HOLOMORPHIC FUNCTIONS ON (GENERALISED) LOOP SPACES

By László Lempert*
Department of Mathematics, Purdue University, West Lafayette, Indiana

[Received 14 December 2001. Read 12 September 2002. Published November 2004.]

Abstract

Let $\mathbb{P}_n$ denote n dimensional complex projective space. Given $k = 0,1,\ldots,\infty$, a map $x_0 : S^1 \to \mathbb{P}_n$ of class C^k, and a closed subset $A \subset S^1$, consider the space Y of those C^k maps $x : S^1 \to \mathbb{P}_n$ that agree with x_0 along A. The space Y can be endowed with the structure of an infinite dimensional complex manifold, and our main result identifies the space of holomorphic functions on Y.

1. Introduction

Let M be a complex manifold, dim $M = n < \infty$ for the sake of simplicity, and fix $k = 0, 1, 2, \ldots, \infty, \omega$. Maps of the circle S^1 into M, of class C^k, constitute the C^k loop space X of M. As usual, C^ω refers to the class of real analytic maps. It is known that X carries the structure of an infinite dimensional complex manifold, modelled on the locally convex topological vector space of C^k maps $S^1 \to \mathbb{C}^n$.

Loop spaces can of course also be defined for real manifolds, and even for topological spaces (when $k = 0$); the study of their topology is an indispensable tool for understanding the topology of the underlying space M. It is suggested, e.g. by [5; 11; 13], that similarly, smooth or complex manifolds should be studied through the differential/complex geometry of their loop spaces. In this paper we shall deal with a few very simple questions concerning complex loop spaces X. Here is an overview of the contents.

In Section 2 we recall how the complex structure is defined on X and discuss its functoriality properties. In Section 3 we fix $x_0 \in X$ and a closed $A \subset S^1$, and introduce the submanifold

$$Y = \{x \in X : x|A = x_0|A\}$$

of 'based' loops. One checks that codim $Y = |A| \dim M \leq \infty$. Thus $Y \subset X$ is a hypersurface when $\dim M = 1$ and A is a singleton; for higher dimensional M, certainly for projective ones, one can construct hypersurfaces as total spaces of finite dimensional families of submanifolds of type Y. One motivation for considering these submanifolds comes precisely from their connection with hypersurfaces, divisors and line bundles.

The main issue in Section 3 is a construction of holomorphic functions on Y. Assuming $k \leq \infty$ we construct a Fréchet space J and a surjective holomorphic map

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*E-mail: lempert@math.perdue.edu

1991 *Mathematics Subject Classification*: 58B12, 58C10, 58D15.
Research partially supported by an NSF grant.

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$j\colon Y \to J$ (Theorem 3.2). Thus holomorphic functions on J give rise to holomorphic functions on Y. The space J is zero-dimensional when $k = 0$ or A is dense in itself, so only constant functions will be obtained in this way; but otherwise $\dim J > 0$ and often $\dim J = \infty$.

In Section 4 we restrict our attention to loop spaces of complex projective spaces $M = \mathbb{P}_n$, and completely describe the space $\mathcal{O}(Y)$ of holomorphic functions on Y. Here is the background. When $A = \emptyset$, the family of compact complex submanifolds of $Y = X$ is large enough to ensure all $f \in \mathcal{O}(Y)$ are constant. When $A = \emptyset$, Y no longer contains compact complex submanifolds. In spite of this, Dineen and Mellon have recently made the surprising discovery that, when $k = 0$, $\mathcal{O}(Y)$ still consists of constants only, see [2].[1] To put this result in perspective, note that in finite dimensions the most general method to prove constancy of all holomorphic functions in a complex manifold Z is to play out the maximum principle against Weierstrass' maximum theorem. Loosely speaking, this approach requires that Z contain a large family of compact complex submanifolds, possibly with boundary; the boundaries, if not empty, should have some pseudoconcavity properties. As said, the loop spaces Y do not have such submanifolds when $A \neq \emptyset$ (see Theorem 3.4), so the Dineen–Mellon result amounts to discovering a new mechanism that forces holomorphic functions to be constant. (In finite dimensions there is also the Bochner/Kodaira method to prove constancy of harmonic tensor fields, based on positivity properties of a Laplacian in some L^2 space. When specialised to holomorphic functions, this approach is less powerful than the one using the maximum principle. Nor does it generalise to infinite dimensions, where appropriate L^2 spaces of functions can not be set up.)

The contribution of our Section 4 to this subject is the treatment of loop spaces Y when $k \geq 1$. We prove that for $k \leq \infty$ all holomorphic functions on Y are gotten by the construction discussed earlier: pulling back holomorphic functions on J by $j\colon Y \to J$ (Theorem 4.1). It follows that Y admits nonconstant holomorphic functions provided $k \geq 1$ and A has isolated points, the bigger k, the more functions; still, $\mathcal{O}(Y)$ will separate points of Y only if $k = \omega$.

The final Section 5 discusses related results and problems.

Most of what we have to say applies, instead of loops $S^1 \to M$, to maps of an arbitrary compact manifold, possibly with boundary, into M. Accordingly this will be our set up in the present paper, and this is what the adjective 'generalised' refers to in our title.

For the fundamentals of real and complex calculus, manifold theory, and complex analysis in infinite dimensions, see [1; 4; 7].

2. Mapping spaces as complex manifolds

Fix $k = 0, 1, \ldots, \infty, \omega$, and if $k \geq 1$, fix also a compact manifold V, possibly with boundary, of class C^k. If $k = 0$, take V an arbitrary compact Hausdorff space. Given furthermore a complex manifold $M, \dim\ M = n < \infty$, denote by $X = C^k(V, M)$ the space of C^k maps $x\colon V \to M$. When $k = 0$, the natural topology on X is the compact

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[1] Dineen and Mellon use a slightly different language and take x_0 constant. The more general result easily follows from their Theorems 7 and 11.

open topology; otherwise endow X with the C^k topology, where a neighbourhood of $x \in X$ is obtained by the following construction. Cover $x(V) \subset M$ by finitely many coordinate charts (M_i, α_i) of M with $\alpha_i: M_i \to \mathbb{C}^n$, and similarly V by coordinate charts (V_i, β_i), making sure that $x(\overline{V}_i) \subset M_i$. In the locally convex vector spaces $C^k(\beta_i(V_i), \mathbb{C}^n)$ choose neighbourhoods $\mathcal{U}_i$ of $\alpha_i \circ x \circ \beta_i^{-1}$, then a neighbourhood of $x \in X$ will be

$$\{y \in X : \alpha_i \circ y \circ \beta_i^{-1} \in \mathcal{U}_i \text{ for all } i\}.$$

It is well known that X can be endowed with the structure of a smooth manifold, see e.g. [4; 6; 9], even for real manifolds M and spaces of maps from regularity classes other than C^k. When M is a complex manifold, the constructions in the above cited works would lead to a complex manifold structure on X, at least when $k \leq \infty$, locally biholomorphic to locally convex vector spaces (i.e. X is a *rectifiable* complex manifold in the language of [7]). A special case, with X the loop space of $\mathbb{P}_1$, is explicitly worked out in [8]. The heart of the matter is the following lemma, where for a set $U \subset V \times M$ and $t \in V$ we write

$$U^t = \{p \in M : (t,p) \in U\} \quad \text{and} \quad \varepsilon^t : U^t \ni p \mapsto (t,p) \in U.$$

Lemma 2.1. *Given $x \in X$, there is a C^k diffeomorphism F between a neighbourhood $U \subset V \times M$ of the graph of x and a neighbourhood of the zero section in x^*TM such that*

(i) *$F(\cdot, x(\cdot))$ is the zero section of x^*TM;*

and for all t

(ii) *$F^t = F \circ \varepsilon^t$ maps U^t biholomorphically on a convex neighbourhood of $0 \in T_{x(t)}M$;*

(iii) *$dF^t(x(t))$ is the identity map $T_{x(t)}U^t = T_{x(t)}M \overset{\approx}{\to} T_{x(t)}M$.*

In (iii) we identified a tangent space of the vector space $T_{x(t)}M$ with $T_{x(t)}M$ itself.

PROOF. When $M = \mathbb{C}^n$ so that TM and $x^*TM = V \times \mathbb{C}^n$ are trivial, take $U = V \times M$ and $F(t,p) = (t, x(t) - p)$. Since a general M is locally biholomorphic to $\mathbb{C}^n$, in general we obtain an open covering $\{V_j\}$ of V, neighbourhoods $U_j \subset V_j \times M$ of the graph of $x | V_j$, and C^k diffeomorphisms F_j of U_j into $(x | V_j)^*TM$ as in the lemma. If $k \leq \infty$, take a C^k partition of unity $\{\chi_j\}$ subordinate to $\{V_j\}$; it is easy to check that the restriction F of $\sum_j \chi_j(t) F_j(t,p)$ to a suitable $U \subset \bigcup_j U_j$ will do.

The case $k = \omega$ is much deeper, but by now standard. Let $\tilde{V}$ be a complex manifold, $\dim_{\mathbb{C}} \tilde{V} = \dim_{\mathbb{R}} V$, that contains V as a totally real, real analytic submanifold with boundary. We can assume x extends to a holomorphic $\tilde{x}: \tilde{V} \to M$ and, according to [3], that $\tilde{V}$ is Stein. Suppose first that M is Stein too. Then $\tilde{V} \times M$ is Stein, hence embeddable into some $\mathbb{C}^N$; so we assume $\tilde{V} \times M \subset \mathbb{C}^N$. Following Siu, construct a Stein neighbourhood $\Omega \subset \mathbb{C}^N$ of $\tilde{V} \times M$ that has a holomorphic retraction on $\tilde{V} \times M$, cf. [10]. Consider the subbundle of $T\Omega | (\tilde{V} \times M)$ consisting of vectors tangent to leaves $\{t\} \times M$, and using the retraction $\Omega \to \tilde{V} \times M$ extend it to a subbundle $E \to \Omega$ of $T\Omega$. Denoting the identity map of $\tilde{V}$ by id, we have $(\mathrm{id}\times\tilde{x})^*E=\tilde{x}^*TM$. As a standard consequence of vanishing theorems on Ω there is a complementary subbundle $E^\perp\subset T\Omega, E\oplus E^\perp=T\Omega$.

Since the fibres of $T\Omega$ can be identified with $\mathbb{C}^N$, we shall also think of the fibres $E_w, E_w^\perp$ as subspaces of $\mathbb{C}^N$ that depend holomorphically on $w\in\mathbb{C}^N$. Note that if $w=(t,\tilde{x}(t))$ with $t\in\tilde{V}$, the affine subspace $w+E_w^\perp$ intersects $\{t\}\times M$ transversely at w. Therefore if w is close to $(t,\tilde{x}(t)), w+E_w^\perp$ will still intersect $\{t\}\times M$ in a unique point $g(t,w)$ close to $(t,\tilde{x}(t))$, which point depends holomorphically on t,w. This allows us to define a holomorphic map G of a neighbourhood of the zero section in $\tilde{x}^*TM=(\mathrm{id}\times\tilde{x})^*E$ into $\tilde{V}\times M$ by setting, for $v\in E_{(t,\tilde{x}(t))}\subset\mathbb{C}^N$ small, $w=(t,\tilde{x}(t))+v$ and $G(v)=g(t,w)\in\{t\}\times M$. The zero section of $(\mathrm{id}\times\tilde{x})^*E$ composed with G is the map $\mathrm{id}\times\tilde{x}$ and, at the points of the zero section, dG is the identity map. This implies G is biholomorphic in a neighbourhood of the zero section, and a suitable restriction of its inverse will be the map F sought.

If we drop the assumption that M is Stein, Siu's theorem still applies to produce a Stein neighbourhood $\tilde{U}\subset\tilde{V}\times M$ of the graph of $\tilde{x}$. Embedding $\tilde{U}$ rather than $\tilde{V}\times M$ into some $\mathbb{C}^N$, the construction above will give an F as required.

In order to define the rectifiable complex manifold structure on X we need to construct holomorphic charts. If $E\to V$ is a complex vector bundle of class C^k, denote by $C^k(E)$ the space of its C^k sections. This is a locally convex topological vector space; Banach (resp. Fréchet) if $k<\infty$ (resp. $k=\infty$). For $x\in X$ choose U, F as in Lemma 2.1. Those $y\in X$ whose graph $\{(t,y(t)):t\in V\}$ is contained in U form a neighbourhood $\mathcal{U}\subset X$ of x. With $y\in\mathcal{U}$ associate a section $\varphi(y)\in C^k(x^*TM)$ by putting $\varphi(y)(t)=F(t,y(t))$; thus $\varphi=\varphi_{x,F}$ is a homeomorphism between $\mathcal{U}$ and an open subset of $C^k(x^*TM)$. To see how another homeomorphism $\varphi'=\varphi_{x',F'}$ associated with $x'\in X$ and a corresponding F' is related to φ, observe that $F'\circ F^{-1}$ defines a C^k bundle morphism between open fibre subbundles of x^*TM and x'^*TM, holomorphic on the fibres. Composition with $F'\circ F^{-1}$ induces the map $\varphi'\circ\varphi^{-1}$ between open subsets of $C^k(x^*TM), C^k(x'^*TM)$; from this description it is straightforward that $\varphi'\circ\varphi^{-1}$ is holomorphic. Thus all the charts $(\mathcal{U},\varphi)$ associated with different x, F are compatible and define a complex manifold structure on X. For example, if $M=\mathbb{C}^n$, the complex structure is the one induced by the locally convex vector space structure of $X=C^k(V,\mathbb{C}^n)$.

Next we shall identify tangent spaces to X. Recall that a tangent vector in T_xX is an equivalence class of smooth curves $\mathbb{R}\ni\tau\mapsto x_\tau\in X, x_0=x$, two curves x_τ, y_τ being equivalent if $d(\varphi\circ x_\tau)/d\tau=d(\varphi\circ y_\tau)/d\tau$ at $\tau=0$, with $(\mathcal{U},\varphi)$ a chart about x. Given a curve x_τ as above, $dx_\tau/d\tau|_{\tau=0}$ defines a C^k section of x^*TM. The following is immediate:

Proposition 2.2. *The section* $dx_\tau/d\tau|_{\tau=0}\in C^k(x^*TM)$ *depends only on the equivalence class of the curve* x_τ. *The resulting map* $T_xX\to C^k(x^*TM)$ *is an isomorphism of locally convex vector spaces.*

A characteristic property of the complex structures introduced above is the following.

Proposition 2.3. *Let M, M' be complex manifolds, $X = C^k(V,M)$ and $X' = C^k(V,M')$ the corresponding complex manifolds of maps. Suppose $\Phi\colon V \times M' \to M$ is a C^k map, and $\Phi(t,\cdot)$ is holomorphic for all $t \in V$. Then the induced map $\Phi_*\colon X' \to X$ defined by $\Phi_*(x')(t) = \Phi(t, x'(t))$ is holomorphic.*

Proof. For $x' \in X'$ and $x = \Phi_*(x') \in X$ let F', F be as in Lemma 2.1, and set $\Psi = F \circ (\mathrm{id}_V \times \Phi) \circ (F')^{-1}$. Thus Ψ is a C^k fibre bundle map between neighbourhoods of the zero section in x'^*TM' (resp. x^*TM), holomorphic on fibres. As noted earlier, composition with such a Ψ induces a map Ψ_* between C^k sections of x'^*TM' and those of x^*TM, holomorphic where defined. Since in the charts $(\mathcal{U}, \varphi), (\mathcal{U}', \varphi')$ associated with x, F (resp. x', F'), the map Φ_* is given by Ψ_*, Φ_* is indeed holomorphic.

The functoriality property expressed in Proposition 2.3 in a sense characterizes the complex structures we introduced. The following is not hard to prove. Suppose for fixed k and V all spaces $X = C^k(V,M)$ are endowed with a complex structure so that the underlying differential structure agrees with the one induced by the complex structure defined above. Assume that when $M = \mathbb{C}^n$, the complex structure on X is the one induced by its locally convex vector space structure; and that for any Φ as in Proposition 2.3 the map Φ_* is holomorphic. Then the complex structure on the spaces X agrees with the one we introduced.

It is quite possible that such a result will be true even without the assumption on the underlying differential structures. A proof of this more general uniqueness result seems to depend on whether the inverse of a holomorphic homeomorphism between complex manifolds is necessarily holomorphic. In finite dimensions this is true, but in infinite dimensions it appears to be unknown.

The complex structures of mapping spaces are functorial in another sense as well. Suppose $l \leq k$, V' is a compact C^l manifold (a compact Hausdorff space if $l = 0$), and $\Psi\colon V' \to V$ a C^l map. Composition with Ψ induces a map Ψ^* between $X = C^k(V, M)$ and $X' = C^l(V', M)$. The following is easy to verify:

Proposition 2.4. *The map $\Psi^*\colon X \to X'$ is holomorphic.*

A special case of this, with V' a singleton, and the fact that the topology of X is finer than the compact open topology, imply

Proposition 2.5. *The evaluation map*

$$e\colon V \times X \ni (t,x) \mapsto x(t) \in M$$

is continuous, and holomorphic in x for fixed t.

3. Submanifolds

We shall consider two kinds of submanifolds of mapping spaces $X = C^k(V, M)$. Fix $x_0 \in X$ and a closed set $A \subset V$. Denote the k-jet of $x \in X$ by $j^k x$ and let

$$\begin{aligned} Y &= Y(k, V, M, A, x_0) = \{x \in X : x|A = x_0|A\}, \\ Z &= Z(k, V, M, A, x_0) = \{x \in X : j^k x|A = j^k x_0|A\}. \end{aligned} \tag{3.1}$$

These are closed subsets of X; and even complex submanifolds. Indeed, if $x \in Y$ resp. Z, and F is as in Lemma 2.1, in the chart $(\mathscr{U}, \varphi_{x,F})$ introduced in Section 2 the set $Y \cap \mathscr{U}$ (resp. $Z \cap \mathscr{U}$) corresponds to the space of those sections in $C^k(x^*TM)$ that vanish on A, (resp. along with their derivatives of order $\leq k$ vanish on A). Denote these spaces by $C^k_A(x^*TM)$ (resp. $C^k_{(A)}(x^*TM)$). Under the isomorphism $T_xX \xrightarrow{\sim} C^k(x^*TM)$ of Proposition 2.2 T_xY and T_xZ correspond to $C^k_A(x^*TM)$ and $C^k_{(A)}(x^*TM)$.

When A is finite, codim $Y = |A| \dim M$. When $k < \infty$ as well, Z also is of finite codimension; in all other cases codim $Y = \infty$ and codim $Z = \infty$.

There are natural maps among the restricted mapping spaces Y, Z that do not come from the maps Φ_*, Ψ^* discussed in Propositions 2.3 and 2.4. Let $V' \subset V$ be a compact region with C^k boundary (when $k = 0$, just a compact subset), and put $A' = A \cap V'$. If the boundary of V' (relative to V) is contained in A' (resp. in the relative interior of $A' \cup (V \backslash V')$), then one can define maps

$$\begin{aligned} &\mu \colon Z' = Z(k, V', M, A', x_0|V') \to Z = Z(k, V, M, A, x_0), \quad \text{resp.} \\ &\lambda \colon Y' = Y(k, V', M, A', x_0|V') \to Y = Y(k, V, M, A, x_0) \end{aligned}$$

by taking $x = \mu(x')$ resp. $\lambda(x')$ to be the continuation of $x' \colon V' \to M$ to V that agrees with x_0 on $V \backslash V'$. The following is straightforward.

Proposition 3.1. *Both λ and μ are holomorphic. Furthermore $\lambda_* T_{x'} Y' \subset T_x Y$ (resp. $\mu_* T_{x'} Z' \subset T_x Z$) corresponds to $C^k_D(x^*TM)$ (resp. $C^k_{(D)}(x^*TM)$), where $D = A' \cup (V \backslash V')$.*

In what follows, first we shall construct holomorphic functions on Y; second we shall make a point concerning compact subvarieties of Y. Assume $k \leq \infty$. The quotient

$$J = J(k, V, M, A, x_0) = C^k_A(x_0{}^*TM)/C^k_{(A)}(x_0{}^*TM), \tag{3.2}$$

a space of k-jets, is a Banach (resp. Fréchet) space when $k < \infty$ (resp. $k = \infty$). There is a holomorphic map $j \colon Y \to J$ defined as follows. Construct $U_0 \subset V \times M$ and $F_0 \colon U_0 \to x_0{}^*TM$ as in Lemma 2.1, with x replaced by x_0. If $x \in Y$ then $(t, x(t)) \in U_0$ for $t \in V$ in a neighbourhood of A, hence $F_0(t, x(t))$ defines a section of $x_0{}^*TM$ over this neighbourhood. Continue this section, if need be from a smaller neighbourhood, to all of V, to obtain a section $\sigma \in C^k_A(x_0{}^*TM)$. The class of σ in J is clearly independent of the continuation, and will be denoted $j(x)$.

Theorem 3.2. *If $k \leq \infty$, the map $j \colon Y \to J$ is an open and surjective holomorphic submersion. Its fibres are the manifolds $Z(\ldots, x)$, for $x \in Y$.*

Proof. Any element of J has a representative $\sigma \in C^k_A(x_0{}^*TM)$ whose graph $\sigma(V)$ is contained in $F_0(U_0)$. This means that $\sigma(t) = F_0(t, x(t))$ for some $x \in Y$, so that $j(x)$ is the class of σ. Therefore j is surjective. Clearly, the fibre through $x \in Y$ is $Z(\ldots, x)$.

Next, given $x_1 \in Y$, choose a neighbourhood $U_1 \subset V \times M$ of the graph of x_1 and $F_1 \colon U_1 \to x_1{}^*TM$ as in Lemma 2.1. Since U_0 is a neighbourhood of the graph of $x_1|A = x_0|A$, hence also of the graph of $x_1|B$ for suitable open neighbourhood $B \subset V$ of A, U_1 can be shrunk so that $U_1 \cap (B \times M) \subset U_0$. Fix a function $\chi \in C^k(V, \mathbb{R})$, supported in B, constant 1 near A, and define a map g from a neighbourhood of $0 \in C^k_A(x_1{}^*TM)$ to $C^k_A(x_0{}^*TM)$ by letting

$$g(s)(t) = \chi(t) F_0(t, F_1^{-1}(t, s(t))), \quad s \in C^k_A(x_1{}^*TM), \tag{3.3}$$

understood to be 0 when $t \notin B$. In the coordinate chart $(\mathscr{U}_1, \varphi_1)$ determined by x_1, F_1, the map j is given by g, followed by the projection $\pi \colon C^k_A(x_0{}^*TM) \to J$. From (3.3) one easily checks that j is indeed a holomorphic submersion.

To see it is also open, note that $\pi \circ g$ factors through $C^k_{(A)}(x_1{}^*TM)$ to induce a holomorphic map $\imath \colon J_1 = J(\dots, x_1) \to J$. Thus the jet map $j_1 \colon Y \to J_1$ can be included in a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & Y \arrow[d, "j_1"] \\ J \arrow[r, "\imath"] & J_1 \end{tikzcd}$$

of holomorphic maps. The roles of x_0, x_1 being symmetric, $\imath$ is a biholomorphism. Now in the chart $(\mathscr{U}_1, \varphi_1)$ j_1 is given by the projection

$$C^k_A(x_1{}^*TM) \to C^k_A(x_1{}^*TM)/C^k_{(A)}(x_1{}^*TM),$$

so $j_1|\mathscr{U}_1$ is open. It follows that $j = \imath^{-1} j_1$ is also open in $\mathscr{U}_1$, hence everywhere, for $x_1 \in Y$ was arbitrary.

The interest of this theorem is that it allows one to construct holomorphic functions on Y. Indeed, assuming $\dim J > 0$, there will be non-constant holomorphic functions on J, which then can be pulled back to non-constant holomorphic functions on Y. For example one has

Theorem 3.3. *If $k = \omega$, and A intersects each component of V then holomorphic functions separate points of Y.*

Proof. If $x, y \in Y$ are different then their infinite order jets along A are also different: $j^\infty x|A \neq j^\infty y|A$. Consider the mapping spaces $X_\infty = C^\infty(V, M) \supset X$ and $Y_\infty = Y(\infty, V, M, A, x_0) \subset X_\infty$, together with the jet space $J = J(\infty, V, M, A, x_0)$ as in (3.2) and the map $j \colon Y_\infty \to J$. Proposition 2.4 implies that the inclusion map $X \hookrightarrow X_\infty$, hence also $i \colon Y \hookrightarrow Y_\infty$, is holomorphic. Since $y \notin Z(\infty, V, M, A, x)$, by Theorem 3.2 $j(x) \neq j(y)$, therefore there is an $f \in \mathcal{O}(J)$ that assumes different values at $j(x)$, $j(y)$. It follows that $f \circ j \circ i \in \mathcal{O}(Y)$ takes different values at x, y, q.e.d.

A consequence is that in the situation of Theorem 3.3 Y contains no positive dimensional subvarieties S biholomorphic to compact complex spaces. This even generalises to $k \leq \infty$, in spite of the fact that, as we shall see later, in this case $\mathcal{O}(Y)$ may fail to separate points. We shall formulate the result without mentioning subvarieties, a subtle notion in the infinite dimensional setting. It should be understood, though, that it applies to holomorphic images of connected compact complex spaces. Below k can be arbitrary.

Theorem 3.4. *Assume A intersects each component of V. Let $\emptyset \neq S \subset Y$ be compact, and suppose that any holomorphic function in a neighbourhood of S is constant on S. Then S is a singleton.*

Proof. Fix $x_1 \in S$ and consider the closed set

$$B = \{t \in V : x(t) = x_1(t) \text{ for all } x \in S\}.$$

We claim B is also open. Indeed, let $t_0 \in B$ and (U, α) be a coordinate chart on $M, x_1(t_0) \in U$. Recall that the evaluation map $e: V \times X \to M$ is continuous (Proposition 2.5). As $e(\{t_0\} \times S) = x_1(t_0)$, and S is compact, it follows that t_0 has a neighbourhood $W \subset V$ such that $e(W \times S) \subset U$. For $t \in W$ the map $e(t, \cdot): X \to M$ is holomorphic, again by Proposition 2.5, hence $\alpha(e(t, \cdot))$ is a holomorphic map of a neighbourhood of $S \subset X$ into $\mathbb{C}^n$. Our assumptions imply the components of this map are constant on S, hence $e(t, \cdot)$ is constant. This means $x(t) = x_1(t)$ for all $x \in S$ and $t \in W$, that is, $W \subset B$.

Thus $B \supset A$ is open and closed; further, it intersects each component of V. This implies $B = V$, i.e. S is the singleton $\{x_1\}$. ■

To conclude this section we note that mapping spaces can also be modelled on smoothness classes other than $C^k, k = 0, 1, \ldots$. Starting with a smooth compact manifold V, one can consider maps $V \to M$ in some Hölder class $C^k, k > 0$ real; or in Sobolev classes $W^{k,p}$ with $kp > \dim V$, or $k \geq \dim V$ when $p = 1$ (to ensure elements of this space can be represented by continuous maps). All our discussion above carries over to the resulting mapping spaces, with the difference that, say in the Hölder setting, $C^k_{(A)}(x^*TM) \subset C^k(x^*TM)$ should be defined as the closure of the space of those sections $\sigma \in C^k(x^*TM)$ that vanish in a neighbourhood of A; and the definition of Z in (3.1) should be changed accordingly.

When $k = 0, 1, 2, \ldots, \infty$, the definition of $C^k_{(A)}(x^*TM)$ and Z in terms of closure is equivalent to the one we gave earlier. Indeed, by Whitney's spectral theorem (cf. [12, chapter 5, théorème 1.3]), in order to decide e.g. whether $\sigma \in C^k(x^*TM)$ can be approximated by sections that vanish in a neighbourhood of A, all one has to do is to check whether for each $a \in V$ there is a $\sigma_a \in C^k(x^*TM)$, vanishing near A, such that $j^k\sigma = j^k\sigma_a$ at a. This property is clearly equivalent to $j^k\sigma|A = 0$.

4. The case of a projective space

In this section we shall restrict our attention to spaces of mappings into complex projective space $M = \mathbb{P}_n$. Accordingly we shall drop reference to M in the notation for these spaces, and write $Y = Y(k, V, A, x_0)$, etc. We shall also assume $k \leq \infty$. With notation introduced in Section 3, the main result is:

Theorem 4.1. *Any holomorphic function on a connected component of the manifold* $Y = Y(k, V, A, x_0)$ *agrees with the pullback* j^*h *of some* $h \in \mathcal{O}(J)$.

This will follow from:

Theorem 4.2. *Any holomorphic function on a connected component of the manifold* $Z = Z(k, V, A, x_0)$ *is constant.*

First a special case will be proved. Represent $\mathbb{P}_n$ as $\mathbb{C}^n \cup \mathbb{P}_{n-1}$ and put $Z^0 = \{x \in Z : x(V) \subset \mathbb{C}^n\}$. If $0 \in \mathbb{C}^n$ denotes the origin, and $x_0 \equiv 0$, the space Z^0 is the locally convex space $C^k_{(A)}(V, \mathbb{C}^n)$ of those C^k maps $x\colon V \to \mathbb{C}^n$ whose jet $j^k x$ vanishes on A.

Proposition 4.3. *Suppose* V *is* C^k *diffeomorphic to a compact region* $V' \subset \mathbb{R}^m$ *with* C^k *boundary,* $x_0 \equiv 0$, *and* $g\colon Z \to \mathbb{C}$ *is continuous. If* $g|Z^0$ *is complex linear then* $g|Z^0 \equiv 0$.

This is the key to Theorems 4.1 and 4.2. The special case when $m = 1$ and $k = 0$ is due to Dineen–Mellon, see [2]. Their proof does not seem to generalise to positive k.

PROOF. We start with the following simple observation. Suppose $\xi\colon \mathbb{R}^m \to \mathbb{C}$ is a function whose partial derivatives up to order k (meaning: of all orders if $k = \infty$) are bounded and uniformly continuous. Let $c_m = \int_{\mathbb{R}^m} (|t|^{2m} + 1)^{-1} dt$, and

$$\xi_\varepsilon(t) = \frac{\varepsilon^m}{c_m} \int_{\mathbb{R}^m} \frac{\xi(t - \tau)d\tau}{|\tau|^{2m} + \varepsilon^{2m}} = \frac{\varepsilon^m}{c_m} \int_{\mathbb{R}^m} \frac{\xi(\tau)d\tau}{|t - \tau|^{2m} + \varepsilon^{2m}}, \quad \varepsilon > 0. \tag{4.1}$$

Then $\lim_{\varepsilon \to 0} \xi_\varepsilon = \xi$, with partial derivatives up to order k converging uniformly.

It suffices to verify this when $k = 0$, for the general case will then follow after repeated differentiation. The substitution $\tau = \varepsilon\sigma$ in (4.1) leads to

$$c_m \xi_\varepsilon(t) = \int_{\mathbb{R}^m} \frac{\xi(t - \varepsilon\sigma)}{|\sigma|^{2m} + 1} d\sigma, \quad \text{and so}$$

$$c_m \sup_t |\xi(t) - \xi_\varepsilon(t)| \leq \int_{\mathbb{R}^m} \sup_t |\xi(t) - \xi(t - \varepsilon\sigma)| \frac{d\sigma}{|\sigma|^{2m} + 1} \to 0$$

as $\varepsilon \to 0$.

Now to the proof of the proposition. We can clearly assume $V = V'$. Since g is linear on Z^0, it will suffice to show $g(z) = 0$ for all $z \in Z^0$ that map V into a coordinate axis of $\mathbb{C}^n$; in other words we can also assume $n = 1$. Decompose $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$, and consider first $z \in Z^0$ that vanishes in a neighbourhood of $A \subset V$. Write $z = \xi\eta|V$ with compactly supported $\xi, \eta \in C^k(\mathbb{R}^m)$, η vanishing in a neighbourhood of $A \subset \mathbb{R}^m$ but not on supp ξ. This can be done as follows. Choose compactly supported $\eta \in C^k(\mathbb{R}^m)$ vanishing in a neighbourhood of $A \subset \mathbb{R}^m$ so that the zero set of $\eta|V$ is in the interior (relative to V) of the zero set of z. Then z/η, defined 0 where $\eta|V = 0$, is in $C^k(V)$, hence by Whitney's (or Tietze's) extension theorem it continues to a compactly supported $\xi' \in C^k(\mathbb{R}^m)$ (see [12, chapter 4, théorèmes 2.2 and 3.1]). Then $z = \xi'\eta|V$ and η does not vanish on $\operatorname{supp}(\xi'|V) = \operatorname{supp} z$. Multiplying ξ' with an appropriate cut off function will give ξ as required.

Define ξ_ε as in (4.1). Then $\xi_\varepsilon\eta|V \in Z^0$, and by our initial observation

$$\xi_\varepsilon\eta|V \to \xi\eta|V = z \quad \text{as} \quad \varepsilon \to 0 \tag{4.2}$$

in Z^0. To compute $g(z)$ consider first $g(\xi_\varepsilon\eta|V)$. Writing

$$x_{\varepsilon,\tau}(t) = \eta(t)/(|t-\tau|^{2m} + \varepsilon^{2m}) \in \mathbb{C} \cup \{\infty\},$$

for $(\varepsilon,\tau) \in [0,1] \times \operatorname{supp}\xi$ and $t \in V$, we have when $\varepsilon > 0$

$$\xi_\varepsilon(t)\eta(t) = \frac{\varepsilon^m}{c_m}\int_{\mathbb{R}^m} \frac{\xi(\tau)\eta(t)d\tau}{|t-\tau|^{2m} + \varepsilon^{2m}} = \frac{\varepsilon^m}{c_m}\int_{\operatorname{supp}\xi} \xi(\tau)x_{\varepsilon,\tau}(t)d\tau, \tag{4.3}$$

and

$$g(\xi_\varepsilon\eta|V) = \frac{\varepsilon^m}{c_m}\int_{\operatorname{supp}\xi} \xi(\tau)g(x_{\varepsilon,\tau})d\tau,$$

since $g|Z^0$ is continuous and linear. Observe that $x_{\varepsilon,\tau} \in Z$ depends continuously on$(\varepsilon,\tau) \in [0,1] \times \operatorname{supp}\xi$ (as $\eta(\tau) \neq 0$). It follows that the integral in (4.3) tends to the finite limit $\int_{\operatorname{supp}\xi} \xi(\tau)g(x_{0,\tau})d\tau$ as $\varepsilon \to 0$, whence $g(z) = \lim_{\varepsilon\to 0} g(\xi_\varepsilon\eta|V) = 0$.

The computation above applies to those $z \in Z^0$ that vanish in a neighbourhood of A. Considering that such maps constitute a dense subset of Z^0, as explained at the end of Section 3, g indeed vanishes on all of Z^0.

Having proved Proposition 4.3 let us turn to Theorem 4.2.

Proof of Theorem 4.2. We need to show that for any $f \in \mathcal{O}(Z)$ the differential df vanishes on each tangent space $T_xZ, x \in Z$. Since $Z(k,V,A,x_0) = Z(k,V,A,x)$, it suffices to prove $df = 0$ on $T_{x_0}Z$; which we shall do by first considering two particular situations. We shall only deal with the case when V is a manifold with boundary. Otherwise k must be 0, and one can borrow the reasoning from the latter part of the proof of Theorem 7 in [2, pp. 759–60].

Step 1: Assume V and $x_0 \equiv 0$ are as in Proposition 4.3. The circle action

$$S^1 \times \mathbb{C}^n \ni (\vartheta, w) \mapsto e^{2\pi i\vartheta}w \in \mathbb{C}^n$$

extends to an action on $\mathbb{P}_n$ that fixes $\mathbb{P}_{n-1} = \mathbb{P}_n \backslash \mathbb{C}^n$, and induces a continuous action ρ: $S^1 \times Z \to Z$; by Proposition 2.3 $\rho_\vartheta = \rho(\vartheta,\cdot)$ is holomorphic for each $\vartheta \in S^1$. The restriction of ρ to the vector space $Z^0 \subset Z$ of Proposition 4.3 is given by $\rho_\vartheta x = e^{2\pi i\vartheta}x$. Expand $f|Z^0$ in a homogeneous series $\sum_{p\geq 0} f_p$, then $f_1 = df|T_{x_0}Z$ (after the customary identification $Z^0 \approx T_{x_0}Z^0 = T_{x_0}Z$). Cauchy's formula

$$f_1(x) = \int_0^1 e^{-2\pi i\vartheta}f(\rho_\vartheta x)d\vartheta, \quad x \in Z^0$$

implies that the linear form f_1 is the restriction to Z^0 of a function $g \in \mathcal{O}(Z)$. Hence $f_1 = 0$ follows from Proposition 4.3, and df indeed vanishes on $T_{x_0}Z$.

Step 2: Next consider the case when V is still diffeomorphic to a compact region with C^k boundary, but x_0 is only assumed to avoid a hyperplane in $\mathbb{P}_n$. Thus $x_0(V) \subset \mathbb{P}_n \backslash \mathbb{P}_{n-1} = \mathbb{C}^n$. The map $V \times \mathbb{C}^n \ni (t,w) \mapsto w + x_0(t) \in \mathbb{C}^n$ extends to a C^k map Φ: $V \times$

$\mathbb{P}_n \to \mathbb{P}_n$ and $\Phi(t,\cdot)$ is biholomorphic for $t \in V$. Hence, by Proposition 2.3, it induces a biholomorphic map $\Phi_*\colon X \to X$. The preimage of $Z(\ldots, x_0)$ is $Z(\ldots, 0)$, on which $f \circ \Phi_*$ is holomorphic. Thus we are in the situation of the first step in this proof, and conclude that $d(f \circ \Phi_*) = 0$ on $T_0Z(\ldots, 0)$, i.e. $df = 0$ on $T_{x_0}Z$.

Step 3: For general A, x_0 construct finitely many $V_i \subset V, i \in I$, each diffeomorphic to a compact region $V_i' \subset \mathbb{R}^m$ with C^k boundary so that the interiors (relative to V) of V_i cover V. Make sure that $x_0(V_i)$ avoids some hyperplane in $\mathbb{P}_n$. Endow V with a metric, for $i \in I$ let $d_i(t)$ denote distance to $V \backslash V_i$, set

$$B_i = \{t \in V_i : d_i(t) \le d_h(t)/2 \text{ for some } h \in I\},$$

and let $A_i = B_i \cup (A \cap V_i)$. Note that any boundary point of V_i (relative to V) is contained in $B_i \subset A_i$. Hence, putting $x^i = x_0|V_i$ there are embeddings

$$\mu_i : Z(k, V_i, A_i, x^i) = Z_i \to Z$$

as in Proposition 3.1. Observe that

$$\bigoplus_i \mu_{i*} T_{x^i} Z_i = T_{x_0} Z. \tag{4.4}$$

Indeed, $V_i \backslash B_i$, $i \in I$, form an open covering of V. Let χ_i be a corresponding C^k partition of unity. Then any $\sigma \in C^k_{(A)}(x_0{}^*T\mathbb{P}_n) \approx T_{x_0}Z$ can be written $\sigma = \sum_i \chi_i \sigma$ with

$$\chi_i \sigma \in C^k_{(A_i \cup [V \backslash V_i])}(x_0{}^*T\mathbb{P}_n) \approx \mu_{i*} T_{x^i} Z_i.$$

By step 2 in this proof $d(f \circ \mu_i) = 0$ on $T_{x^i}Z_i$. Thus $df = 0$ on $\mu_{i*}T_{x^i}Z_i$, hence also on $T_{x_0}Z$ by (4.4).

PROOF OF THEOREM 4.1. Let H be a connected component of Y. The main point is to show that the fibres of $j|H$, i.e. the manifolds $H \cap Z(\ldots, x)$, are connected, $x \in H$. It suffices to do this for $x = x_0$; then $Z(\ldots, x) = Z$. Select first $x_1 \in H \cap Z$ that agrees with x_0 in a neighbourhood of $A \subset V$. There is a continuous curve $x_\vartheta \in H$ $(0 \le \vartheta \le 1)$ connecting x_0 and x_1; we shall produce a connecting curve $y_\vartheta \in H \cap Z$.

For this purpose construct a neighbourhood $U \subset V \times M$ of the graph of x_0 and $F\colon U \to x_0{}^*TM$ as in Lemma 2.1. Since the graphs of $x_\vartheta|A = x_0|A$ are contained in U, the same will hold for the graphs of $x_\vartheta|B$ with $B \subset V$ some fixed neighbourhood of A. Choose B so small that $x_1|B = x_0|B$; let $\chi \in C^k(V, [0,1])$ vanish on a neighbourhood of A and be 1 on a neighbourhood of $V \backslash B$. Then

$$y_\vartheta(t) = \begin{cases} F^{-1}(t, \chi(t)F(t, x_\vartheta(t))) & \text{for } t \in B \\ x_\vartheta(t) & \text{for } t \in V \backslash B, \end{cases}$$

$0 \le \vartheta \le 1$, defines a continuous curve $y_\vartheta \in Z \subset Y$ connecting x_0 and x_1. Since this curve must be contained in H as well, x_0, x_1 are in the same component of $H \cap Z$.

On the other hand, as explained at the end of Section 3, an arbitrary $x \in H \cap Z$ can be approximated by maps x_1 that agree with x_0 in a neighbourhood of A. Since the components of $H \cap Z$ are closed, it follows there is just one component: $H \cap Z$ is indeed connected.

Now any $f \in \mathcal{O}(H)$ will be constant on the fibres of $j|H$ by virtue of Theorem 4.2. Hence $f = j^* h$ with some function h on J; and the properties of j listed in Theorem 3.2 imply $h \in \mathcal{O}(J)$, as claimed.

5. Concluding remarks

The description of holomorphic functions on mapping spaces of $\mathbb{P}_n$ given in Theorem 4.1 immediately generalises to mapping spaces of Grassmannians, but not far beyond that. If M can be embedded in some $\mathbb{C}^N$ then X embeds in the locally convex space $C^k(V, \mathbb{C}^N)$, hence holomorphic functions separate points of X, and *a fortiori*, points of $Y \subset X$. This latter can even happen for compact M. Suppose for example that M is covered by a manifold M' that embeds in $\mathbb{C}^N$. For simplicity take $V = S^1$ and $A = \{a\}$ a singleton. Lifting loops in M to paths in M' induces an embedding of $Y = Y(k, S^1, M, \{a\}, x_0)$ into $C^k([0,1], M') \subset C^k([0,1], \mathbb{C}^N)$. Again it follows that $\mathcal{O}(Y)$ separates points of Y.

However, for more general M, even for projective ones, $\mathcal{O}(X)$ is not understood at all. It would be of interest to compute $\mathcal{O}(X)$, $\mathcal{O}(Y)$, and more generally the analytic cohomology groups of X, Y, especially for loop spaces. In [14] Zhang took the first step in the computation of the groups $H^{0,q}(X, L)$, $q = 0,1$, for X the loop space of $\mathbb{P}_1$ and various line bundles $L \to X$, including the trivial one.

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