SUMS OF AUTOMORPHISMS OF FREE ABELIAN GROUPS AND MODULES

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Abstract

We consider endomorphisms of M, a free $\mathbf{R}$-module of arbitrary rank, where $\mathbf{R}$ is an associative unital ring with the property that every element of $\mathbf{R}$ is a sum of two units of $\mathbf{R}$. We prove that every endomorphism of M is a sum of two automorphisms of M.

1. Introduction

The question of when the automorphism group of a module or abelian group additively generates its full endomorphism ring has been of interest for many years (see [3]). Of more specific interest has been the determination for various modules M of n, the least positive integer (if such exists), such that every endomorphism of M is a sum of exactly n automorphisms of M. There is a considerable body of literature on this topic including [1; 2; 6–16]. To aid our discussion we recall the following definitions from [9]: an associative ring $\mathbf{R}$ is said to have the *n-sum property* (for a positive integer n) if every element of $\mathbf{R}$ can be written as the sum of exactly n units of $\mathbf{R}$. Clearly, if this property holds for an integer n, then it also holds for any integer $k > n$, and so we can make the following definition of the unit sum number of a ring $\mathbf{R}$: $\text{usn}(\mathbf{R}) := \min\{n \mid \mathbf{R} \text{ has the } n\text{-sum property}\}$. If there is an element of $\mathbf{R}$ which is not a sum of units we set the unit sum number to be ∞ while if every element of $\mathbf{R}$ is a sum of units but $\mathbf{R}$ does not have the n-sum property for any n, we set $\text{usn}(\mathbf{R}) = \omega$. The unit sum number of an abelian group or module is defined to be equal to that of its endomorphism ring.

In 1954 Zelinsky showed for a vector space V over a field F that $\text{usn}(V) = 2$ unless V is one-dimensional and F is the field of two elements, in which case $\text{usn}(V) = \omega$ (see [16]). In 1985 Goldsmith [7] considered unit sum numbers for reduced torsion-free modules over a complete discrete valuation ring. This was further developed by Scott in [13] and by Goldsmith *et al.* in [9]. In [15] Wans considered free $\mathbf{R}$-modules, where $\mathbf{R}$ is a *PID*, and showed that if the rank of M is finite and greater than 1, then $\text{usn}(M) = 2$, while, in the case where M has infinite rank, Wans showed that M has the 3-sum property but did not determine $\text{usn}(M)$. Note that *PID*s may have unit sum numbers of 2 or ω and certain rational groups have been shown to have a finite unit sum number greater than 2 (see [8; 11]).

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In this work we consider free **R**-modules, M, of countable rank, where **R** is an arbitrary associative unital ring with usn(**R**) $= 2$. We prove that usn$(M) = 2$. The result is then extended to uncountable rank using a method of Castagna [1].

All groups other than automorphism groups are assumed to be abelian and our terminology is standard and may be found in Fuchs [4; 5]; an exception is that we write maps on the right. We also adopt the standard practice, where necessary, of distinguishing a ring from a module by using bold face characters for the former.

2. Definitions and properties of certain endomorphisms

We begin by recalling definitions and results regarding locally nilpotent endomorphisms.

Definition 2.1. *Let G be a group and ϕ an endomorphism of G. If for each $g \in G$ there is some $n \in \mathbb{N}$, n depending on g, such that $g\phi^n = 0$, then ϕ is said to be locally nilpotent.*

Lemma 2.2. *Let M be a free* **R***-module of arbitrary rank over a ring* **R** *and let η be any locally nilpotent endomorphism of M. Then $(\eta + 1)$ is an automorphism of M.*

Proof. Obviously $\eta + 1$ is also an endomorphism of M. To show that $\eta + 1$ is bijective let $m \neq 0$ be an arbitrary element of M. Since η is locally nilpotent there exists an integer $n \geq 1$ such that $m\eta^n = 0$ and $m\eta^{n-1} \neq 0$, where we agree that $\eta^0 = 1$.

Now, $m = m(1 - \eta + \eta^2 - \eta^3 + \ldots \eta^{n-1})(\eta + 1)$, so $\eta + 1$ is surjective: as $m(\eta + 1)\eta^{n-1} = m\eta^{n-1} \neq 0$ it follows that $m \notin \ker(\eta + 1)$, implying that $(\eta + 1)$ is injective since m was arbitrary. ■

Lemma 2.3. *Let M be a free* **R***-module of arbitrary rank. Moreover, let ϕ, ψ be locally nilpotent endomorphisms of M satisfying $\phi\psi = 0$. Then the endomorphism $\phi + \psi$ is locally nilpotent.*

Proof. First we show by induction on n that the following holds for any $n \in \mathbb{N}$:

$$(\phi + \psi)^n = \sum_{i=0}^{n} \psi^i \phi^{n-i} \qquad (*)$$

The statement (*) is certainly true for $n = 1$. Assume that (*) is true for all $1 \leq n \leq k$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}(\phi + \psi)^{k+1} &= \left(\sum_{i=0}^{k} \psi^i \phi^{k-i}\right)(\phi + \psi)\\ &= \left(\phi^k + \sum_{i=1}^{k} \psi^i \phi^{k-i}\right)(\phi + \psi)\\ &= \phi^{k+1} + \sum_{i=1}^{k} \psi^i \phi^{(k+1)-i} + \phi^k \psi + \sum_{i=1}^{k} \psi^i \phi^{k-i} \psi.\end{aligned}$$

However, since $\phi\psi = 0$ the last two summands reduce to ψ^{k+1} and thus we deduce

$$(\phi+\psi)^{k+1} = \phi^{k+1} + \sum_{i=1}^{k} \psi^i \phi^{(k+1)-i} + \psi^{k+1} = \sum_{i=0}^{k+1} \psi^i \phi^{(k+1)-i},$$

as required.

Using (*), it is easy to show that $\phi+\psi$ is locally nilpotent. Let $m \neq 0$ be an arbitrary element of M. Since ψ is locally nilpotent there exists $u \in \mathbb{N}$ such that $m\psi^u = 0$. Since ϕ is also locally nilpotent we can find $v \in \mathbb{N}$ so that $(m\psi^i)\phi^v = 0$ for all $0 \leq i \leq u$. Therefore, letting $n = u+v$, it follows that $m\psi^i\phi^{n-i} = 0$ for all $0 \leq i \leq n = u+v$, and thus $m(\phi+\psi)^n = m\sum_{i=0}^{n}\psi^i\phi^{n-i} = 0$.

As this is true for all elements $m \in M$ we can conclude that $\phi+\psi$ is locally nilpotent. ■

There are certain useful kinds of endomorphisms of free modules. Recall that M is a *free module* if M is generated by a linearly independent set of elements, called a *basis* of M; that is $M = \bigoplus_{i\in I} \mathbf{R}e_i$ for some index set I, where $|I|$ is the rank of M. Furthermore, we define the *support* of elements $m = \sum_{i\in I} r_i e_i$ of M by $[m] = \{i \in I | r_i \neq 0\}$ and the support of $0 \in M$ is the empty-set $\emptyset$. Note that the support depends on the choice of basis and that $[m]$ is finite for any $m \in M$; we shall not, however, vary bases in our discussions so that the support of an element will always refer to the same fixed basis.

Definition 2.4. *Let* $\mathbf{R}$ *be a ring and* $M = \bigoplus_{i\in I} \mathbf{R}e_i$ *a free* $\mathbf{R}$*-module of arbitrary rank with* I *a linearly ordered set and* ϕ *an endomorphism of* M*. We define the following*:

(i) ϕ *is an* α*-endomorphism of* M *if* $[e_i\phi] \subseteq \{j \in I | j > i\}$, *for each* $i \in I$;

(ii) ϕ *is a* β*-endomorphism of* M *if* $[e_i\phi] \subseteq \{j \in I | j < i\}$, *for each* $i \in I$;

(iii) ϕ *is a* d*-endomorphism of* M *if* $[e_i\phi] \subseteq \{i\}$, *for each* $i \in I$.

Note, if ϕ is represented as a matrix, then an α-endomorphism has non-zero entries only in the upper triangle, a β-endomorphism has non-zero entries only in the lower triangle and a d-endomorphism has non-zero entries only on the diagonal.

Lemma 2.5. *Let* $\mathbf{R}$ *be a ring and* $M = \bigoplus_{i<\omega} \mathbf{R}e_i$ *a free* $\mathbf{R}$*-module. If* ϕ *is a* β*-endomorphism of* M *then* ϕ *is locally nilpotent.*

PROOF. Note that it suffices to show that a power of ϕ annihilates each basis element e_k. Since ϕ is a β-endomorphism of M we have $[e_k\phi] \subseteq \{0, \ldots, k-1\}$, $[e_k\phi^2] \subseteq \{0, \ldots, k-2\}, \ldots, [e_k\phi^k] \subseteq \{0\}$, and so it is clear that $e_k\phi^{k+1} = 0$. Therefore ϕ is locally nilpotent. ■

3. Free modules over rings with the 2-sum property

In this paper we set out to show for any free **R**-module, M, that $\operatorname{usn}(M) = 2$ provided that $\operatorname{usn}(\mathbf{R}) = 2$. The finite rank case is well known, and indeed much more general results, omitting the assumption $\operatorname{usn}(\mathbf{R}) = 2$, are also known (see [15]).

Lemma 3.1. *Let* $M = \bigoplus_{i<n} \mathbf{R}e_i$ *be a free* **R***-module of finite rank* $n \in \mathbb{N}$. *If* $\operatorname{usn}(\mathbf{R}) = 2$ *then* $\operatorname{usn}(M) = 2$.

Proof. Noting that the endomorphism ring of M is the ring of $n \times n$ matrices over **R**, $M_n(\mathbf{R})$, we refer the reader to [9, proposition 1.2(c)]. ■

We proceed next to the countably infinite case. The following proposition is due to Freedman [2]; it provides useful sequences for 'breaking' endomorphisms into pieces.

Proposition 3.2. *Let* $M = \bigoplus_{i<\omega} \mathbf{R}e_i$ *be a free* **R***-module of countably infinite rank and let* ϕ *be an endomorphism of* M. *Then there exists a strictly increasing sequence of natural numbers* $0 = r_0 < r_1 < \dots < r_s \dots$ $(s \in \omega)$ *such that if* $i < n$ *and* $r_s \le i < r_{s+1}$, *then*

$$[e_i\phi] \subseteq \{0, 1, \dots, r_{s+2} - 1\} \tag{**}$$

Proof. Choose r_1 to be any positive integer greater than 0. Clearly $[e_i\phi]$ is finite for any $i \le r_1$ and so we may choose $r_2 \in \mathbb{N}$ to be bigger than the maximum of the finite set $\{r_1\} \cup \bigcup_{i=r_0}^{r_1-1} [e_i\phi]$. Therefore the property (**) is satisified for $s = 0$.

We continue inductively in the same way. Suppose r_{s+1} is given for some $s \ge 1$, then we obtain r_{s+2} as an integer bigger than the maximum of the finite set $\{r_{s+1}\} \cup \bigcup_{i=r_s}^{r_{s+1}-1} [e_i\phi]$. Then $[e_i\phi] \subseteq \{1, \dots, r_{s+2} - 1\}$ for each $r_s \le i < r_{s+1}$. Therefore the property (**) is satisified for s.

Hence, $0 = r_0 < r_1 < \dots$ is a strictly ascending sequence which has the desired property (**). ■

Applying the above proposition to an α-endomorphism we obtain:

Corollary 3.3. *Let* $M = \bigoplus_{i<\omega} \mathbf{R}e_i$ *be a free* **R***-module of countably infinite rank and let* ϕ *be an* α*-endomorphism of* M. *Then there exists a strictly increasing sequence of natural numbers* $0 = r_0 < r_1 \dots < r_s \dots$ $(s \in \omega)$, *such that if* $i < n$ *and* $r_s \le i < r_{s+1}$, *then* $[e_i\phi] \subseteq \{i+1, \dots, r_{s+2} - 1\}$.

Proof. The proof follows directly from Proposition 3.2 and the definition of an α-endomorphism. ■

The next lemma describes, for a free **R**-module of countably infinite rank, how any α-endomorphism may be broken into two locally nilpotent α-endomorphisms. This lemma is also due to Freedman [2].

Lemma 3.4. *Let* $M = \bigoplus_{i<\omega} \mathbf{R}e_i$ *be a free* $\mathbf{R}$*-module of countably infinite rank and let* η *be an* α*-endomorphism of* M*. Then* η *is a sum of two locally nilpotent* α*-endomorphisms.*

PROOF. By Corollary 3.3 there is a strictly ascending sequence of integers $0 = r_0 < r_1 < r_2 \ldots$ such that $[e_i\eta] \subseteq \{i+1, \ldots, r_{s+2}-1\}$, for each $r_s \le i < r_{s+1}$ ($s \in \omega$).

Using this sequence we now define endomorphisms η_1, η_2, of M as follows (for $t = 0,1,\ldots$):

$$e_i\eta_1 = \begin{cases} e_i\eta & \text{for } r_{2t} \le i < r_{2t+1} \\ 0 & \text{for } r_{2t+1} \le i < r_{2t+2} \end{cases}$$

and

$$e_i\eta_2 = \begin{cases} 0 & \text{for } r_{2t} \le i < r_{2t+1} \\ e_i\eta & \text{for } r_{2t+1} \le i < r_{2t+2} \end{cases}$$

Clearly, η_1 and η_2 are α-endomorphisms of M with $\eta = \eta_1 + \eta_2$. Thus it remains to show that η_1 and η_2 are locally nilpotent. Obviously, by symmetry, it is enough to consider η_1. Moreover, we only need to consider the base elements e_i ($i < n$). If $r_{2t+1} \le i < r_{2t+2}$ for some $t \in \omega$ then $e_i\eta_1 = 0$ by the above definition.

So, let $r_{2t} \le i < r_{2t+1}$, for some $t \in \omega$. Then also by definition, $e_i\eta_1 = e_i\eta$ and therefore we have $[e_i\eta_1] \subseteq \{i+1, \ldots, r_{2t+2}-1\}$. Now, since $e_j\eta_1 \subseteq \{j+1, \ldots, r_{2t+1}-1\}$ or $e_j\eta_1 = 0$ for all $j = r_{2t}, \ldots, r_{2t+2}-1$, then $[e_i\eta_1^2] \subseteq \{i+2, \ldots, r_{2t+2}-1\}$ or $e_i\eta_1^2 = 0$ and $[e_i\eta_1^3] \subseteq \{i+3, \ldots, r_{2t+2}-1\}$ or $e_i\eta_1^3 = 0$, and continuing in this way we have $[e_i\eta_1^m] \subseteq \{r_{2t+1}+1, \ldots, r_{2t+2}-1\}$ or $e_i\eta_1^m = 0$ for some $m \in \omega$. Hence $e_i\eta_1^{m+1} = 0$ and so η_1 is locally nilpotent as required. ■

Now we are ready to prove the main theorem.

Theorem 3.5. *Let* $M = \bigoplus_{i<\omega} \mathbf{R}e_i$ *be a free* $\mathbf{R}$*-module of countably infinite rank. If* $\mathrm{usn}(\mathbf{R}) = 2$ *then* $\mathrm{usn}(M) = 2$.

PROOF. Let M be as above and assume $\mathrm{usn}(\mathbf{R}) = 2$. Moreover, let ϕ be an arbitrary endomorphism of M. Then ϕ can obviously be expressed as

$$\phi = \eta + \rho + \delta, \tag{1}$$

where η is an α-endomorphism, ρ is a β-endomorphism and δ is a d-endomorphism. By Corollary 3.3 there is a strictly ascending sequence of integers $0 = r_0 < r_1 < r_2 \ldots$ such that $[e_i\eta] \subseteq \{i+1, \ldots, r_{s+2}-1\}$ or $e_i\eta = 0$, for all $r_s \le i < r_{s+1}$ ($s \in \omega$).

Now, by Lemma 3.4 there are locally nilpotent α-endomorphisms η_1 and η_2 such that $\eta_1 + \eta_2 = \eta$.

Recall that $e_i\eta_1 = 0 = e_i\eta_2$ for $j \in I_1 := \bigcup_{t \in w} \{k \in \omega \mid r_{2t} \le k < r_{2t+1}\}$ and $i \in I_2 := \bigcup_{t \in w} \{k \in \omega \mid r_{2t+1} \le k < r_{2t+2}\}$, while $e_i\eta_1 = e_i\eta$ for $i \in I_1$ and $e_j\eta_2 = e_j\eta$ for $j \in I_2$.

Notice that, for any d-endomorphism γ of M and any arbitrary element m of M, the support of $m\gamma$ is contained in the support of m. Therefore we have, in fact, obtained that $\gamma\eta_1$ and $\gamma\eta_2$ are locally nilpotent α-endomorphisms for any d-endomorphism γ of M (see proof of Lemma 3.4).

Next we decompose the β-endomorphism ρ. For each $i<\omega$ we write $e_i\rho$ as

$$e_i\rho = \sum_{j<i} e_j b_{ij} = \sum_{\substack{j<i\\ j\in I_1}} e_j b_{ij} + \sum_{\substack{j<i\\ j\in I_2}} e_j b_{ij}$$

and we define ρ_1, ρ_2 correspondingly, i.e.

$$e_i\rho_1 = \sum_{\substack{j<i\\ j\in I_1}} e_j b_{ij} \quad e_i\rho_2 = \sum_{\substack{j<i\\ j\in I_2}} e_j b_{ij}.$$

Clearly, $\rho_1+\rho_2=\rho$ and $\gamma\rho_1$ and $\gamma\rho_2$ are also β-endomorphisms of M for any d-endomorphism of M. Note that any β-endomorphism is locally nilpotent by Lemma 2.5. Moreover, the definitions of η_1, η_2, ρ_1, ρ_2 imply immediately that $\rho_1\eta_2=0=\rho_2\eta_1$. In fact, given any d-endomorphism γ of M we have $\gamma\rho_1\gamma\eta_2=0=\gamma\rho_2\gamma\eta_1$.

Now we consider the d-endomorphism δ. For each $i<\omega$ there is an element a_i of $\mathbf{R}$ such that $e_i\delta=a_ie_i$. Since usn($\mathbf{R}$) $=2$, there are units u_{i1}, u_{i2} of $\mathbf{R}$ $(i<n)$ such that $a_i=u_{i1}+u_{i2}$. Putting $e_i\delta_1=u_{i1}e_i$ and $e_i\delta_2=u_{i2}e_i$ for each $0\le i<\omega$ we obtain d-endomorphisms δ_1 and δ_2 of M, which are, in fact, automorphisms of M and satisfy $\delta=\delta_1+\delta_2$.

Finally, we rewrite Equation (1) as follows:

$$\begin{aligned}\phi &= \eta+\rho+\delta = \eta_1+\eta_2+\rho_1+\rho_2+\delta_1+\delta_2 = (\eta_1+\rho_2+\delta_1)+(\eta_2+\rho_1+\delta_2)\\ &= \delta_1(\delta_1^{-1}(\eta_1+\rho_2)+1)+\delta_2(\delta_2^{-1}(\eta_2+\rho_1)+1) \qquad (2)\end{aligned}$$

By Lemma 2.3, $\delta_1^{-1}(\eta_1+\rho_2)$ is locally nilpotent since $\delta_1^{-1}\eta_1$ and $\delta_1^{-1}\rho_2$ are locally nilpotent and satisfy $\delta_1^{-1}\rho_2\delta_1^{-1}\eta_1=0$. Therefore, by Lemma 2.2, $\delta_1^{-1}(\eta_1+\rho_2)+1$ is an automorphism of M and so $\delta_1(\delta_1^{-1}(\eta_1+\rho_2)+1)$ is also an automorphism.

Moreover using the same argument, we deduce that $\delta_2(\delta_2^{-1}(\eta_2+\rho_1)+1)$ is an automorphism. Therefore ϕ is a sum of two automorphisms of M as required. ■

Applying a method due to Castagna (see [1]), we can extend the above result to free modules of uncountable rank. We begin with the following lemma.

Lemma 3.6. *Let* $M=\bigoplus_{\alpha<\kappa}\mathbf{R}e_\alpha$ *be a free* $\mathbf{R}$*-module of uncountable rank* κ*, and* ϕ *any endomorphism of M. Then M can be written as the union of a smooth ascending chain* $\{H_\beta \mid \beta<\kappa\}$ *of submodules* H_β *of M of rank* $\beta<\kappa$ *such that:*

(i) $(H_\beta)\phi \subseteq H_\beta$ *for all* $\beta<\kappa$;

(ii) $H_{\beta+1}=H_\beta\oplus C_\beta$ *where* $0<\mathrm{rk}(C_\beta)\le\aleph_0$, *for all* $\beta<\kappa$;

(iii) $[H_\beta]\phi \subseteq H_\beta$ *for all* $\beta<\kappa$.

PROOF. See theorem 2.2 in [1]. ■

Theorem 3.7. *Let* $M = \bigoplus_{\beta<\kappa} \mathbf{R}e_\beta$ *be a free* **R**-*module of uncountable rank* κ. *If every free* **R**-*module of countable rank has unit sum number equal to* 2 *then* $\operatorname{usn}(M) = 2$.

PROOF. Let $\phi \in E(M)$ and write $M = \bigcup_{\beta<\kappa} H_\beta$ as in Lemma 3.6. Inductively we define automorphisms $\theta_{i,\beta}$ of H_β for each $\beta < \kappa$ and for each $i = 1, 2$ such that $\phi\restriction_{H_\beta} = \sum_{i=1}^{2}\theta_{i,\beta}$ and if $\alpha < \beta$, then $\theta_{i,\beta}\restriction_{H_\alpha} = \theta_{i,\alpha}$ for $i = 1, 2$.

For $\beta = 0$, $H_0 = 0$ and therefore $\phi\restriction_{H_0} = 0 = \sum_{i=1}^{2} 0$. Since $H_0 = 0$ the endomorphism 0 is injective and surjective and so is an automorphism of H_0. For $\alpha < \beta$ assume that $\{\theta_{i,\alpha} \mid i = 1, 2\}$ has been suitably defined.

Let β be a limit ordinal. For any $i = 1, 2$ define $\theta_{i,\beta} = \bigcup_{\alpha<\beta} \theta_{i,\alpha}$, which is well defined since each $\theta_{i,\alpha}$ is an extension of $\theta_{i,\delta}$, for each $\delta < \alpha$. Moreover, for any $i =$ 1, 2, we have that $\theta_{i,\beta}$ is an automorphism of H_β as each $\theta_{i,\alpha}$ is an automorphism of H_α for each $\alpha < \beta$. Now assume β is not a limit ordinal.

Let $\beta = \alpha + 1$. By Lemma 3.6 we have $H_{\alpha+1} = H_\alpha \oplus C_\alpha$ where $0 < \operatorname{rk}(C_\alpha) \leq \aleph_0$. Define π_1 and π_2 as the projections of $H_{\alpha+1}$ onto H_α and C_α, respectively. Then $(\phi\restriction_{C_\alpha})\pi_2$ is an endomorphism of C_α. Since $0 < \operatorname{rk}(C_\alpha) \leq \aleph_0$ there exist $\{\psi_i\}_{i=1,2}$ such that $\psi_i \in Aut(C_\alpha)$ for each $i = 1, 2$ and such that $(\phi\restriction_{C_\alpha})\pi_2 = \sum_{i=1}^{2}\psi_i$.

For each $c \in C_\alpha$ define $v_c \in H_\alpha$ as $v_c = (c\phi)\pi_1$. Note that $\phi\pi_1$ is a mapping from C_α to H_α.

For each $i = 1, 2$ define $\theta_{i,\alpha+1}$ on $H_{\alpha+1}$ by

$$(x + c)\theta_{1,\alpha+1} = x\theta_{1,\alpha} + c\psi_1 + v_c,$$

$$(x + c)\theta_{2,\alpha+1} = x\theta_{2,\alpha} + c\psi_2,$$

where $x \in H_\alpha$ and $c \in C_\alpha$. For each $i = 1, 2$, it is clear that $\theta_{i,\alpha+1}$ is a homomorphism which is an extension of $\theta_{i,\alpha}$.

Next we show that $\theta_{1,\alpha+1}$ and $\theta_{2,\alpha+1}$ are automorphisms of $H_{\alpha+1}$.

Consider the kernel of $\theta_{1,\alpha+1}$. For $x \in H_\alpha$, $c \in C_\alpha$ with $x + c \in \ker \theta_{1,\alpha+1}$ we have $0 = (x + c)\theta_{1,\alpha+1} = x\theta_{1,\alpha} + c\psi_1 + v_c = (x\theta_{1,\alpha} + v_c) + c\psi_1$. Now, $(x\theta_{1,\alpha} + v_c) \in H_\alpha$ and $c\psi_1 \in C_\alpha$. Since ψ_1 is an automorphism of C_α then $c = 0$ and hence $v_c = 0$. Since, by assumption, $\theta_{1,\alpha}$ is an automorphism of H_α, $x = 0$. Therefore $\ker \theta_{1,\alpha+1} = 0$.

We now show that $\theta_{1,\alpha+1}$ is surjective. Let $a+b$ be an arbitrary element of $H_{\alpha+1} = H_\alpha \oplus C_\alpha$, where $a \in H_\alpha$ and $b \in C_\alpha$. Then $a + b = ((a - v_c)\theta_{1,\alpha}^{-1})\theta_{1,\alpha} + (b\psi_1^{-1})\psi_1 + v_c$, where $c = b\psi_1^{-1} \in C_\alpha$ and $v_c = (c\phi)\pi_1$. Therefore, letting $x = (a - v_c)\theta_{1,\alpha}^{-1} \in H_\alpha$, we have $a+b = (x+c)\theta_{1,\alpha+1}$. Therefore $\theta_{1,\alpha+1}$ is an automorphism of $H_{\alpha+1}$.

A similar argument shows $\theta_{2,\alpha+1}$ is also an automorphism of $H_{\alpha+1}$.

It remains to show that $\phi\restriction_{H_{\alpha+1}} = \sum_{i=1}^{n}\theta_{i,\alpha+1}$. Let $(x+c)\in H_{\alpha+1}$ $(x\in H_\alpha,\ c\in C_\alpha)$. Then

$$\begin{aligned}(x+c)\left(\sum_{i=1}^{2}\theta_{i,\alpha+1}\right) &= x\left(\sum_{i=1}^{2}\theta_{i,\alpha}\right)+c\left(\sum_{i=1}^{2}\psi_i\right)+v_c\\ &= x(\phi\restriction_{H_\alpha})+(c\phi)\pi_2+(c\phi)\pi_1\\ &= x\phi+(c\phi)(\pi_2+\pi_1)=x\phi+c\phi=(x+c)\phi.\end{aligned}$$

Finally we conclude that $\phi=\sum_{i=1}^{2}\theta_i$, where $\theta_i=\bigcup_{\beta<\kappa}\theta_{i,\beta}$ is an automorphism for $i=1,2$. ■

Corollary 3.8. *Let M be a free* **R***-module of arbitrary rank. If* $\mathrm{usn}(\mathbf{R})=2$*, then* $\mathrm{usn}(M)=2$.

Proof. The proof follows directly from Theorem 3.5 and Theorem 3.7. ■

Since local rings such as J_p, the ring of p-adic integers, and $\mathbb{Z}_{(p)}$, the ring of integers localised at the prime p, have unit sum numbers of 2, our result above extends known results in [7; 9; 13; 15; 16].

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