

A NOTE ON UNSTEADY INHOMOGENEOUS EXTENSIONS OF A CLASS OF NEO-HOOKEAN ELASTIC SOLIDS

By K.R. RAJAGOPAL*

Department of Mechanical Engineering, Texas A&M University, College Station,
Texas 77843

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ABSTRACT

In this note we determine explicit new exact solutions for unsteady inhomogeneous motions of a neo-Hookean solid due to the application of boundary traction that engenders a displacement along a coordinate direction that is proportional to the coordinate. The problem is the counterpart of a problem in fluid mechanics concerning the boundary layer flow due to a flexible boundary being stretched in such a manner that the velocity in a coordinate direction is proportional to its coordinate.

1. Introduction

There have been many studies devoted to the dynamics of isotropic incompressible solids in general (Hayes [1], Hayes and Rivlin [2; 3], Carroll [4–7]) and the neo-Hookean and Mooney–Rivlin solid in particular (Boulanger and Hayes [8–11], Currie and Hayes [14]). In incompressible materials, by virtue of the presence of the indeterminate part of the stress, it is possible to engender motions by assigning the appropriate structure to the indeterminate part. Such a possibility notwithstanding, there are few explicit exact solutions available in the case of non-linear elastic materials.

In this note we study a special class of dynamic motions of an incompressible neo-Hookean solid wherein a particle at an x coordinate undergoes a displacement along the X direction that is proportional to its x coordinate. Such a motion can be engendered in a half-plane, that above the X – Z plane, say, by applying boundary tractions to the X – Z plane such that the X -component of the displacement is proportional to the X coordinate of the particle. We find that a rich class of motions are possible that correspond to a variety of pressure fields. Exact solutions serve a dual purpose. In addition to providing an explicit solution which can be analysed in detail they provide a means for checking the accuracy of complicated computational schemes that are used to study complex non-linear equations.

The steady flow of a Navier–Stokes fluid due to the stretching of a sheet so that the velocity at a particular coordinate is proportional to its location has been studied in much detail within the context of the classical linearly viscous Navier–Stokes fluid (see Chang and Kazarinoff [13], Danberg and Fansler [15], McLeod and Rajagopal [16], Troy *et al.* [18]), and non-Newtonian fluids (see McLeod and Rajagopal [16],

*E-mail: krajagopal@mengr.tamu.edu

Rajagopal and Gupta [17] and many other related papers), and it serves to illustrate the development of boundary layers in such fluids.

The exact counterpart of the problem that is being studied here, namely unsteady plane flow solutions with the X -component of the velocity being proportional to the x -coordinate, was studied within the context of the Navier–Stokes theory by Boulanger *et al.* [12]. They consider a velocity field $\mathbf{v} = (v_1, v_2, v_3)$ of the form (note the similarity in the structure between the assumption below for the velocity field in [12] and the assumption (2.5) for the displacement field in this study):

$$v_1 = x \frac{\partial \hat{f}}{\partial y}, \quad (1.1)$$

$$v_2 = -\hat{f}(y, t), \quad (1.2)$$

$$v_3 = 0. \quad (1.3)$$

In addition to general unsteady solutions of the above form in full space, Boulanger *et al.* [12] exhibit interesting solutions due to the unsteady flow of a layer of a Navier–Stokes fluid between two elastic stretching sheets. Here we consider unsteady displacement fields that have a structure similar to that for the velocity (1.1)–(1.3) and show that many interesting exact solutions can be established for a large class of boundary value problems.

2. Governing equations

Let $\mathbf{X} \in \kappa(\mathbf{B})$ denote a particle belonging to the reference configuration $\kappa(\mathbf{B})$ of a body \mathbf{B} . By the motion χ of the body we mean a one-to-one mapping that assigns to each $\mathbf{X} \in \kappa(\mathbf{B})$ a point \mathbf{x} belonging to a three dimensional euclidean space, at time t :

$$\mathbf{x} = \chi(\mathbf{X}, t). \quad (2.1)$$

In general the mapping χ depends upon κ but we shall suppress the dependence for the sake of convenience.

The deformation gradient \mathbf{F} is defined through

$$\mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}, \quad (2.2)$$

and the Cauchy–Green stretch tensor \mathbf{B} through

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T. \quad (2.3)$$

We are interested in considering motions of a neo-Hookean elastic solid, and as this is an incompressible material we require that the body can undergo only isochoric motions, i.e.

$$\det \mathbf{F} = 1. \quad (2.4)$$

We consider time-dependent motion of two bodies, the half space defined through

$$\{(X, Y, Z) | -\infty < X < \infty, 0 \leq Y < \infty, -\infty < Z < \infty\},$$

and a slab of finite thickness H defined through

$$\{(X, Y, Z) | -\infty < X < \infty, 0 \leq Y \leq H, -\infty < Z < \infty\}$$

due to the boundary, the $Y = 0$ plane being displaced proportionally in the X direction in a time-dependent manner. We suppose that the plane $Y = 0$ is moved in the following manner:

$$u_1(X, 0, Z, t) = Xq(t), \quad u_2(X, 0, Z, t) = 0, \quad u_3(X, 0, Z, t) = 0,$$

where u_1, u_2, u_3 are the X, Y and Z components of the displacement, and $q(t)$ is an arbitrary function of time.

In view of the boundary displacement, we seek a special motion of the following form:

$$\begin{aligned} x &= Xf(Y, t), \\ y &= g(Y, t), \\ z &= Z. \end{aligned} \tag{2.5}$$

We are using a semi-inverse method; the tractions that are necessary to produce the motion can be calculated after the solution that is being sought has been determined. For the motion (2.5), the deformation gradient \mathbf{F} and Cauchy–Green tensor \mathbf{B} have the following representations:

$$\mathbf{F} = \begin{pmatrix} f & Xf_Y & 0 \\ 0 & g_Y & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

$$\mathbf{B} = \begin{pmatrix} f^2 + X^2 f_Y^2 & Xf_Y & 0 \\ Xf_Y & f & 0 \\ f & f^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

where a subscript denotes partial derivative with respect to that variable. It follows from (2.4) that

$$fg_Y = 1. \tag{2.8}$$

We consider the motion of an incompressible neo-Hookean material. The Cauchy stress in such a material is given by

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{B}, \tag{2.9}$$

where $-p\mathbf{1}$ is the indeterminate part of the stress due to the constraint of incompressibility and μ is the shear modulus that is a constant. On entering the above expression for the stress into the balance of linear momentum

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt}, \tag{2.10}$$

where \mathbf{b} is the external body force, ρ the density and \mathbf{v} the velocity, and making use of the fact that

$$\frac{\partial}{\partial x} = \frac{1}{f} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = -Xf_Y \frac{\partial}{\partial X} + f \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}, \quad (2.11)$$

we obtain (on the neglect of the body forces)

$$-\frac{1}{f} \frac{\partial p}{\partial X} + \mu X f_{YY} = \rho X f_{tt}, \quad (2.12)$$

$$\mu \frac{f_Y}{f^2} + X \frac{\partial p}{\partial X} f_Y - \frac{\partial p}{\partial Y} f + \mu f \frac{\partial}{\partial Y} \left(\frac{1}{f^2} \right) = \rho g_{tt}. \quad (2.13)$$

A straightforward calculation leads to

$$\mu(f_Y f_{YY} - f f_{YYY}) + \rho(f f_{tt} - f_Y f_{tt}) = 0. \quad (2.14)$$

Thus, if $f \neq 0$ anywhere, then (2.14) can be integrated to yield

$$\left(\frac{\mu}{\rho} \right) (f_{YY}) - f_{tt} = h(t)f, \quad (2.15)$$

where $h(t)$ is an arbitrary function that is related to the pressure gradient in the material.

It follows from (2.12) and (2.15) that

$$\frac{\partial p}{\partial X} = \rho X [h(t)] f^2. \quad (2.16)$$

Once an f is determined that satisfies (2.15) and the appropriate initial and boundary conditions, g can be determined through (2.8). The pressure p can then be determined through (2.12) and (2.13).

3. Special solutions

The partial differential equation (2.15) admits many interesting solutions, depending on the nature of the boundary conditions. First, let us consider the case when $h(t)$ is zero, i.e. when there is no pressure gradient in the X direction. In this case, (2.15) reduces to the classical wave equation

$$f_{YY} - \left(\frac{\rho}{\mu} \right) f_{tt} = 0, \quad (3.1)$$

which admits solutions of the form

$$f(Y, t) = \hat{f}(Y + \alpha t), \quad (3.2)$$

where $\alpha^2 = (\rho/\mu)$. Thus any initial value for f propagates as a wave. Such solutions would be of interest to us if we were interested in how the initial data prescribed along the Y-axis propagates with time.

We now turn our attention to a discussion of several boundary value problems in a slab of finite thickness H . Here we are interested in the manner in which a slab of

thickness H in the Y direction, and infinite in the other two directions, deforms the boundary conditions being prescribed on the planes $Y=0$ and $Y=H$.

3a. Unsteady motions in a slab in which one plane is held fixed while the other is proportionately oscillated in the absence of a pressure gradient

Let us suppose that the plane $Y=0$ moves in the following manner:

$$\begin{aligned} u_1(X, 0, Z, t) &= UX \sin \omega t, \\ u_2(X, 0, Z, t) &= 0, \\ u_3(X, 0, Z, t) &= 0. \end{aligned} \quad (3.3)$$

Thus the displacement along the X directional is proportional to the X coordinate.

Then it follows from (2.5) and (2.19) that

$$f(0, t) = 1 + U \sin \omega t, \quad (3.4)$$

and

$$g(0, t) = 0. \quad (3.5)$$

Furthermore, let us suppose that

$$u_1(X, H, Z, t) = u_3(X, H, Z, t) = 0, \quad (3.6)$$

that is the slab is fixed to a rigid plane and cannot move relative to this plane at $Y=H$. We will see that u_2 at $Y=H$ is determined as a part of the solution, i.e. due to the proportional stretching of the plane along the X direction on the plane $Y=0$, the plane $Y=H$ could displace along the Y direction.

Then

$$f(H, t) = 1. \quad (3.7)$$

We recall that $h(t)$ essentially determines the pressure gradient along the X direction. We make a special choice for $h(t)$ and determine solutions, if any, that correspond to our choice of $h(t)$. Equation (2.16) will then determine the pressure¹ p that ensures the solution to (2.15). We have already seen that the choice of $h(t)=0$ leads to the wave equation and we can study the propagation of waves in the half-space.

We turn our attention to solving boundary value problems in the finite slab. Let us seek a solution of the form

$$f(y, t) = 1 + F(y) \sin \omega t. \quad (3.8)$$

It immediately follows that

$$F(0) = U, \quad F(H) = 1. \quad (3.9)$$

¹ It is incorrect to associate the indeterminate part of the stress with the mean normal stress, which is referred to as the pressure, in general. While the mean normal stress is indeed the pressure (i.e. $p = -\frac{1}{3}tr\mathbf{T}$), in the case of an incompressible Navier–Stokes fluid, $p \neq -\frac{1}{3}tr\mathbf{T}$ in a neo-Hookean solid as $tr\mathbf{B} \neq 0$, in general. We will however call the constitutively indeterminate part of the stress the pressure.

Then, a straightforward computation yields

$$f(Y, t) = \frac{U}{\sin \beta H} [\sin \beta(H - Y)] \sin \omega t + 1, \quad (3.10)$$

where

$$\beta = \omega \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}},$$

and thus by (3.3)

$$u_1 = \frac{UX}{\sin \beta H} [\sin \beta(H - Y)] \sin \omega t. \quad (3.11)$$

At this juncture it would be appropriate to compare and contrast the form of the solution for the displacement and pressure fields obtained here with that of the velocity and pressure fields for the counterpart of our problem within the context of the Navier–Stokes theory. Boulanger *et al.* [12, eq. (1.1)] show that $v_x(y, t)$ can be one of the following forms (with the pressure field taking on the appropriate structure to render such flows possible):

$$(i) \quad v_1(y, t) = -kx[A \exp\{\Omega t - ky\} - B \exp\{\Omega t + ky\}]$$

$$p = p_0 - \frac{3}{2}[A^2 \exp\{2(\Omega t - ky)\} + B^2 \exp\{2(\Omega t + ky)\}]$$

$$+ 2\rho ABk^2 x^2 \exp\{2\Omega t\};$$

$$(ii) \quad v_2(y, t) = kCx \exp\{-\Omega t\} \sin(ky + d)$$

$$p = p_0 + \frac{\rho}{2} C^2 \{-2\Omega t\} \{\sin^2(ky + d) - k^2 x^2\};$$

$$(iii) \quad v_3(y, t) = a(t)x$$

$$p = p_0 - \frac{\rho}{2} \left\{ \left(a^2 - \frac{da}{dt} \right) y^2 + 2 \left(ab - \frac{db}{dt} \right) y + \left(a^2 + \frac{da}{dt} \right) x^2 \right\};$$

where a, b and k are arbitrary functions of time, A, B and C are constants, and $\Omega = vk^2$, where v is the kinematic viscosity. However, solutions of the type (i)–(iii) are not possible for the problem under consideration. In any event, the form of such solutions for the velocity field share little in common with the form of the solution (3.11) for the displacement field.

Next we need to compute $g(Y, t)$ by solving

$$g_Y(Y, t) = \frac{1}{1 + \frac{U}{\sin \beta H} [\sin \beta(H - Y)] \sin \omega t}, \quad (3.12)$$

subject to the boundary condition

$$g(0, t) = 0. \quad (3.13)$$

We cannot impose a displacement u_2 at $Y = H$. It is determined by g which is obtained by integrating (3.12) with respect to y and using the boundary condition (3.13).

Let us define

$$A(t) = \frac{U \sin \omega t}{\sin \beta H}. \tag{3.14}$$

If $-1 < A(t) < 1$ for all t , it follows that

$$g(Y, t) = \frac{-2}{\beta \sqrt{1 - A^2}} \tan^{-1} \left\{ \frac{\tan \left[\frac{\beta}{2} (H - Y) \right] + A}{\sqrt{1 - A}} \right\} + \frac{2}{\beta \sqrt{1 - A^2}} \tan^{-1} \left\{ \frac{\tan \left(\frac{\beta}{2} H \right) + A}{\sqrt{1 - A^2}} \right\}. \tag{3.15}$$

If $U/\sin \beta H > 1$, the solution is given in terms of different explicit forms for $A(t) > 1$, $A(t) < -1$ and $A(t) = 1$. We do not record them here.

3b. Unsteady motion in a slab in which one plane is held fixed and the other is proportionately oscillated in the presence of a pressure gradient

Now, let us investigate other possibilities. Suppose $h(t) = \gamma$, where γ is a constant. In this case, (2.15) reduces to

$$\left(\frac{\mu}{\rho} \right) f_{YY} - f_u = \gamma f. \tag{3.16}$$

Our choice for the form of the solution is motivated by the general structure for the velocity field in the case of a Navier–Stokes fluid that is subject to a pulsating pressure gradient about a mean. Here, $h(t)$ is a constant and γ is the basic pressure gradient, a time dependent part being superposed onto it.

Let us seek a separable solution of the form

$$f(Y, t) = F(Y)T(t) + G(Y). \tag{3.17}$$

However, we see that the specific form of (3.17) bears no similarity to any of the solutions (i)–(iii) for the Navier–Stokes fluid. It is in fact surprising that such semi-inverse choices for the displacement, based on an analogy with the Navier–Stokes theory, work in the first place.

It trivially follows that

$$\alpha^2 F'' T + \alpha^2 G'' - FT'' = \gamma FT + \gamma G, \alpha^2 = \frac{\mu}{\rho}. \tag{3.18}$$

We seek solutions to

$$\alpha^2 F''T - FT'' = \gamma FT, \quad (3.19)$$

and

$$\alpha^2 G'' - \gamma G = 0. \quad (3.20)$$

We consider solutions corresponding to the boundary conditions (3.4) and (3.7), which lead to

$$F(0) = U, \quad G(0) = 1, \quad (3.21)$$

$$F(H) = 0, \quad G(H) = 1. \quad (3.22)$$

In view of the boundary conditions, we make the choice

$$T(t) = \sin \omega t. \quad (3.23)$$

Suppose $\omega^2 > \gamma$. Let

$$\beta = \frac{\omega^2 - \gamma}{\alpha^2}. \quad (3.24)$$

It immediately follows from (3.19) and (3.23) that

$$F(Y) = \frac{U}{\sin \beta H} [\sin \beta(H - Y)], \quad (3.25)$$

while the solution for G is given by

$$G(Y) = C_1 e^{\delta Y} + C_2 e^{-\delta Y}, \quad (3.26)$$

where

$$\delta = \left(\frac{\gamma}{\alpha^2} \right)^{\frac{1}{2}}, \quad C_1 = \frac{[e^{-\delta H} - 1]}{(e^{-\delta H} - e^{\delta H})}, \quad C_2 = \frac{[e^{\delta H} - 1]}{(e^{\delta H} - e^{-\delta H})}. \quad (3.27)$$

Next suppose $\gamma > \omega^2$. Then let

$$\lambda^2 = \frac{\gamma - \omega^2}{\alpha^2}. \quad (3.28)$$

We find that

$$F(Y) = K_1 e^{\lambda Y} + K_2 e^{-\lambda Y}, \quad (3.29)$$

where

$$K_1 = \frac{U e^{-\lambda H}}{e^{-\lambda H} - e^{\lambda H}}, \quad K_2 = \frac{U e^{\lambda H}}{e^{\lambda H} - e^{-\lambda H}}. \quad (3.30)$$

The solution for $G(Y)$ is given by

$$G(Y) = D_1 e^{\eta Y} + D_2 e^{-\eta Y}, \quad (3.31)$$

where

$$\eta^2 = \frac{\gamma}{\alpha^2}, \tag{3.32}$$

and

$$D_1 = \frac{e^{-\eta H} - 1}{(e^{-\eta H} - e^{\eta H})}, \quad D_2 = \frac{e^{\eta H} - 1}{(e^{\eta H} - e^{-\eta H})}. \tag{3.33}$$

Having determined $f(Y, t)$, we can once again determine g by using (2.8) subject to the boundary condition (3.32).

3c. Arbitrary unsteady motion of a boundary of a slab while the other is held fixed

Next suppose the boundary condition (3.3) is replaced by

$$u_1(X, 0, Z, t) = Xr(t), \tag{3.34}$$

$$u_2(X, 0, Z, t) = 0, \tag{3.35}$$

$$u_3(X, 0, Z, t) = 0. \tag{3.36}$$

This implies that

$$f(0, t) = r(t) + 1. \tag{3.37}$$

We now seek a separable solution of the form

$$f(Y, t) = F(Y)r(t). \tag{3.38}$$

It immediately follows that

$$\frac{(\mu/\rho)F''}{F} = \frac{r'' + h(t)r}{r} = \delta, \tag{3.39}$$

and the pressure field adjusts itself appropriately for the motion in question.

Since we have the flexibility to pick $h(t)$, which is tantamount to picking the requisite pressure field, we require that

$$h(t) = \frac{-r''(t) + \delta r(t)}{r(t)}. \tag{3.40}$$

Depending on whether the constant δ is positive or negative, we will obtain solutions for $F(Y)$ that are either sinusoidal or exponential. A choice of the form (3.40) can be made provided $r(t)$ does not become zero for any time. This choice of $h(t)$, and the corresponding pressure automatically meet the boundary condition, and we are left with determining the solution to

$$(\mu/\rho)F'' - \delta F = 0, \tag{3.41}$$

subject to (3.9), which is a trivial problem. We do not record the solutions here. Once F and thus f is determined, again we use (2.8) and the boundary condition $g(0, t) = 0$ to determine $g(Y, t)$.

Finally we consider a rather interesting feature of Equation (2.14) that has been obscured by integrating it, under the assumption that $f \neq 0$ anywhere. We first note that a separable solution

$$f(Y, t) = F(Y)T(t), \quad (3.42)$$

automatically satisfies

$$ff_{tt} - f_Y f_{tY} = 0. \quad (3.43)$$

Thus, such a separable solution would meet (2.14), provided

$$\mu(f_Y f_{YY} - ff_{YY}) = 0, \quad (3.44)$$

i.e. if

$$F'F'' - FF''' = 0, \quad (3.45)$$

which can be integrated to yield

$$F'' + \sigma F = 0, \quad (3.46)$$

where σ is a constant.

We notice that T does not appear in (3.44), and F is determined as the solution to (3.44). Thus, a separable solution of the form (3.41) is possible provided $T(t)$ is such that $f(Y, t)$ satisfies the boundary condition. This would be satisfied if $T(t)F(0)$ is chosen as $r(t)$. Thus, given any arbitrary displacement of the boundary of the form (3.34), we are guaranteed that the pressure field will adjust itself and a separable solution of the form (3.38) is possible, with F being a solution to Equation (3.41). Depending on the value of σ we would have solutions that are oscillatory or exponential in structure.

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