

ON A LOCALISED SINGLE-VALUED EXTENSION PROPERTY

By

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ABSTRACT

In this article, we investigate, a certain localised version of the single-valued extension property for a bounded linear operator on a Banach space. We show that this condition behaves canonically under the Riesz functional calculus, and derive a number of characterisations in terms of kernel-type and range-type spaces for the operator and its adjoint. The theory is exemplified in the case of isometries, analytic Toeplitz operators, invertible composition operators on Hardy spaces, and weighted shifts.

1. Introduction

The *single-valued extension property*, SVEP for brevity, dates back to the early days of local spectral theory. Following Dunford, a bounded linear operator $T \in L(X)$ on a complex Banach space X is said to have SVEP if, for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \rightarrow X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$ is the zero function $f \equiv 0$ on U ; see [5; 6].

The basic importance of SVEP arises in connection with the following notion. For arbitrary $x \in X$, the *local spectrum* $\sigma_T(x)$ of T at x is defined to be the complement in \mathbb{C} of the set $\rho_T(x)$ of all $\omega \in \mathbb{C}$ for which there exists an analytic function $f_\omega : U_\omega \rightarrow X$ on some open neighbourhood U_ω of ω such that $(T - \lambda I)f_\omega(\lambda) = x$ for all $\lambda \in U_\omega$. Evidently, SVEP ensures the consistency of these local solutions, and hence leads to a globally defined analytic function $f : \rho_T(x) \rightarrow X$ for which $(T - \lambda I)f(\lambda) = x$ for each $\lambda \in \rho_T(x)$. Thus, not surprisingly, SVEP plays a leading role in the investigation of normal and spectral operators [6] and their generalisations, the generalised spectral and decomposable operators; see [5; 8; 18]. In fact, all of these operators have SVEP, and they enjoy spectral decompositions by means of the *local spectral subspaces* given by $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$ for all subsets F of \mathbb{C} .

On the other hand, simple examples show that quotients of decomposable operators may fail to have SVEP. Hence, as already observed by Bishop [4], in a general duality theory for restrictions and quotients of decomposable operators, it is

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more appropriate to work with another kind of spectral subspaces: for a closed set $F \subseteq \mathbb{C}$, let the *glocal spectral subspace* $\mathcal{X}_T(F)$ consist of all $x \in X$ for which there exists an analytic function $f: \mathbb{C} \setminus F \rightarrow X$ such that $(T - \lambda I)f(\lambda) = x$ for each $\lambda \in \mathbb{C} \setminus F$; see [8]. Evidently, $\mathcal{X}_T(F) \subseteq X_T(F)$. Moreover, by proposition 3.3.2 of [8], equality occurs for all closed sets $F \subseteq \mathbb{C}$ precisely when T has SVEP. We emphasise that neither the local nor the glocal spectral subspaces have to be closed.

In this paper, the spaces $K(T) := X_T(\mathbb{C} \setminus \{0\})$ and $H_0(T) := \mathcal{X}_T(\{0\})$ are of particular importance. Both spaces were thoroughly studied by Mbekhta [9; 10], Schmoegeer [15], and Vrbová [19]. By proposition 3.3.7 of [8], and also by proposition 1.3 of [10], $K(T)$ coincides with the set of all $x \in X$ for which there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that

$$x_0 = x, \quad Tx_n = x_{n-1}, \quad \text{and} \quad \|x_n\| \leq c^n \|x\| \quad \text{for all } n \in \mathbb{N}.$$

The latter set is also known as the *analytic core of T* . By the open mapping theorem, the preceding characterisation of $K(T)$ entails, in particular, that $K(T) = X$ if and only if T is surjective; see also proposition 1.3.2 of [8]. On the other hand, by proposition 3.3.13 of [8], the space $H_0(T)$ consists precisely of all $x \in X$ for which

$$\|T^n x\|^{1/n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For this reason, $H_0(T)$ is often called the *quasi-nilpotent part of T* .

The spaces $H_0(T)$ and $K(T)$ are related to the kernel $N(T)$ and the range $R(T)$ of T as follows. With the notation

$$\mathcal{N}(T) := \bigcup_{n=1}^{\infty} N(T^n) \quad \text{and} \quad \mathcal{R}(T) := \bigcap_{n=1}^{\infty} R(T^n)$$

for the *generalised kernel and range of T* , we have, by propositions 1.2.16 and 3.3.1 of [8], the increasing chain of kernel-type spaces

$$N(T) \subseteq N(T^n) \subseteq N(T) \subseteq H_0(T) \subseteq X_T(\{0\})$$

and the decreasing chain of range-type spaces

$$R(T) \supseteq R(T^n) \supseteq \mathcal{R}(T) \supseteq K(T) \supseteq X_T(\emptyset)$$

for arbitrary $n \in \mathbb{N}$. In this paper, we shall explore the extent to which the geometric position of the kernel-type spaces with respect to the range-type spaces is related to a certain localised version of SVEP for the operator T and its adjoint T^* .

An operator $T \in L(X)$ is said to have SVEP at a point $\lambda \in \mathbb{C}$ if, for every open disc U centered at λ , the only analytic solution $f: U \rightarrow X$ of the equation $(T - \mu I)f(\mu) = 0$ for all $\mu \in U$ is the zero function $f \equiv 0$ on U . This notion dates back to Finch [7], and was pursued further, for instance, in [1; 2; 3; 10; 15]. Evidently T has SVEP at λ precisely when $T - \lambda I$ has SVEP at 0, while SVEP for T is equivalent to SVEP for T at λ for each $\lambda \in \mathbb{C}$.

In the next three sections, we shall employ local spectral theory to establish a variety of characterisations of this localised version of SVEP that involve the kernel-type and range-type spaces introduced above. Our main tools will be a spectral mapping theorem for this condition and a certain duality result for the analytic core

and the quasi-nilpotent part of an operator. We shall also correct a basic description of the localised SVEP from [10] and an application of this result provided in [15].

For an arbitrary operator $T \in L(X)$, let

$$\mathfrak{S}(T) := \{\lambda \in \mathbb{C} : T \text{ fails to have SVEP at } \lambda\}.$$

Obviously $\mathfrak{S}(T)$ is empty precisely when T has SVEP. Moreover, it follows readily from the identity theorem for analytic functions that $\mathfrak{S}(T)$ is open, and therefore contained in the interior of the spectrum $\sigma(T)$. In the last two sections of this article, we shall compute the set $\mathfrak{S}(T)$ for several classes of operators. Thus, although SVEP is, in general, not preserved under quotients and duality, in many important examples it is possible to determine all points at which SVEP holds.

We finally mention that $\mathfrak{S}(T)$ is closely related to the *analytic spectral residuum* $\mathcal{S}(T)$ of the operator T , as considered in [18; 19]. $\mathcal{S}(T)$ is defined to be the complement in \mathbb{C} of the largest open set $U \subseteq \mathbb{C}$ with the property that, for each open subset V of U , the only analytic solution $f: V \rightarrow X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in V$ is $f \equiv 0$ on V . It is easily verified that $\mathfrak{S}(T) \subseteq \mathcal{S}(T)$. However, equality occurs only in the trivial case when T has SVEP, since $\mathfrak{S}(T)$ is open and $\mathcal{S}(T)$ is closed. On the other hand, it is not difficult to see that $\mathfrak{S}(T)$ is always dense in $\mathcal{S}(T)$; see also lemma 4.3.11 of [18].

2. Basic characterisations

Throughout the following, let $T \in L(X)$ denote a bounded linear operator on an arbitrary complex Banach space X . Our starting point is the following characterisation from theorem 1.9 of [2]. The result shows, in particular, that every injective operator $T \in L(X)$ has SVEP at 0, and may be viewed as a local version of the classical fact that T has SVEP if and only if $X_T(\emptyset) = \{0\}$; see proposition 1.2.16 of [8].

Theorem 1. *T has SVEP at 0 precisely when $N(T) \cap X_T(\emptyset) = \{0\}$.*

Since $N(T) \cap K(T) \subseteq X_T(\{0\}) \cap X_T(\mathbb{C} \setminus \{0\}) = X_T(\emptyset)$, it is clear that $N(T) \cap K(T) = N(T) \cap X_T(\emptyset)$ for every $T \in L(X)$. Since, as noted above, $K(T) = X$ whenever T is surjective, Theorem 1 leads to the following characterisation, which extends a classical result due to Finch [7]; see also corollary 1.11 of [2].

Corollary 2. *T has SVEP at 0 if and only if $N(T) \cap K(T) = \{0\}$. In particular, if T is surjective, then T has SVEP at 0 precisely when T is injective.*

As another immediate consequence of Theorem 1, we obtain the following result:

Corollary 3. *If T satisfies either $H_0(T) \cap K(T) = \{0\}$ or $\mathcal{N}(T) \cap \mathcal{R}(T) = (0)$, then T has SVEP at 0.*

As we shall see, the preceding implications cannot be reversed in general. We complement the first part of Corollary 3 by an observation which has the same

flavour as the following classical result: if $X_T(\emptyset)$ is closed, then actually $X_T(\emptyset) = \{0\}$, and hence T has SVEP; see proposition 1.2.16 of [8]. The next result is also reminiscent of the fact that T has SVEP if $\mathcal{X}_T(F)$ is closed for each closed subset F of \mathbb{C} ; see proposition 3.3.4 of [8].

Corollary 4. *Suppose that $H_0(T) \cap K(T)$ is closed. Then $H_0(T) \cap K(T) = \{0\}$, and therefore T has SVEP at 0.*

PROOF. Evidently, the space $Y := H_0(T) \cap K(T)$ is invariant under T , so that we may consider the restriction $S := T|_Y \in L(Y)$. Because

$$\|S^n u\|^{1/n} = \|T^n u\|^{1/n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for each $u \in Y$, we infer that $H_0(S) = Y$. By proposition 3.3.14 of [8], it follows that S is quasi-nilpotent, and hence has SVEP. Thus $\sigma_S(u) = \{0\}$ for all non-zero $u \in Y$.

On the other hand, it turns out that $K(S) = Y$. Indeed, let $u \in Y$ be given, and recall that $K(T)$ coincides with the analytic core of T . Hence there exist a $c > 0$ and elements $x_n \in X$ such that $x_0 = u$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$. From $u \in H_0(T)$ we conclude that $x_n \in H_0(T)$ for all $n \in \mathbb{N}$. Since $x_n \in K(T)$, we obtain that $x_n \in Y$ for all $n \in \mathbb{N}$, and therefore $u \in K(S)$, as desired. Thus $0 \notin \sigma_S(u)$ for all $u \in Y$. By the result of the preceding paragraph, we conclude that $Y = \{0\}$. ■

Along the same line, we note that $H_0(T) \cap K(T) = H_0(T) \cap X_T(\emptyset) \subseteq X_T(\emptyset)$ for arbitrary $T \in L(X)$. Thus SVEP for T entails that $H_0(T) \cap K(T) = \{0\}$, but, contrary to a claim made in theorem 1.4 of [10], the next example will show that SVEP for T at 0 does not necessarily imply that $H_0(T) \cap K(T) = \{0\}$. On the other hand, by theorem 2.7 of [1], the latter condition characterises SVEP at 0 for a large class of operators, namely for all operators with a generalised Kato decomposition in the sense of [1]. Our counter-example is based on the theory of weighted shifts [16]. Further information on the local spectral properties of weighted shifts is obtained in Section 6.

Example 1. Let $\beta := (\beta_n)_{n \in \mathbb{Z}}$ be the weight sequence defined by $\beta_n := 1 + |n|$ if $n < 0$, and $\beta_n := e^{-n^2}$ if $n \geq 0$. As in section 3 of [16], we consider the corresponding Hilbert space $X := L^2(\beta)$ of all formal Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{for which} \quad \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty,$$

where X is endowed with the canonical norm $\|\cdot\|_\beta$.

It is easily seen that the bilateral right shift T given by

$$T \left(\sum_{n=-\infty}^{\infty} a_n z^n \right) := \sum_{n=-\infty}^{\infty} a_n z^{n+1}$$

is bounded on $L^2(\beta)$. In fact, by propositions 2 and 7 of [16],

$$\|T\| = \sup\{\beta_{n+1}/\beta_n : n \in \mathbb{Z}\} = 1,$$

and T is unitarily equivalent to the bilateral weighted right shift on the two-sided sequence space $\ell^2(\mathbb{Z})$ with respect to the weight sequence $(\beta_{n+1}/\beta_n)_{n \in \mathbb{Z}}$.

Since T is injective, we conclude from Theorem 1 that T has SVEP at 0. On the other hand, the following argument will show that $1 \in H_0(T) \cap K(T)$, and will therefore provide the announced counter-example to theorem 1.4 of [10].

Since $\|z^n\|_\beta = \beta_n$ for all $n \in \mathbb{Z}$, we obtain that

$$\lim_{n \rightarrow \infty} \|z^{n-1}\|_\beta^{1/n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z^{-n-1}\|_\beta^{1/n} = 1.$$

By the standard formula for the radius of convergence of a power series with values in a Banach space, we conclude that the two infinite series

$$f(\lambda) := -\sum_{n=1}^{\infty} z^{n-1} \lambda^{-n} \quad \text{and} \quad g(\lambda) := \sum_{n=0}^{\infty} z^{-n-1} \lambda^n$$

converge in $L^2(\beta)$ for all $\lambda \in \mathbb{C}$ with $|\lambda| > 0$ and $|\lambda| < 1$, respectively. Moreover, the function f is analytic on $\mathbb{C} \setminus \{0\}$, and satisfies

$$(T - \lambda I)f(\lambda) = -\sum_{n=1}^{\infty} z^n \lambda^{-n} + \sum_{n=1}^{\infty} z^{n-1} \lambda^{-(n-1)} = 1 \quad \text{for all } \lambda \neq 0,$$

while g is analytic on the open unit disc \mathbb{D} , and satisfies

$$(T - \lambda I)g(\lambda) = \sum_{n=0}^{\infty} z^{-n} \lambda^n - \sum_{n=0}^{\infty} z^{-n-1} \lambda^{n+1} = 1 \quad \text{for all } \lambda \in \mathbb{D}.$$

Thus $1 \in \mathcal{X}_T(\{0\}) \cap \mathcal{X}_T(\mathbb{C} \setminus \{0\})$, and in particular, $1 \in H_0(T) \cap K(T)$ as claimed.

We note that the adjoint of the operator T from the preceding example is hyponormal, since the weight sequence β was chosen so that $(\beta_{n+1}/\beta_n)_{n \in \mathbb{Z}}$ is decreasing; see section 7 of [16] for the characterisation of hyponormal weighted shifts. By proposition 2.4.9 and theorems 2.4.14 and 2.5.5 of [8], it follows that T is the quotient of some decomposable operator by a closed invariant subspace. By contrast, restrictions of decomposable operators have SVEP, and hence cannot serve as a counter-example in this context. In particular, if $(\beta_{n+1}/\beta_n)_{n \in \mathbb{Z}}$ is increasing, then T is hyponormal, so that, in this case, T has SVEP, and therefore satisfies $H_0(T) \cap K(T) = \{0\}$.

We finally mention that theorem 1.4 of [10] was employed in [15] in the study of isolated points in the spectrum of an arbitrary operator $T \in L(X)$. In view of Example 1, the proof of proposition 4 of [15] requires an adjustment. Fortunately, a correct approach may be based on the following characterisation due to Mbekhta. By theorem 1.6 of [9], 0 is an isolated point of $\sigma(T)$ if and only if $H_0(T)$ is non-trivial and the decomposition $X = H_0(T) \oplus K(T)$ holds as a topological direct sum.

3. A spectral mapping theorem

For an arbitrary operator $T \in L(X)$ and an analytic function $f: U \rightarrow \mathbb{C}$ on an open neighbourhood U of $\sigma(T)$, let $f(T) \in L(X)$ denote the operator given by the Riesz

functional calculus

$$f(T) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

where Γ is a contour in U that surrounds $\sigma(T)$; see [6; 18]. The classical spectral mapping asserts that $\sigma(f(T)) = f(\sigma(T))$. In the same vein, we now obtain the following result.

Theorem 5. *Let $T \in L(X)$, let $U \subseteq \mathbb{C}$ be an open neighbourhood of $\sigma(T)$, and let $f : U \rightarrow \mathbb{C}$ be an analytic function that is non-constant on each connected component of U . Then $\mathfrak{S}(f(T)) = f(\mathfrak{S}(T))$. Moreover, the operator $f(T)$ has SVEP at a point $\lambda \in \mathbb{C}$ if and only if T has SVEP at each point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. In particular, $f(T)$ has SVEP precisely when T has SVEP.*

PROOF. Evidently it suffices to establish the stipulated characterisation of SVEP at λ for $f(T)$, since this equivalence is nothing but a reformulation of the identity $\mathfrak{S}(f(T)) = f(\mathfrak{S}(T))$.

Suppose first that $f(T)$ has SVEP at a point $\lambda \in \mathbb{C}$, and consider an arbitrary point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. To show that T has SVEP at μ , we shall employ Theorem 1. Hence, let $x \in N(T - \mu I) \cap X_{T - \mu I}(\emptyset)$ be given. Since $f(\mu) = \lambda$, we obtain a factorisation of the form $f - \lambda = g(Z - \mu)$, where Z denotes the identity function, and g is analytic on U . Since the Riesz functional calculus preserves multiplication, we conclude that $f(T) - \lambda I = g(T)(T - \mu I)$, and therefore $x \in N(f(T) - \lambda I)$. Moreover, from $\sigma_{T - \mu I}(x) = \emptyset$ we obtain that $\sigma_T(x) = \emptyset$, and therefore, by the spectral mapping property for the local spectrum from theorem 3.3.8 of [8], $\sigma_{f(T)}(x) = f(\sigma_T(x)) = f(\emptyset) = \emptyset$. Thus

$$N(T - \mu I) \cap X_{T - \mu I}(\emptyset) \subseteq N(f(T) - \lambda I) \cap X_{f(T) - \lambda I}(\emptyset),$$

so that, by Theorem 1, T has SVEP at μ .

Conversely, let $\lambda \in \mathbb{C}$ be given, and suppose that T has SVEP at each point $\mu \in \sigma(T)$ for which $f(\mu) = \lambda$. To show that $f(T)$ has SVEP at λ , we may, by the classical spectral mapping theorem for the Riesz functional calculus, assume that $\lambda \in f(\sigma(T))$. Since f is non-constant on each connected component of U , it follows from the identity theorem for analytic functions that the function $f - \lambda$ has only finitely many zeros in $\sigma(T)$, and that all these zeros are of finite multiplicity. Hence we obtain the factorisation $f - \lambda = gp$, where g is analytic on U and without zeros in $\sigma(T)$, while p is a polynomial of the form $p = (Z - \mu_1) \cdots (Z - \mu_m)$ with not necessarily distinct elements $\mu_1, \dots, \mu_m \in \sigma(T)$. Now, to show, via Theorem 1, that $f(T)$ has SVEP at λ , we consider an arbitrary element

$$x \in N(f(T) - \lambda I) \cap X_{f(T) - \lambda I}(\emptyset).$$

Since $f(T) - \lambda I = g(T)p(T)$ and $g(T)$ is, again by the classical spectral mapping theorem, invertible, it follows that $p(T)x = 0$, and therefore $(T - \mu_1 I)y = 0$, where

$y := q(T)x$ and $q := (Z - \mu_2) \cdots (Z - \mu_m)$. On the other hand, because $x \in X_{f(T)-\lambda I}(\emptyset)$, another application of theorem 3.3.8 of [8] ensures that $f(\sigma_T(x)) = \sigma_{f(T)}(x) = \emptyset$, and therefore, by proposition 1.2.17 of [8], $\sigma_T(y) = \sigma_T(q(T)x) \subseteq \sigma_T(x) = \emptyset$. Since T has SVEP at μ_1 , Theorem 1 implies that $y = 0$. An obvious repetition of this argument for μ_2, \dots, μ_m leads to the desired conclusion that $x = 0$. Consequently, $f(T)$ has SVEP at λ . ■

In Theorem 5, the additional condition on the analytic function is essential. Indeed, if f is constant on U , then $\Xi(f(T))$ is empty, while $f(\Xi(T))$ is empty only when T has SVEP. The preservation of SVEP under the Riesz functional calculus is originally due to Colojoară and Foiaş; see [5, theorem 1.1.5; 8, theorem 3.3.9].

It is interesting to note that Theorem 5 yields the spectral mapping theorem for the analytic spectral residuum; see [18, theorem 4.3.14; 19, theorem 1.6]. In fact, since $\Xi(T)$ is dense in $\mathcal{S}(T)$, the identity $\mathcal{S}(f(T)) = f(\mathcal{S}(T))$ follows immediately from the corresponding result for $\Xi(T)$.

As another consequence of Theorem 5, we can now show that, in the characterisations of Theorem 1 and Corollary 2, the kernel may be replaced by the generalised kernel.

Corollary 6. *For every operator $T \in L(X)$, the following assertions are equivalent:*

- (a) T has SVEP at 0;
- (b) T^n has SVEP at 0 for each $n \in \mathbb{N}$;
- (c) $N(T) \cap X_T(\emptyset) = \{0\}$;
- (d) $N(T) \cap K(T) = \{0\}$;
- (e) $\mathcal{N}(T) \cap X_T(\emptyset) = \{0\}$;
- (f) $\mathcal{N}(T) \cap K(T) = \{0\}$.

Moreover, these equivalent conditions hold if and only if $f(T)$ has SVEP at 0 for every analytic function f on an open neighbourhood U of $\sigma(T)$ with the property that f is non-constant on each component of U and 0 is the only zero of f in $\sigma(T)$.

PROOF. We observe that $\mathcal{N}(T) \cap X_T(\emptyset) = \mathcal{N}(T) \cap K(T)$ and that $\mathcal{N}(T) \cap X_T(\emptyset) = \{0\}$ if and only if $N(T^n) \cap X_T(\emptyset) = \{0\}$ for each $n \in \mathbb{N}$. Hence the result is immediate from Theorem 1, Corollary 2 and Theorem 5. ■

4. Duality results

As already observed by Mbekhta [9], the analytic core and the quasi-nilpotent part of an operator are, in a sense, dual concepts. This idea will now be employed, in the spirit of [1; 3], to obtain certain dual characterisations of the localised SVEP.

For a linear subspace M of a complex Banach space X , let

$$M^\perp := \{\varphi \in X^* : \varphi(x) = 0 \text{ for all } x \in M\}$$

as usual, denote the *annihilator of M* in the dual space X^* , and for a linear subspace N of X^* , let

$${}^\perp N := \{x \in X : \varphi(x) = 0 \text{ for all } \varphi \in N\}$$

denote the *preannihilator of N* in X . By the bipolar theorem, ${}^\perp(M^\perp)$ is the norm-closure of M , and $({}^\perp N)^\perp$ is the weak- $*$ -closure of N . Moreover, for every $T \in L(X)$, it is well known that $N(T^*) = R(T)^\perp$ and $N(T) = {}^\perp R(T^*)$, while $R(T)$ is a norm-dense subspace of ${}^\perp N(T^*)$, and $R(T^*)$ is a weak- $*$ -dense subspace of $N(T)^\perp$.

Proposition 7. *For every operator $T \in L(X)$, the following assertions hold:*

- (a) $K(T) \subseteq {}^\perp H_0(T^*)$ and $K(T^*) \subseteq H_0(T)^\perp$;
- (b) if $H_0(T) + R(T)$ is norm-dense in X , then T^* has SVEP at 0;
- (c) if $H_0(T^*) + R(T^*)$ is weak- $*$ -dense in X^* , then T has SVEP at 0.

PROOF. In the Hilbert space case, the two claims of assertion (a) are, of course, equivalent to the inclusion $H_0(T) \subseteq {}^\perp K(T^*)$ from proposition 1.8 of [9]. The argument from [9] could be transferred to the general case of Banach spaces, but instead we offer the following short proof based on local spectral theory.

By proposition 2.5.1 of [8], the inclusion $\mathcal{X}_T(F) \subseteq {}^\perp \mathcal{X}_{T^*}^*(G)$ holds for every pair of disjoint closed sets $F, G \subseteq \mathbb{C}$. Now, given an arbitrary $x \in K(T)$, we have $0 \in \rho_T(x)$, and therefore $x \in \mathcal{X}_T(F)$ for some closed set $F \subseteq \mathbb{C}$ for which $0 \notin F$. Thus, taking $G = \{0\}$, we infer that $x \in {}^\perp H_0(T^*)$. This establishes the first inclusion of assertion (a), and the second one follows by a similar argument.

By standard duality theory, we conclude that

$$K(T^*) \cap N(T^*) \subseteq H_0(T)^\perp \cap R(T)^\perp = (H_0(T) + R(T))^\perp.$$

If $H_0(T) + R(T)$ is norm-dense in X , then the last annihilator is zero, and therefore $K(T^*) \cap N(T^*) = \{0\}$. By Corollary 2, this means precisely that T^* has SVEP at 0, and thus establishes assertion (b). To prove the last claim, we note that

$$K(T) \cap N(T) \subseteq {}^\perp H_0(T^*) \cap {}^\perp R(T^*) = {}^\perp (H_0(T^*) + R(T^*)).$$

Hence, by the Hahn–Banach theorem, the density condition of assertion (c) entails that $K(T) \cap N(T) = \{0\}$, so that T has SVEP at 0, again by Corollary 2. ■

Since $\mathcal{N}(T) \subseteq H_0(T)$ and $K(T) \subseteq \mathcal{R}(T) \subseteq R(T)$ for every $T \in L(X)$, we obtain immediately the following result. Corollary 8 is, in a sense, dual to Corollary 3, and extends results from theorem 1.10 of [3] and theorem 1.6 of [1].

Corollary 8. *If either $H_0(T) + K(T)$ or $\mathcal{N}(T) + \mathcal{R}(T)$ is norm-dense in X , then T^* has SVEP at 0. Moreover, if either $H_0(T^*) + K(T^*)$ or $\mathcal{N}(T^*) + \mathcal{R}(T^*)$ is weak- $*$ -dense in X^* , then T has SVEP at 0.*

In Proposition 17, we shall provide a class of examples involving unilateral weighted shifts to illustrate that, even in the Hilbert space setting, the implications of

parts (b) and (c) in Proposition 7 cannot be reversed, in general. Here the problem is that the inclusions provided in part (a) need not be identities. In fact, a glance at the proof of Proposition 7 shows that equivalences occur in (b) and (c) whenever identities hold in (a). One of the main obstructions is, of course, that the analytic core of an operator is not necessarily closed. However, for suitable classes of operators, we now establish that the results of Proposition 7 hold with identities and equivalences.

Operators that satisfy one of the conditions (i) or (ii) below are known as *semi-Fredholm operators*, while the conjunction of (i) and (ii) defines the class of *Fredholm operators*. On the other hand, by lemma 2.3 of [9], condition (v) characterises the *semi-regular operators*; see [8; 11] for a discussion of this class of operators.

Theorem 9. *Suppose that the operator $T \in L(X)$ satisfies one of these conditions:*

- (i) $N(T)$ is finite-dimensional and $R(T)$ is closed;
- (ii) $R(T)$ is of finite codimension in X ;
- (iii) $N(T^m) = N(T^{m+1})$ and $R(T^{m+1})$ is closed for some $m \in \mathbb{N}$;
- (iv) $R(T^m) = R(T^{m+1})$ and $R(T^m)$ is closed for some $m \in \mathbb{N}$;
- (v) $\mathcal{N}(T) \subseteq \mathcal{R}(T)$ and $R(T)$ is closed.

Then the following assertions hold:

- (a) $\mathcal{R}(T) = K(T) = {}^\perp H_0(T^*) = {}^\perp \mathcal{N}(T^*)$;
- (b) $\mathcal{R}(T^*) = K(T^*) = H_0(T)^\perp = \mathcal{N}(T)^\perp$;
- (c) $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\} \Leftrightarrow T$ has SVEP at 0;
- (d) $\mathcal{N}(T^*) \cap \mathcal{R}(T^*) = \{0\} \Leftrightarrow T^*$ has SVEP at 0;
- (e) $\overline{\mathcal{N}(T) + R(T)} = X \Leftrightarrow H_0(T) + R(T) = X \Leftrightarrow T^*$ has SVEP at 0;
- (f) $\overline{\mathcal{N}(T^*) + R(T^*)}^{w^*} = X^* \Leftrightarrow \overline{H_0(T^*) + R(T^*)}^{w^*} = X^* \Leftrightarrow T$ has SVEP at 0,

where w^* indicates the closure with respect to the weak- $*$ -topology.

PROOF. (a) It is well known that each of the three conditions (i), (ii) and (v) ensures that all powers of T have closed range; see, for instance, proposition 3.1.5 of [8] in the semi-regular case. Also, by proposition 4.10.4 of [8], conditions (iii) and (iv) imply that $R(T^n)$ is closed for each $n \geq m$. In particular, it follows that, in all five cases, $\mathcal{R}(T)$ is closed in X . On the other hand, by a purely algebraic observation, theorem 2.3 of [2], each of the conditions (i)–(v) implies that $T\mathcal{R}(T) = \mathcal{R}(T)$. Consequently, the restriction $S := T|_{\mathcal{R}(T)}$ is a surjection on the Banach space $\mathcal{R}(T)$. From a basic property of the analytic core mentioned above we conclude that $\mathcal{R}(T) = K(S) \subseteq K(T)$, and hence, by Proposition 7, $\mathcal{R}(T) \subseteq K(T) \subseteq {}^\perp H_0(T^*) \subseteq {}^\perp \mathcal{N}(T^*)$. Finally, since $R(T^n)$ is closed for all but finitely many $n \in \mathbb{N}$, we obtain that $R(T^n) = {}^\perp N(T^{*n})$ for almost all $n \in \mathbb{N}$, and therefore $\mathcal{R}(T) = {}^\perp \mathcal{N}(T^*)$.

(b) By the closed range theorem, in all five cases, the adjoint of T^n has weak- $*$ -closed range for all sufficiently large $n \in \mathbb{N}$; see theorem A.1.10 of [8]. Moreover, it is well known and easily seen that T^* satisfies one of the conditions (i)–(v) precisely when T does. In fact, conditions (i) and (ii) are dual to each other, and so are (iii) and (iv). Hence, to establish assertion (b), one may proceed exactly as in the preceding paragraph.

(c) and (d) are clear from (a) and (b) and Corollary 6.

(e) By Proposition 7, it remains to be seen that $\mathcal{N}(T) + R(T)$ is dense in X provided that T^* has SVEP at 0. But this is clear from the Hahn–Banach theorem, since $\mathcal{N}(T)^\perp \cap R(T)^\perp = K(T^*) \cap N(T^*) = \{0\}$, by part (b) and Corollary 2.

(f) The proof of the last assertion is similar and therefore omitted. ■

To shed further light on the assertions (a) and (e) of the preceding theorem, we consider, as in example 1.4.12 of [8], the Volterra operator T on the Banach space $C([0, 1])$, given by

$$(Tf)(t) := \int_0^t f(s) \, ds \quad \text{for all } f \in C([0, 1]) \text{ and } t \in [0, 1].$$

Being quasi-nilpotent, T has SVEP and satisfies $K(T) = \{0\}$. However, since $\mathcal{R}(T)$ consists precisely of all $f \in C^\infty([0, 1])$ for which $f^{(n)}(0) = 0$ for every integer $n \geq 0$, it follows that $\mathcal{R}(T)$ is non-closed and strictly larger than $K(T)$. Moreover, $\mathcal{N}(T) + R(T)$ fails to be dense in $C([0, 1])$, although T^* is quasi-nilpotent, and hence has SVEP.

Further developments in connection with semi-Fredholm and semi-regular operators may be found in [1–3; 7; 10; 15]. Here we shall only need a simple consequence of Theorem 9 for semi-regular operators, which was already noted in [2]. For $T \in L(X)$, let $\rho_K(T)$ consist of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is semi-regular. The *Kato spectrum* $\sigma_K(T) := \mathbb{C} \setminus \rho_K(T)$ is a closed subset of $\sigma(T)$ and contains $\partial\sigma(T)$; see [8; 11] for details.

Corollary 10. *Let $T \in L(X)$ be semi-regular. Then T has SVEP at 0 precisely when T is injective, while T^* has SVEP at 0 precisely when T is surjective.*

Moreover, for arbitrary $T \in L(X)$, each connected component Ω of $\rho_K(T)$ satisfies either $\Omega \subseteq \mathfrak{S}(T)$ or $\Omega \cap \mathfrak{S}(T) = \emptyset$.

PROOF. Since $\mathcal{N}(T) \cap \mathcal{R}(T) = \mathcal{N}(T)$ and $\mathcal{N}(T) + R(T) = R(T) = \overline{R(T)}$ whenever T is semi-regular, the first assertions are clear from parts (c) and (e) of Theorem 9. To establish the last claim, it remains to be shown that injectivity of $T - \lambda I$ for some $\lambda \in \Omega$ implies that $T - \mu I$ is injective for all $\mu \in \Omega$. But this is immediate, since, by part (b) of Theorem 9, $\mathcal{N}(T - \mu I) = {}^\perp \mathcal{R}(T^* - \mu I)$ and, by propositions 3.1.6 and 3.1.11 of [8], $\mathcal{R}(T^* - \mu I) = \mathcal{R}(T^* - \lambda I)$ for all $\mu \in \Omega$. ■

5. Isometries, analytic Toeplitz and composition operators

In this section, we identify, for some general and concrete classes of operators, the set of points at which SVEP occurs. It will be useful to focus on certain parts of the spectrum. As usual, let $\sigma_{\text{ap}}(T)$ denote the approximate point spectrum of T , and let

$$\sigma_{\text{su}}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$$

be the *surjectivity spectrum* of T , also known as the approximate defect spectrum. These notions are dual to each other, in the sense that $\sigma_{\text{ap}}(T) = \sigma_{\text{su}}(T^*)$ and

$\sigma_{\text{su}}(T) = \sigma_{\text{ap}}(T^*)$; see proposition 1.3.1 of [8]. Moreover, by proposition 1.3.2 and corollary 3.1.7 of [8], we have $\sigma_{\text{su}}(T) = \sigma(T)$ and $\sigma_{\text{ap}}(T) = \sigma_{\text{K}}(T)$ if T has SVEP, and $\sigma_{\text{ap}}(T) = \sigma(T)$ and $\sigma_{\text{su}}(T) = \sigma_{\text{K}}(T)$ if T^* has SVEP. We obtain the following local version of these results.

Proposition 11. *For every operator $T \in L(X)$, the following assertions hold:*

- (a) *if $\lambda \in \sigma(T) \setminus \sigma_{\text{ap}}(T)$, then T has SVEP at λ , but T^* fails to have SVEP at λ ;*
- (b) *if $\lambda \in \sigma(T) \setminus \sigma_{\text{su}}(T)$, then T^* has SVEP at λ , but T fails to have SVEP at λ .*

PROOF. By a standard characterisation of the approximate point spectrum, the condition that $\lambda \in \sigma(T) \setminus \sigma_{\text{ap}}(T)$ means precisely that the operator $T - \lambda I$ has closed range, and is injective, but not surjective; see proposition 1.2.3 of [8]. In particular, it follows that $T - \lambda I$ is semi-regular. Hence assertion (a) is clear from Corollary 10. Similarly, if $\lambda \in \sigma(T) \setminus \sigma_{\text{su}}(T)$, then $T - \lambda I$ is surjective and hence semi-regular, but not injective. Consequently, assertion (b) follows again from Corollary 10. ■

Proposition 12. *Let $T \in L(X)$ be an operator for which $\sigma_{\text{ap}}(T) \subseteq \partial\sigma(T)$, and hence $\sigma_{\text{ap}}(T) = \partial\sigma(T)$. Then T has SVEP, while $\mathfrak{S}(T^*)$ coincides with the interior of $\sigma(T)$. On the other hand, if $\sigma_{\text{su}}(T) \subseteq \partial\sigma(T)$, then T^* has SVEP, while $\mathfrak{S}(T)$ is the interior of $\sigma(T)$.*

PROOF. Since both T and T^* have SVEP at each point of $\overline{\rho(T)}$, it remains to consider what happens at a point λ in the interior of $\sigma(T)$. If $\sigma_{\text{ap}}(T) \subseteq \partial\sigma(T)$, then it follows that $\lambda \in \sigma(T) \setminus \sigma_{\text{ap}}(T)$. By part (a) of Proposition 11, we conclude that T has SVEP at λ , while T^* does not have SVEP at λ . Thus T has SVEP at every point $\lambda \in \mathbb{C}$, while T^* has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $\lambda \in \overline{\rho(T)}$. Similarly, the last claim follows easily from part (b) of Proposition 11. ■

In the following, let $\nabla(\lambda, r)$ denote the closed disc with center $\lambda \in \mathbb{C}$ and radius $r \geq 0$, and let $V(\lambda, r)$ denote the corresponding open disc.

The last result has a number of interesting applications. As one of the less obvious ones, we mention the Cesàro operator on the classical Hardy space $H^p(\mathbb{D})$ for $1 < p < \infty$. Indeed, as established in [12], the approximate point spectrum of this operator is the boundary of $\nabla(p/2, p/2)$, while the spectrum is the entire disc $\nabla(p/2, p/2)$. Hence, by Proposition 12, the adjoint of the Cesàro operator does not have SVEP at any point in the open disc $V(p/2, p/2)$. Moreover, it follows that the Cesàro operator has SVEP, but, as shown in [12], this operator is actually the restriction of some decomposable operator.

To obtain further information, let $r(T)$ denote the spectral radius of an operator $T \in L(X)$, let $\kappa(T) := \inf\{\|Tx\| : x \in X \text{ with } \|x\| = 1\}$ be the lower bound of T , and let

$$i(T) := \lim_{n \rightarrow \infty} \kappa(T^n)^{1/n} = \sup_{n \in \mathbb{N}} \kappa(T^n)^{1/n}.$$

It is immediate that $i(T) \leq r(T)$.

Proposition 13. *Let $T \in L(X)$ be an arbitrary operator, and suppose that $\lambda \in \mathbb{C}$ is a point for which $|\lambda| < i(T)$. Then T has SVEP at λ , while T^* has SVEP at λ precisely when T is invertible. In particular, if $i(T) = r(T)$, then the following dichotomy holds:*

- (a) *if T is invertible, then both T and T^* have SVEP;*
- (b) *if T is non-invertible, then T has SVEP, while T^* has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $|\lambda| \geq r(T)$. Moreover, $\sigma(T) = \nabla(0, r(T))$ and $\sigma_{\text{ap}}(T) = \partial\sigma(T)$.*

PROOF. From proposition 1.6.2 of [8] we know that $\sigma_{\text{ap}}(T)$ is contained in the possibly degenerate annulus $\{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\}$. By the same result, we also have $\nabla(0, i(T)) \subseteq \sigma(T)$ if T is non-invertible, while $V(0, i(T)) \subseteq \rho(T)$ if T is invertible. Hence the assertions follow from Proposition 11. ■

The preceding result applies, in particular, to an arbitrary isometry $T \in L(X)$, since, in this case, $i(T) = r(T) = 1$. We conclude that every isometry has SVEP, while the adjoint of a non-invertible isometry has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $|\lambda| \geq 1$. We mention in passing that an invertible isometry is, in fact, decomposable, and that each isometry is, by a result due to Douglas, the restriction of some invertible isometry, see proposition 1.6.6 of [8]. Thus all isometries are restrictions of decomposable operators.

The question of the extent to which an operator or its adjoint may have SVEP at the points inside the approximate point or surjectivity spectrum is a more delicate issue. In the next section, we shall provide a complete answer in the case of unilateral and bilateral weighted shifts. The following result will allow us to settle the question in the case of certain analytic Toeplitz operators.

We shall need the fact that the approximate point spectrum behaves canonically under the Riesz functional calculus. It is interesting to note that this result follows readily from local spectral theory. Indeed, let $T \in L(X)$ be an arbitrary operator, let $f : U \rightarrow \mathbb{C}$ be analytic on an open neighbourhood U of $\sigma(T)$, and suppose that f is non-constant on each connected component of U . Then, as noted above, theorem 3.3.8 of [8] ensures that $\sigma_{f(T)}(x) = f(\sigma_T(x))$ for all $x \in X$. Taking the union over all $x \in X$, we conclude from proposition 1.3.2 of [8] that $\sigma_{\text{su}}(f(T)) = f(\sigma_{\text{su}}(T))$. Thus, by duality, $\sigma_{\text{ap}}(f(T)) = \sigma_{\text{su}}(f(T)^*) = \sigma_{\text{su}}(f(T^*)) = f(\sigma_{\text{su}}(T^*)) = f(\sigma_{\text{ap}}(T))$, as desired.

Proposition 14. *Let $T \in L(X)$ be a non-invertible isometry, and let $f : U \rightarrow \mathbb{C}$ be a non-constant analytic function on some connected open neighbourhood U of the closed unit disc. Then the following assertions hold:*

- (a) $\sigma(f(T)) = f(\overline{\mathbb{D}})$ and $\sigma_{\text{ap}}(f(T)) = f(\partial\mathbb{D})$;
- (b) $f(T)$ is the restriction of a decomposable operator, and hence has SVEP;
- (c) $f(T)^*$ has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $\lambda \notin f(\mathbb{D})$;
- (d) $f(\partial\mathbb{D}) \cap f(\mathbb{D}) = \{\lambda \in \sigma_{\text{ap}}(f(T)) : f(T)^* \text{ does not have SVEP at } \lambda\}$.

PROOF. Since $\sigma(T) = \overline{\mathbb{D}}$ and $\sigma_{\text{ap}}(T) = \partial\mathbb{D}$, assertion (a) is clear from the spectral mapping properties of the spectrum and the approximate point spectrum, while (b) follows from the corresponding result for isometries and the fact that the Riesz

functional calculus preserves decomposability; see theorem 3.3.6 of [8]. To establish (c), we recall that $f(T)^* = f(T^*)$. Hence Theorem 5 ensures that $f(T)^*$ has SVEP at a point $\lambda \in \mathbb{C}$ if and only if T^* has SVEP at each point $\mu \in U$ for which $f(\mu) = \lambda$. By the consequence of Proposition 13 for non-invertible isometries mentioned above, the latter condition holds precisely when $\lambda \notin f(\mathbb{D})$. The final assertion is clear from (a) and (c). ■

Assertion (d) of the preceding result leads to many simple examples in which SVEP for the adjoint fails to hold at points of the approximate point spectrum. For instance, $f(\partial \mathbb{D}) \cap f(\mathbb{D})$ is non-empty whenever $f(\lambda) = (\lambda - \gamma)(\lambda - \omega)g(\lambda)$ for all $\lambda \in U$, where $\gamma \in \partial \mathbb{D}$, $\omega \in \mathbb{D}$, and g is an arbitrary analytic function on U .

Proposition 14 applies canonically to the case of analytic Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ for a large class of symbols. Indeed, if f is a non-constant analytic function on some connected open neighbourhood of the closed unit disc, and if M_f denotes the operator of multiplication by f on $H^2(\mathbb{D})$, then it is clear that $M_f = f(T)$, where T denotes the operator of multiplication by the independent variable, that is $(Tu)(\lambda) := \lambda u(\lambda)$ for all $u \in H^2(\mathbb{D})$ and $\lambda \in \mathbb{D}$. Since T is unitarily equivalent to the unilateral right shift on $\ell^2(\mathbb{N})$, and hence a non-invertible isometry, Proposition 14 shows, in particular, that $\sigma(M_f) = f(\overline{\mathbb{D}})$ and $\sigma_{\text{ap}}(M_f) = f(\partial \mathbb{D})$, and that M_f^* has SVEP at a point $\lambda \in \mathbb{C}$ exactly when $\lambda \notin f(\mathbb{D})$.

Similar results hold for Toeplitz operators with arbitrary bounded analytic symbols. More precisely, if $f \in H^\infty(\mathbb{D})$, then M_f is subnormal, $\sigma_{\text{ap}}(M_f)$ coincides with the essential range of the boundary function that is obtained by taking non-tangential limits of f almost everywhere on the unit circle, and M_f^* does not have SVEP at any point $\lambda \in f(\mathbb{D})$. These results may be established using standard tools from the theory of Hardy spaces; the details are left to the interested reader.

We close this section with an example from the theory of composition operators on Hardy spaces; see [13; 17]. It is well known that every analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ induces, by composition, a bounded linear operator on $H^2(\mathbb{D})$, and that this operator is invertible if and only if φ is an automorphism of the unit disc, i.e. a mapping of the form

$$\varphi(\lambda) = \frac{a\lambda + \bar{b}}{b\lambda + \bar{a}} \quad \text{for all } \lambda \in \mathbb{D},$$

where a and b are complex numbers for which $|a|^2 - |b|^2 = 1$. These automorphisms are classified as follows: φ is *elliptic* if $|\operatorname{Im} a| > |b|$, *parabolic* if $|\operatorname{Im} a| = |b|$ and *hyperbolic* if $|\operatorname{Im} a| < |b|$. The next result follows easily from an inspection of the proofs of theorem 6 of [13] and theorems 1.4 and 2.3 of [17].

Proposition 15. *Let φ be a hyperbolic automorphism of the unit disc, and let T denote the corresponding composition operator on $H^2(\mathbb{D})$ given by $Tu := u \circ \varphi$ for all $u \in H^2(\mathbb{D})$. Then $\sigma(T) = \{\lambda \in \mathbb{C} : 1/r \leq |\lambda| \leq r\}$ for some $r > 1$, and $\mathfrak{S}(T)$ coincides with the interior of this annulus, while T^* is subnormal, and hence has SVEP.*

By contrast, if an automorphism of \mathbb{D} is either elliptic or parabolic, then, as shown in [17], the corresponding composition operator is generalised scalar on $H^p(\mathbb{D})$ for arbitrary $1 \leq p < \infty$. Thus, in the elliptic and the parabolic case, both the composition operator and its adjoint have SVEP.

6. Unilateral and bilateral weighted shifts

Evidently, Proposition 14 applies to the unilateral right shift on the sequence space $\ell^p(\mathbb{N})$ for arbitrary $1 \leq p \leq \infty$. In this section, we explore the question of SVEP for both unilateral and bilateral weighted shifts. For a thorough discussion of the basic theory of weighted shifts in the Hilbert space setting, we refer to [16].

Let $1 \leq p < \infty$ be given, and let $\omega = (\omega_n)_{n \in \mathbb{N}}$ be a bounded sequence of strictly positive real numbers. The corresponding *unilateral weighted right shift* on $\ell^p(\mathbb{N})$ is the operator given by

$$Tx := \sum_{n=1}^{\infty} \omega_n x_n e_{n+1} \quad \text{for all } x = (x_n)_{n \in \mathbb{N}} \in \ell^p(\mathbb{N}),$$

where $(e_n)_{n \in \mathbb{N}}$ stands for the canonical basis of $\ell^p(\mathbb{N})$. Since T has no eigenvalues, it is immediate that T has SVEP. Moreover, from proposition 1.6.15 of [8], a slight extension of a classical result due to Ridge [14], we know that

$$\sigma(T) = \nabla(0, r(T)) \quad \text{and} \quad \sigma_{\text{ap}}(T) = \{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\},$$

where $i(T)$ and $r(T)$ may be computed as

$$i(T) = \lim_{n \rightarrow \infty} \inf_{k \in \mathbb{N}} (\omega_k \cdots \omega_{k+n-1})^{1/n} \quad \text{and} \quad r(T) = \lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} (\omega_k \cdots \omega_{k+n-1})^{1/n}.$$

Also, as shown in example 3.7.7 of [8], the essential spectrum $\sigma_e(T)$ and the Kato spectrum $\sigma_K(T)$ both coincide with the annulus $\sigma_{\text{ap}}(T)$.

It is well known and easily seen that the adjoint of T is the unilateral weighted left shift on $\ell^q(\mathbb{N})$ given by

$$T^*x = (\omega_n x_{n+1})_{n \in \mathbb{N}} \quad \text{for all } x = (x_n)_{n \in \mathbb{N}} \in \ell^q(\mathbb{N}),$$

where, as usual, $1/p + 1/q = 1$, and $\ell^q(\mathbb{N})$ is canonically identified with the dual space of $\ell^p(\mathbb{N})$. From Proposition 13 we infer that T^* does not have SVEP whenever $i(T) > 0$. To discuss the question of SVEP for T^* more thoroughly, we need the quantity

$$c(T) := \lim_{n \rightarrow \infty} \inf (\omega_1 \cdots \omega_n)^{1/n}.$$

Evidently, $i(T) \leq c(T) \leq r(T)$.

Theorem 16. *Let T be a unilateral weighted right shift on $\ell^p(\mathbb{N})$ for some $1 \leq p < \infty$. Then T has SVEP, while T^* has SVEP at a point $\lambda \in \mathbb{C}$ precisely when $|\lambda| \geq c(T)$. In particular, T^* has SVEP if and only if $c(T) = 0$.*

PROOF. By the standard formula for the radius of convergence of a vector-valued power series, it is clear that, for each $\lambda \in V(0, c(T))$, the infinite series

$$f(\lambda) := \sum_{n=1}^{\infty} e_n \lambda^{n-1} / (\omega_1 \cdots \omega_{n-1})$$

converges in $\ell^q(\mathbb{N})$. Moreover, this series defines an analytic function f on the disc $V(0, c(T))$. Since it is easily checked that $(T^* - \lambda I)f(\lambda) = 0$ for all $\lambda \in V(0, c(T))$, we conclude that $\mathfrak{S}(T^*) \subseteq V(0, c(T))$.

On the other hand, it is not difficult to verify that T^* has no eigenvalues outside the closed disc $\mathbb{V}(0, c(T))$. Evidently this implies that T^* has SVEP at every point λ for which $|\lambda| \geq c(T)$, and hence completes the argument. ■

The significance of the preceding result arises from the fact that, for every triple of real numbers i , c and r for which $0 \leq i \leq c \leq r$, it is possible to find a unilateral weighted right shift T on $\ell^p(\mathbb{N})$ such that $i(T) = i$, $c(T) = c$ and $r(T) = r$; see lemma 7 of [14] for an outline of the construction of a suitable weight sequence ω .

In particular, it follows that there exists a unilateral weighted right shift T on $\ell^p(\mathbb{N})$ for which both $i(T) = 0$ and $c(T) > 0$. From the preceding results we conclude that T provides an example of an operator with SVEP for which

$$\sigma(T) = \sigma_{\text{ap}}(T) = \sigma_{\text{su}}(T) = \sigma_{\text{e}}(T) = \sigma_{\text{K}}(T),$$

although T^* fails to have SVEP.

Moreover, for every unilateral weighted right shift T , we have $e_1 \in N(T^*) \cap \mathcal{R}(T^*)$. Thus $\mathcal{N}(T^*) \cap \mathcal{R}(T^*)$ is always non-trivial, whereas T^* has SVEP at 0 precisely when $c(T) = 0$. In particular, if $r(T) = 0$, then T^* is decomposable and has SVEP, but $\mathcal{N}(T^*) \cap \mathcal{R}(T^*) \neq \{0\}$. This illustrates that the second implication of Corollary 3 cannot be reversed in general, and that the closed range condition is essential in parts (c) and (d) of Theorem 9.

As another consequence of Theorem 16, we can now exemplify that the converse of the results provided in Proposition 7 fails to be true in general. In fact, the following result shows that the implications from assertions (b) and (c) of Proposition 7 need not be equivalences, and therefore that the inclusions from part (a) need not be identities.

Proposition 17. *Let $1 \leq p < \infty$, and let T be an arbitrary weighted right shift on $\ell^p(\mathbb{N})$ with weight sequence ω . Then $H_0(T) + R(T)$ is norm-dense in $\ell^p(\mathbb{N})$ if and only if*

$$\limsup_{n \rightarrow \infty} (\omega_1 \cdots \omega_n)^{1/n} = 0,$$

while T^* has SVEP at 0 if and only if

$$\liminf_{n \rightarrow \infty} (\omega_1 \cdots \omega_n)^{1/n} = 0.$$

PROOF. In view of Theorem 16, it remains to establish the first equivalence. Since $\|T^n e_1\| = \omega_1 \cdots \omega_n$ for all $n \in \mathbb{N}$, the identity

$$\limsup_{n \rightarrow \infty} (\omega_1 \cdots \omega_n)^{1/n} = 0$$

holds precisely when $e_1 \in H_0(T)$. Hence this condition entails that $H_0(T) + R(T)$ is dense in $\ell^p(\mathbb{N})$, because $e_n \in R(T)$ for all $n \geq 2$.

Conversely, suppose that the sum $H_0(T) + R(T)$ is dense in $\ell^p(\mathbb{N})$, and choose elements $u_k \in H_0(T)$ and $v_k \in R(T)$ for all $k \in \mathbb{N}$ such that $u_k + v_k \rightarrow e_1$ as $k \rightarrow \infty$. If P denotes the projection on $\ell^p(\mathbb{N})$ given by $P(x) := x_1 e_1$ for all $x \in \ell^p(\mathbb{N})$, then it is clear that P vanishes on $R(T)$ and leaves $H_0(T)$ invariant. Moreover, the space $H_0(T) \cap R(P)$ is closed, since its dimension is at most one. Because $P(u_k) = P(u_k + v_k) \rightarrow P(e_1) = e_1$ as $k \rightarrow \infty$, we thus conclude that $e_1 \in H_0(T)$, as desired. ■

Since $\mathcal{N}(T) = \mathcal{R}(T) = K(T) = \{0\}$ for every unilateral weighted right shift T on $\ell^p(\mathbb{N})$, we see that, for this class of operators, the implications from Corollary 8 are considerably weaker than those provided in Proposition 7. Again, it follows that the closed range condition is essential in Theorem 9. Note that, for a unilateral weighted right shift T , it is easy to check when the range is closed. Indeed, since T is injective, the annulus formula for $\sigma_{\text{ap}}(T)$ mentioned above ensures that the range of T , and hence of all of its powers, is closed precisely when $i(T) > 0$.

We note in passing that decomposability is a rare phenomenon among unilateral weighted shifts. In fact, by proposition 1.6.14 and theorem 2.5.7 of [8], a unilateral weighted right shift on $\ell^p(\mathbb{N})$, or, equivalently, the corresponding unilateral weighted left shift on $\ell^q(\mathbb{N})$, is decomposable if and only if $r(T) = 0$. However, the characterisation of those unilateral weighted shifts which are similar to restrictions of decomposable operators remains a challenging open problem. For partial results, see [8, §1.6].

We conclude with a brief discussion of the bilateral case. For a two-sided bounded sequence $\omega = (\omega_n)_{n \in \mathbb{Z}}$ of strictly positive real numbers, the corresponding *bilateral weighted right shift* on $\ell^p(\mathbb{Z})$ for $1 \leq p \leq \infty$ is defined by

$$Tx := (\omega_{n-1} x_{n-1})_{n \in \mathbb{Z}} \quad \text{for all } x = (x_n)_{n \in \mathbb{Z}} \in \ell^p(\mathbb{Z}).$$

Contrary to the unilateral case, T may well have eigenvalues. In fact, we shall see that T need not have SVEP. To formulate the bilateral counterpart of Theorem 16, we define $\alpha_0 := 1$, $\alpha_n := \omega_0 \cdots \omega_{n-1}$, and $\alpha_{-n} := \omega_{-n} \cdots \omega_{-1}$ for all $n \in \mathbb{N}$, and let

$$c^\pm(T) := \liminf_{n \rightarrow \infty} \alpha_{\pm n}^{1/n} \quad \text{and} \quad d^\pm(T) := \limsup_{n \rightarrow \infty} \alpha_{\pm n}^{1/n}.$$

Theorem 18. *Let T be a bilateral weighted right shift on $\ell^p(\mathbb{Z})$ for some $1 \leq p \leq \infty$. Then $\mathfrak{S}(T) = \{\lambda \in \mathbb{C} : d^+(T) < |\lambda| < c^-(T)\}$. In particular, T has SVEP precisely when $c^-(T) \leq d^+(T)$.*

PROOF. First, suppose that λ is an eigenvalue of T , and consider a corresponding non-zero eigenvector $x \in \ell^p(\mathbb{Z})$. Then a simple computation shows that $\lambda \neq 0$ and that the equations $x_n = x_0 \alpha_n / \lambda^n$ and $x_{-n} = x_0 \lambda^n / \alpha_{-n}$ hold for every $n \in \mathbb{N}$. Because $x \in \ell^p(\mathbb{Z})$, it follows that $d^+(T) \leq |\lambda| \leq c^-(T)$.

On the other hand, if $d^+(T) < c^-(T)$, then, as in the proof of Theorem 16, the classical formula for the radius of convergence guarantees that the definition

$$f(\lambda) := \sum_{n=0}^{\infty} e_n \alpha_n / \lambda^n + \sum_{n=1}^{\infty} e_{-n} \lambda^n / \alpha_{-n}$$

for all $\lambda \in \mathbb{C}$ with $d^+(T) < |\lambda| < c^-(T)$ yields an analytic solution of the equation $(T - \lambda I)f(\lambda) = 0$ on the annulus $\{\lambda \in \mathbb{C} : d^+(T) < |\lambda| < c^-(T)\}$. ■

Corollary 19. *For every bilateral weighted right shift T on $\ell^p(\mathbb{Z})$ for some $1 \leq p < \infty$, the following assertions hold:*

- (a) $\mathfrak{S}(T^*) = \{\lambda \in \mathbb{C} : d^-(T) < |\lambda| < c^+(T)\}$;
- (b) T^* has SVEP precisely when $c^+(T) \leq d^-(T)$;
- (c) at least one of the operators T or T^* has SVEP.

PROOF. Evidently the adjoint of T is the bilateral weighted left shift on $\ell^q(\mathbb{Z})$ given by

$$T^*x = (\omega_n x_{n+1})_{n \in \mathbb{Z}} \quad \text{for all } x = (x_n)_{n \in \mathbb{Z}} \in \ell^q(\mathbb{Z}),$$

where $1/p + 1/q = 1$. Moreover, with the choice $\hat{\omega} := (\omega_{-n-1})_{n \in \mathbb{Z}}$ and $S(x) := (x_{-n})_{n \in \mathbb{Z}}$ for all $x = (x_n)_{n \in \mathbb{Z}} \in \ell^q(\mathbb{Z})$, it follows that

$$(ST^*S)(x) = (\hat{\omega}_{n-1} x_{n-1})_{n \in \mathbb{Z}} \quad \text{for all } x = (x_n)_{n \in \mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

This shows that T^* is similar to the bilateral weighted right shift on $\ell^q(\mathbb{Z})$ with weight sequence $\hat{\omega}$. In the sense of the right shift representation of T^* , we obtain the identities $c^\pm(T^*) = c^\mp(T)$ and $d^\pm(T^*) = d^\mp(T)$, because $\hat{\omega}_n = \alpha_{-n}$ for all $n \in \mathbb{Z}$. Hence assertion (a) is clear from Theorem 18, and (b) is an immediate consequence of (a).

Finally, to prove (c), assume that both T and T^* fail to have SVEP. Then the preceding results entail that $d^+(T) < c^-(T)$ and $d^-(T) < c^+(T)$. But this leads to an obvious contradiction, since $c^-(T) \leq d^-(T)$ and $c^+(T) \leq d^+(T)$. ■

In part (c) of the preceding result, it is possible that both T and T^* have SVEP. In fact, there are many examples of decomposable bilateral weighted shifts beyond the quasi-nilpotent ones. However, the precise characterisation of those weight sequences for which the corresponding bilateral shift is decomposable on $\ell^p(\mathbb{Z})$, or, alternatively, similar to the restriction of some decomposable operator, appears to be an interesting open problem.

Finally, the results obtained so far seem to indicate that the local spectral properties of weighted shifts do not depend on the parameter p for $1 \leq p < \infty$, but the theory beyond SVEP is far from being settled.

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