it is a dense subspace of a proper (LB)-space, namely, the inductive limit of the Banach spaces ℓ^n (cf. [15]). Alternatively, the unit balls of the steps form a closed bornivorous sequence (Makarov [18, 8.5.20]) that proves that D is not b-Baire-like. At the same time, D cannot contain φ, since D is dominated by a metrisable space and φ is surely not.

In fact, D is not even sequential [13], since it is an (LM)-space (and a (DF)-space) that is neither metrisable nor Montel. We may now correct three mistakes in the literature concerning φ.

Firstly, D is a direct counter-example to the claim of [12, theorem] that an (LM)-space is metrisable if and only if it does not contain φ. (Most of [12, theorem] remains valid, constituting (1)–(3) of Theorem 2.1 above.)

Secondly, D is bornological, hence quasi-barrelled, and contradicts the claim of [18, 8.2.11(a)] that every quasi-barrelled space not containing φ must be b-Baire-like. The two errors are closely linked: Kąkol's would-be proof [12, theorem] relies on Pérez Carreras and Bonet's [18] observation 8.2.11(a).

Thirdly, observation 8.2.11(b) of [18], that any Mackey space E not containing φ must be Baire-like if E' is locally complete in the weak $\sigma(E', E)$-topology, is contradicted by the Mackey space $(\ell^\infty, \tau(\ell^\infty, \ell^1))$: it does not contain φ, since it is dominated by a metrisable space; it is evidently dual ℓ^1-complete, hence dual locally complete (cf. [22]); and yet it is not Baire-like, nor even barrelled, since the unit vectors in the dual are not equicontinuous.

3. Products and three-space properties

It seems to be still open as to whether the square of a Fréchet–Urysohn lcs is always Fréchet–Urysohn, although Todorčević [23] answered an Arkhangel'skii question by finding two Fréchet–Urysohn spaces $C_p(X)$ and $C_p(Y)$ whose product is not Fréchet–Urysohn, having uncountable tightness. Since three-space properties are always preserved by finite products, it follows that Fréchet–Urysohn is not a three-space property, i.e. there exists an lcs E with a closed Fréchet–Urysohn subspace F such that E/F is also Fréchet–Urysohn but E itself is not. A special example of this is provided below.

Michael [14] proved that, in contrast to the Todorčević example, if X is a first-countable topological space and Y is a Fréchet–Urysohn topological space with property (**), then $X \times Y$ *is* Fréchet–Urysohn. For topological vector spaces, where metrisable is equivalent to first-countable, we have a stronger three-space type analogue.

Theorem 3.1. *If F is a closed metrisable subspace of a* tvs *E such that the quotient E/F is a Fréchet–Urysohn space, then E itself is a Fréchet–Urysohn space.*

PROOF. By hypothesis, there exists a sequence $\{V_n\}_n$ of neighbourhoods of zero in E such that $\{V_n \cap F\}_n$ is a base of neighbourhoods of zero in F, and we may assume that each $V_{n+1} + V_{n+1} \subset V_n$. Let A be a subset of E such that $0 \in \overline{A}$. Let $q : E \longrightarrow E/F$ denote the quotient map. For each $n \in \mathbb{N}$ and each neighbourhood U of zero in E there exists $x_{U,n} \in A \cap U \cap V_n$. Hence $q(0)$ is in the closure of $\{q(x_{U,n}) : U$ is a 0-neighbourhood in $E\}$, and, since E/F is Fréchet–Urysohn, there exists a sequence $\{U_k(n)\}_k$ of 0-neighbourhoods in E such that

$$\lim_{k\to\infty} q(x_{U_k(n),n}) = q(0)$$

for each $n \in \mathbb{N}$. By (**) there exist sequences $\{k_p\}_p$ and $\{n_p\}_p$ of natural numbers such that

$$\lim_{p\to\infty} n_p = \infty \quad \text{and} \quad \lim_{p\to\infty} q(x_{U_{k_p}(n_p),n_p}) = q(0).$$

For each $p \in \mathbb{N}$ put $z_p = x_{U_{k_p}(n_p),n_p}$. For W any balanced neighbourhood of zero in E there exists $K \in \mathbb{N}$ such that $(V_K + V_K) \cap F \subset W$, and there exists $N > K$ such that $p > N$ implies that

$$[z_p + (W \cap V_K)] \cap F \neq \varnothing.$$

Therefore there exist $y \in W \cap V_K$ and $u \in F$ such that

$$z_p + y = u.$$

Hence $u \in W$ and $z_p = u - y \in W + W$. It follows that $\{z_p\}_p$ is a sequence in A that converges to 0 in E, and E is Fréchet–Urysohn. ■

Corollary 3.2. (Michael). *Let E, F be topological vector spaces with E Fréchet–Urysohn and F metrisable. Then $E \times F$ is Fréchet–Urysohn.*

Let $E = \prod_{\alpha\in A} E_\alpha$ be the product of topological spaces and let $y = (y_\alpha)_{\alpha\in A} \in E$. By a $\sum$*-product at* y (of E) we mean the space

$$E(y) = \{x = (x_\alpha)_{\alpha\in A} \in E \ : \text{card}\{\alpha : x_\alpha \neq y_\alpha\} \leq \aleph_0\}.$$

Noble [16] proved that every $\sum$-product is Fréchet–Urysohn if each E_α is a first-countable topological space. If the E_α are vector spaces, then $E(0)$ is a linear subspace of E.

Our final example proves false the alternative three-space analogue to Michael's result; i.e. Theorem 3.1 is no longer true when the hypotheses on F and E/F are interchanged.

Example 3.1. The Fréchet–Urysohn property is not a three-space property. In fact, there is a locally convex space E that is not Fréchet–Urysohn even though it has a closed Fréchet–Urysohn subspace F such that E/F is a Banach space. We take $\mathbb{R}_0 = \{(x_\alpha)_\alpha \in \mathbb{R}^{\mathbb{R}} \ : \text{card}\{\alpha : x_\alpha \neq 0\} \leq \aleph_0\}$ and fix $x_0 \in \mathbb{R}^{\mathbb{R}} \backslash \mathbb{R}_0$. The $\sum$-product $\mathbb{R}_0$ is Fréchet–Urysohn; $G = \mathbb{R}_0 + [x_0]$ is not. Hence, by Theorem 3.1, $G/[x_0]$ is not Fréchet–Urysohn. Choose a dense hyperplane H of ℓ^2 and take $y \in \ell^2 \setminus H$. Applying Roelcke and Dierolf's lemma 1.4 of [19], we obtain the following. There exists a locally convex space E containing a closed subspace F such that F is isomorphic to $\mathbb{R}_0$, the space E/F is isomorphic to the Banach space $\ell^2/[y]$, and E has a quotient isomorphic to $G/[x_0]$. As $G/[x_0]$ is not Fréchet–Urysohn, neither is E, since the Fréchet–Urysohn property is preserved by quotients. ■

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