SOME VARIANTS OF WEBER'S THEOREM

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Abstract

Weber's theorem says that if $A : H \to H$ is bounded and linear on a separable Hilbert space, then any operator that is compact, commutes with A and lies in the weak closure of the range of the inner derivation induced by A must also be quasinilpotent. In this note we consider related problems for generalised inner derivations associated with operators A and B on H.

1. Introduction

For bounded linear operators $T : X \to Y$ and $S : Y \to Z$ on Banach spaces the condition

$$S^{-1}(0) \cap T(X) = 0 \tag{1.1}$$

is equivalent to equality $(ST)^{-1}(0) = T^{-1}(0)$; when $X = Y = Z$ and $T = S^n$ this is the familiar condition that the operator S 'has ascent $\leq n$'. Stronger conditions would replace the range $T(X)$ by its closure, either in the norm or in some weaker topology; weaker conditions would ask that the intersection of $S^{-1}(0) \cap T(X)$ with some subspace of Y was in some sense nearly zero. Thus Kleinecke [3] showed that if $X = Y = Z = \mathscr{A}$ for a Banach algebra $\mathscr{A}$ and $S = T = \delta_a : x \mapsto ax - xa$ is an inner derivation on $\mathscr{A}$ then

$$S^{-1}(0) \cap T(X) \subseteq Q, \tag{1.2}$$

where $Q = QN(\mathscr{A})$ is the quasinilpotents in $\mathscr{A}$. Weber [6] showed for the same S and T that when $\mathscr{A} = B(H)$ for separable Hilbert space H, then

$$S^{-1}(0) \cap cl_\tau T(X) \cap J \subseteq Q, \tag{1.3}$$

where cl_τ represents the closure in $B(H)$ with respect to the weak operator topology $\tau = w$ and $J = K(H)$ is the compact operator. In this note we consider more generally $S = \delta_{AB} : U \mapsto AU - UB$ with either $T = S$ or $T = \delta_{A^*B^*}$, and we find that, for example, (1.3) holds for $Q = \{0\}$ and $S = \Delta_{AB}$ and $T = \Delta_{A^*B^*}$ when J is the finite rank operators and $\tau = w$, the weak operator topology, and also when J is the trace class and $\tau = w^*$, the ultra weak operator topology.

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2. Preliminaries

Let C_1 be the ideal of trace class operator, that is, all compact operators $A \in B(H)$ for which the eigenvalues of $(TT^*)^{\frac{1}{2}}$ counted according to multiplicity are summable. The ideal C_1 admits a complex valued function $tr(T)$ that has the characteristic properties of the trace for matrices. The trace function is defined by $tr(T) = \sum_n (Te_n, e_n)$, where (e_n) is any complete orthonormal sequence in H. As Banach spaces, C_1 may be identified with the conjugate space of the ideal K(H) of compact operators by means of the linear isometry $T \mapsto f_T$, where $f_T(X) = tr(XT)$. Moreover, $B(H)$ is the conjugate space of C_1. The ultraweak continuous linear functionals on $B(H)$ are those of the form f_T for some $T \in C_1$. Furthermore, the weak continuous linear functionals on $B(H)$ are those of the form f_T, where T is of finite rank.

We begin with the following result of Weber [6].

Thorem 2.1. *Let $A \in B(H)$ and let T be a trace class operator which commutes with A. If either*

(1) *T is the weak limit of a sequence of elements in $R(\delta_A)$, or*

(2) *T is of finite rank and in the weak closure of $R(\delta_A)$,*

then $tr\,(T^n) = 0$ for all $n \geq 2$.

Theorem 2.2. *Let T be an operator for which there exists a projector $P \in \{T\}''$ (bicommutant of T) such that $T|_{R(P)}$ is of finite rank and $tr(T^n) \neq 0$. Then $T \notin \overline{R(\delta_A)}^w \cap \{A\}'$, where $\{A\}'$ is the commutant of A.*

Proof. Suppose that $AX_\alpha - X_\alpha A \xrightarrow{w} T \in \{A\}'$, since $P \in \{T\}''$; also we have $P \in \{T\}'$. Consequently, the subspace $R(P)$ reduces T and A. Hence we can write with respect to the decomposition $H = R(P) \oplus R(I-P)$ topologic direct sum

$$T = \begin{bmatrix} S & 0 \\ 0 & * \end{bmatrix}, \quad A = \begin{bmatrix} B & * \\ 0 & * \end{bmatrix}, \quad X_\alpha = \begin{bmatrix} Y_\alpha & * \\ * & * \end{bmatrix}.$$

Since $AX_\alpha - X_\alpha A \xrightarrow{w} T \in \{A\}'$, then $BY_\alpha - Y_\alpha B \xrightarrow{w} S \in \{B\}'$. Since $S = T|_{R(P)}$, hence it follows from Theorem 2.1, that $tr(T^n) = 0$, which contradicts the hypothesis on T. ■

Corollary 2.1. (*Weber's theorem*). *Let $A \in B(H)$, if K is a compact operator in $\overline{R(\delta_A)}^w \cap \{A\}'$, then K is quasinilpotent.*

Proof. Since the Riesz projection associated with a non-vanishing point of the spectrum of the compact operator K, is of finite rank and belongs to the bicommutant of K, hence it suffices to apply Theorem 2.2. ■

3. Quasinilpotent operators in $\overline{R(\delta_A)} \cap \{A^*\}'$

Let $\wp$ denote a class of operators, satisfying the following properties:

(1) If $A \in \wp$ and M is an invariant subspace of A, then $A|_M \in \wp$;

(2) If $A \in \wp$ and the restriction of A to an invariant subspace is normal, then M reduces A;
(3) If $A|_M \in \wp$ and M is finite dimensional, then $A|_M$ is normal. As a trivial example of the class $\wp$ one can consider $\wp = \{0\}$; an intersting class is $\wp$, the class of all hyponormal operators.

An operator $A \in B(H)$ is said to be *k-quasihyponormal*, for $k \geq 1$ some integer, if

$$\|A^*A^k x\| \geq \|A^{k+1}x\|$$

for all x in H, or equivalently $A^{*k}(A^*A - AA^*)A^k \geq 0$. A 1-*quasihyponormal* is quasihyponormal. An operator $A \in B(H)$ is called *dominant* by Stampfli and Wadhwa [5] if, for all complex λ, $range(A-\lambda) \subseteq range(A-\lambda)^*$, or equivalently, if there is a real number $M_\lambda \geq 1$ such that

$$\|(A-\lambda)^* f\| \leq M_\lambda \|(A-\lambda)f\|,$$

for all $f \in H$. If there exists a real number M such that $M_\lambda \leq M$ for all λ, the dominant operator A is said to be M-*hyponormal*. A 1-*hyponormal* is hyponormal.

Theorem 3.1. *Let $A \in B(H)$. If $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then*

$$A \in \wp \Rightarrow \{\lambda \in \sigma_p(T^*) : \dim \ker (T^* - \overline{\lambda}) < \infty\} \subset \{0\},$$
$$A^* \in \wp \Rightarrow \{\lambda \in \sigma_p(T) : \dim \ker (T - \lambda) < \infty\} \subset \{0\},$$

where $\sigma_p(T)$ is the point spectrum of T.

PROOF. We start with the second assertion. Suppose that $A^* \in \wp$ and $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$. Let $\lambda \in \sigma_p(T)$ such that $E = \ker(T-\lambda)$ is finite dimensional, then the subspace E is invariant under T and A^*. Since $A^* \in \wp$, E reduces A^*. Let $H = E \oplus E^\perp$; hence we can write

$$A^* = \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix}, \quad T = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}.$$

Since $T \in \overline{R(\delta_A)}^w$, $\lambda I_E \in R(\delta_{A_1})$ and this implies that $\lambda = 0$. This completes the proof of the second assertion.

Remark that if $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$, then

$$T^* \in \overline{R(\delta_{A^*})}^w \cap \{A\}'.$$

Then the first assertion of the theorem follows in exactly the same way as the second. ■

Corollary 3.1. *If A or $A^* \in \wp$. Then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent.*

PROOF. Suppose that $T \in \overline{R(\delta_A)}^w \cap \{A^*\}'$ with T compact and $\lambda \in \sigma(T) \backslash \{0\}$. Then, $\lambda \in \sigma_p(T)$ with $\dim \ker(T-\lambda) < \infty$ and $\overline{\lambda} \in \sigma_p(T^*)$ with $\dim \ker(T^* - \overline{\lambda}) < \infty$. Thus, T is quasinilpotent by Theorem 3.1. ■

Let $\mathscr{S}$, $\mathscr{M}$, $\mathscr{D}$ and $\mathscr{Q}(k)$ be respectively the class of subnormal, M-hyponormal, dominant and k-quasihyponormal operators.

Corollary 3.2. *Let $A \in B(H)$. Then every compact operator in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ is quasinilpotent in each of the following cases*:

(1) $A \in \mathscr{S}$ *or* $A^* \in \mathscr{S}$;
(2) $A \in \mathscr{M}$ *or* $A^* \in \mathscr{M}$;
(3) $A \in \mathscr{D}$ *or* $A^* \in \mathscr{D}$;
(4) $A \in \mathscr{Q}(k)$ *or* $A^* \in \mathscr{Q}(k)$.

Proof. Adapted from Duggal [2], if we have (1), (2), (3) and (4), then A, $A^* \in \wp$. It suffices to apply Corollary 3.1. ■

Let $\mathfrak{I}$ and C_1 be respectively the ideal of finite rank and the ideal of trace class operators on H. The weakly continuous linear form (*resp*. the ultra-weakly continuous linear form) on $B(H)$ is defined by; Φ_T, where $T \in \mathfrak{I}$ (resp. $T \in C_1$) with $\Phi_T = tr(TX) = tr(XT)$ for every $X \in B(H)$ (see [1, p. 23]).

Theorem 3.2. *Let $A, B \in B(H)$. Then*

(1) *every finite rank operator in* $\overline{R(\delta_{A,B})}^w \cap \ker \delta_{A^*,B^*}$ *vanishes*,
(2) *every trace class operator in* $\overline{R(\delta_{A,B})}^{w*} \cap \ker \delta_{A^*,B^*}$ *vanishes.*

Proof. (1) Let $T \in \overline{R(\delta_{A,B})}^w \cap \ker \delta_{A^*,B^*} \cap \mathfrak{I}$. Then,

$$T^* \in \ker \delta_{A,B} \cap \mathfrak{I};\ T^*A = BT^*.$$

Since $T \in \overline{R(\delta_{A,B})}^w$, there exists a generalised sequence $\{X_\alpha\}$ such that

$$AX_\alpha - X_\alpha B \xrightarrow{w} T,$$

hence,

$$\Phi_{T^*}(AX_\alpha - X_\alpha B) \xrightarrow{w} \Phi_{T^*}(T)$$

and we have

$$\Phi_{T^*}(AX_\alpha - X_\alpha B) = tr(T^*(AX_\alpha - X_\alpha B)) = tr(T^*AX_\alpha) - tr(T^*X_\alpha B).$$

So,

$$0 = \Phi_{T^*}(AX_\alpha - X_\alpha B) \xrightarrow{w} \Phi_{T^*}(T).$$

Then, $\Phi_{T^*}(T) = tr(TT^*) = 0$, that is, $TT^* = 0$ and thus $T = 0$.

(2) It suffices to replace $\mathfrak{I}$ by C_1 in the above proof. ■

Remark 3.1. *The problem of classifying all non-compact operators in $\overline{R(\delta_A)}^w \cap \{A^*\}'$ or in $\overline{R(\delta_A)}^w \cap \{A\}'$ for a given $A \in \mathscr{B}(H)$ is still an open problem.*

Theorem 3.3. *Let* $A, B \in B(H)$. *Then*

(1) $\overline{R(\delta_{A,B})}^{w} = B(H) \Leftrightarrow \ker \delta_{B,A} \cap \mathfrak{I}_1 = 0$,

(2) $\overline{R(\delta_{A,B})}^{w*} = B(H) \Leftrightarrow \ker \delta_{B,A} \cap C_1 = 0$.

Proof. (1) $\Rightarrow$) Suppose that $\overline{R(\delta_{A,B})}^{w} = B(H)$ and $T \in \ker \delta_{B,A} \cap \mathfrak{I}$, then

$$T^* \in \overline{R(\delta_{A,B})}^{w} \cap \ker \delta_{A^*,B^*},$$

hence $T = 0$ by Theorem 3.2.

$\Leftarrow$) Suppose that there exists $T \in B(H) \backslash \overline{R(\delta_{A,B})}^{w}$, then there exists $S \in \mathfrak{I}$ such that $tr\,(ST) \neq tr\,(SX) = 0$, for every $X \in R(\delta_{A,B})$. So, $T \in \ker \delta_{B,A} \cap \mathfrak{I}$ and $S \neq 0$.

(2) It suffices to replace $\mathfrak{I}$ by C_1 in the above proof. ■

4. Cases where $\overline{R(\delta_{A,B})}^{w} \cap \ker \delta_{A^*,B^*} = \{0\}$

Definition 4.1. *An operator* $A \in B(H)$ *is called algebraic if* $P(A) = 0$ *for some non-trivial polynomial P. If P is of degree* ≤ 2 *we shall call A quadratic*.

Lemma 4.1. *Let* $A \in B(H)$ *and P,Q be two positive operators. If* $AP + QA = 0$, *then* $AP = QA = 0$.

PROOF. Since $AP + QA = 0$, then $AP^2 = Q^2A$, and so $A\sqrt{P^2} = \sqrt{Q^2}A$, that is, $AP = QA$. ■

Theorem 4.1. *Let* $A \in B(H)$. *If every positive operator in* $\overline{R(\delta_A)}^{w}$ *vanishes, then*

$$\overline{R(\delta_{A,B})}^{w} \cap \ker\delta_{A^*,B^*} = \{0\}$$

for every operator $B \in B(H)$.

PROOF. Suppose that $T^* \in \overline{R(\delta_{A,B})}^{w} \cap \ker \delta_{A^*,B^*}$, then $TT^* \in \overline{R(\delta_A)}^{w}$, since TT^* is a positive operator, it follows that, $TT^* = 0$ and $T = 0$. ■

Corollary 4.1. *Let* $A \in B(H)$ *be quadratic. Then*

$$\overline{R(\delta_{A,B})}^{w} \cap \ker \delta_{A^*,B^*} = \{0\}$$

for every operator $B \in B(H)$.

PROOF. It suffices to prove that every positive operator in $\overline{R(\delta_A)}^{w}$ vanishes. Indeed, since trivially $\delta_{A^2-\lambda} = \delta_{A^2}$ for every complex λ, it suffices to consider $P(A) = A^2 - 2\beta A$. Let S be a positive operator, if $S \in \overline{R(\delta_A)}^{w}$, then $AS + SA \in \overline{R(\delta_{A^2})}^{w}$. Therefore $AS + SA - 2\beta S \in \overline{R(\delta_{A^2-2\beta A})}^{w}$. So $AS + SA - 2\beta S = 0$, or $S(A-\beta) + (A-\beta)S = 0$. Since S is positive, it results from Lemma 4.1, that $AS = SA = \beta S$.

Since $S \in \overline{R(\delta_A)}^w$, then there exists a generalised sequence $\{X_\alpha\}$ such that $AX_\alpha - X_\alpha A \xrightarrow{w} S$. Then $0 = SAX_\alpha S - SX_\alpha AS \xrightarrow{w} S^3$, hence $S^3 = 0$, and so $S = 0$. ■

Remark 4.1. *If A is quadratic, then*

$$\overline{R(\delta_A)}^w \cap \ker \delta_{A^*} = \{0\},$$

in particular if A is nilpotent of order 2, $\overline{R(\delta_A)}^w \cap \ker \delta_{A^*} = \{0\}$.

We can ask the following questions:

(1) *Which positive operators belong to* $\overline{R(\delta_A)}^w$?
(2) *Is* $\overline{R(\delta_A)}^w \cap \ker \delta_{A^*} = \{0\}$, *if A is nilpotent of order 3*?

Corollary 4.2. *Let* $A \in B(H)$. *Suppose that* $A = A_1 \oplus A_2$ *on* $H_1 \oplus H_2$ (*orthogonal direct sum*), *where* A_1 *is such that* $P(A_1) = 0$ *for some quadratic polynomial* $P(z)$ *and* A_2 *is such that* $\overline{R(\delta_{A_2})}^w \cap \ker \delta_{A_2^*} = \{0\}$. *Then*

$$\overline{R(\delta_A)}^w \cap \ker \delta_{A^*} = \{0\}.$$

Proof. Let $T^* \in \overline{R(\delta_A)}^w \cap \ker \delta_{A^*}$. Then there exists a generalised sequence $\{X_\alpha\}$ in $B(H)$ such that

$$(A_1 \oplus A_2)X_\alpha - X_\alpha(A_1 \oplus A_2) \xrightarrow{w} T^*; \ (A_1 \oplus A_2)T = T(A_1 \oplus A_2).$$

Then we can write respectively A, X_α and T on $H_1 \oplus H_2$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}; \quad X_\alpha = \begin{bmatrix} X_\alpha^1 & X_\alpha^2 \\ X_\alpha^3 & X_\alpha^4 \end{bmatrix}; \quad T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

and we have

$$A_1X_\alpha^1 - X_\alpha^1 A_1 \xrightarrow{w} T_1^*; A_1T_1 = T_1A_1,$$

$$A_2X_\alpha^4 - X_\alpha^4 A_2 \xrightarrow{w} T_4^*; A_2T_4 = T_4A_2,$$

$$A_1X_\alpha^2 - X_\alpha^2 A_2 \xrightarrow{w} T_2^*; A_1T_2 = T_2A_2,$$

$$A_2X_\alpha^3 - X_\alpha^3 A_1 \xrightarrow{w} T_3^*; A_2T_3 = T_3A_1.$$

Hence

$$T_1^* \in \overline{R(\delta_{A_1})}^w \cap \ker \delta_{A_1^*},$$

$$T_4^* \in \overline{R(\delta_{A_2})}^w \cap \ker \delta_{A_2^*},$$

$$T_2 \in \overline{R(\delta^w_{A_1^*, A_2^*}) \cap \ker \delta_{A_1, A_2}},$$

$$T_3^* \in \overline{R(\delta_{A_1, A_2})}^w \cap \ker \delta_{A_1^*, A_2^*}.$$

By applying Corollary 4.1 and Remark 4.1, we obtain $T_1 = T_2 = T_3 = T_4 = 0$, hence $T = 0$. ■

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