GENERALISED WEYL'S THEOREM FOR A CLASS OF OPERATORS SATISFYING A NORM CONDITION

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Abstract

A bounded linear operator $A \in \mathcal{B}(X)$, X a Banach space, is *heredetarily normaloid* if, whenever $M \subseteq X$ is a closed invariant suspace of A, the restriction $A_{|M}$ of A to M is normaloid. It is shown that the generalised Weyl's theorem holds for heredetarily normaloid operators on Banach spaces, in particular, for paranormal operators.

1. Introduction

Throughout this note X will denote a Banach space, and $\mathcal{B}(X)$ will denote the algebra of bounded linear operators acting on X. If $A \in \mathcal{B}(X)$ we shall write $N(A)$ and $R(A)$ for the null space and range of A. We say that the operator A is *Weyl* if it is Fredholm of index 0; the *Weyl spectrum* $\sigma_w(A)$ of A is the set

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm of index zero}\}.$$

Suppose $A \in \mathcal{B}(X)$. We say [11] that Weyl's theorem holds for A provided that

$$\sigma(A) \backslash \sigma_w(A) = \pi_{00}(A):$$

the complement in the spectrum of the Weyl spectrum consists of the isolated points of the spectrum of finite multiplicity. More generally, Berkani in [2] says that the generalised Weyl's theorem holds for A provided

$$\sigma(A) \backslash \sigma_{Bw}(A) = E(A):$$

here $E(A)$ denotes the isolated points of the spectrum which are eigenvalues (no restriction on multiplicity), while $\sigma_{Bw}(A)$ is the set of complex numbers λ for which $A - \lambda I$ fails to be B-Weyl. Berkani [2] has called an operator $A \in \mathcal{B}(X)$ B-Fredholm if there exists $n \in \mathcal{N}$ for which the induced operator $A_n : A^n(X) \to A^n(X)$ is Fredholm in the usual sense, and B-Weyl if in addition A_n has index zero. As Berkani has shown in [2], if the generalised Weyl's theorem holds for A, then so does Weyl's theorem. Our

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main result in this note is that the generalised Weyl's theorem holds for the operators $A \in \mathcal{B}(X)$, which are in a certain sense heredetarily normaloid (see Definition 1 below).

2. Class of operators satisfying norm condition

Recall that the numerical range $V(\mathcal{B}(X), A)$ of $A \in \mathcal{B}(X)$ is the closed convex subset

$$V(\mathcal{B}(X), A) = \{f(A) : f \in \mathcal{B}(X)^*, \|f\| = \|f(I)\| = 1\}$$

of $\mathcal{C}$; when this is real then A is said to be *hermitian*, and *normal* operators are generated in the usual way by commuting pairs of hermitians. The bounded linear operator $A \in \mathcal{B}(X)$ is *normaloid* if there is equality

$$\|A\| = r(A) = v(A),$$

where $\|A\|$ is the usual operator bound of A, $r(A)$ is its spectral radius and $v(A)$ is its numerical radius. For our class of heredetarily normaloid operators, this must hold for all restrictions to invariant subspaces.

Definition 2.1. The class HN of *heredetarily normaloid* operators between Banach spaces consists of those operators $A \in \mathcal{B}(X)$ for which, whenever $M \subseteq X$ is a (closed) invariant subspace of A, the restriction $A_{|M}$ is normaloid. We say that the operator $A \in HN$ is *totally heredetarily normaloid*, or $A \in THN$, if whenever $A_{|M} \in \mathcal{B}(M)^{-1}$ is invertible the inverse $A_{|M}^{-1}$ is normaloid.

For heredetarily normaloid operators, isolated points of the spectrum are simple poles of the resolvent.

Lemma 2.1. *If $A \in \mathcal{B}(X)$ is heredetarily normaloid and $\lambda \in \mathcal{C}$ is an isolated point of $\sigma(A)$, then λ is a simple pole of the resolvent $R_z(A) = (zI - A)^{-1} : \mathcal{C} \backslash \sigma(A) \to \mathcal{B}(X)$.*

PROOF. If $\lambda \in \text{iso}\ \sigma(A)$, write $P_\lambda(A)$ for the spectral projection, given in [12; 20] by the familiar Cauchy integral. Then

$$R(P_\lambda(A)) = \{x \in X : \lim_{n\to\infty} \|(A-\lambda)^n x\|^{1/n} = 0\},$$

$(A-\lambda)|_{\mathcal{N}}$ is invertible and $X = \mathcal{M} \oplus \mathcal{N}$ [12; 18], where we have the set $R(P_\lambda(A)) = \mathcal{M}$ and $N(P_\lambda(A)) = \mathcal{N}$. Now, if in addition $A \in HN$, then there are two cases: if $\lambda = 0$, then $A_1 = A_{|\mathcal{M}} \in HN$ and $\sigma(A_1) = \{0\}$. Since A_1 is normaloid, $A_1 = 0$ and $A(\mathcal{M}) = 0$. This implies that 0 is a simple pole of $P_0(A)$.

If $\lambda \neq 0$, then $A_1 = \frac{1}{\lambda} A_{|\mathcal{M}} \in H\mathcal{N}$, $\sigma(A_1) = \{1\}$, $\|A_1\| = r(A_1) = v(A_1) = 1$, and A_1 is invertible. This by [14, theorem 2] implies that T_1 is similar to an invertible isometry B (on an equivalent normed linear space) with $\sigma(B) = \{1\}$. Applying [14, theorem 5] it follows that 1 is an eigenvalue of B. A_1 and B being similar, 1 is an eigenvalue of A_1. Hence $(A-\lambda)(\mathcal{M}) = 0$, and λ is a simple pole of the resolvent of A. ■

Our totally heredetarily normaloid operators inherit some of the properties of normal operators on a Banach space; for example, eigenspaces corresponding to distinct (non-zero) eigenvalues are mutually orthogonal. Here we write $x \perp y$ to mean that

$$\lambda \in \mathbb{C} \Rightarrow \|x\| \leq \|x + \lambda y\|,$$

and $M \perp N$ to mean that

$$(x, y) \in M \times N \Rightarrow x \perp y.$$

Note that in general this is not a symmetric relation, although it reduces to the usual (symmetric) orthogonality when X is a Hilbert space.

Lemma 2.2. *If $A \in THN$, then $N(A - \alpha) \perp N(A - \beta)$ for distinct scalars $\alpha(\neq 0)$ and β. In particular, if $A \in$ HN then eigenspaces corresponding to different non-zero eigenvalues of A are orthogonal.*

PROOF. Let M denote the subspace generated by $x, y \in X$ such that $(A - \alpha)x = 0 = (A - \beta)y$. Let $A_1 = A_{|M}$; then $\sigma(A_1) = \{\alpha, \beta\}$, $r(A_1) = |\alpha|$ if $|\beta| \leq |\alpha|$ and $r(A_1) = |\beta|$ if $|\alpha| < |\beta|$. But if $0 < |\alpha| < |\beta|$ then A_1 is invertible with $\sigma(A_1^{-1}) = \{\alpha^{-1}, \beta^{-1}\}$, $r(A_1^{-1}) = |\alpha|^{-1}$ and $(A_1^{-1} - \alpha^{-1})x = 0 = (A_1^{-1} - \beta^{-1})y$. Hence to prove the lemma it will suffice to consider the case $|\beta| \leq |\alpha|$ and show that $\|x\| \leq \|x + y\|$.

If $|\beta| \leq |\alpha|$, then (by the normaloid property of A_1)

$$r(A_1) = v(A_1) = |\alpha| \text{ and } \alpha \in \partial V(\mathcal{B}(M), A_1),$$

where $\partial V(\mathcal{B}(M), A_1)$ denotes the boundary of the numerical range of $A_1 \in \mathcal{B}(M)$. It follows from a result of Sinclair (that $N(A_1 - \alpha) \perp R(A_1 - \alpha)$, see [9; 19], and hence) that

$$\|(A_1 - \alpha)w + x\| \geq \|x\|$$

for all $x \in N(A_1 - \alpha)$ and $w \in M$. Let $P_\alpha(A_1)$ denote the Riesz projection associated with α and A_1. Then

$$R(A_1 - \alpha) = R(1 - P_\alpha(A_1)) = R(P_\beta(A_1)) = N(A_1 - \beta),$$

which implies that $\|y + x\| \geq \|x\|$. ∎

Remark. Lemma 3 fails for the case in which $\alpha = 0$. To see this, we let $X = H$ be a Hilbert space in Lemma 3, whence it follows that if Lemma 3 holds for all $\alpha \in \mathbb{C}$, then α is a normal eigenvalue of A (i.e. the eigenspace corresponding to α reduces A). Now let $A \in \mathcal{B}(H)$ be an operator such that $|A^k|^2 - 2\lambda|A|^k + \lambda^2 \geq 0$ for all $\lambda > 0$ and some even integer $k \geq 2$. Then $A \in HN$ (indeed, $A \in THN$ if $k = 2$) and the non-zero eigenvalues of A are normal (see [7, pp. 94–5]): however, there exists an operator A satisfying the above positivity condition such that $Ax = 0$ but $A^*x \neq 0$ (see [7, p. 96]).

Lemma 2.3. *If $A \in$ HN, then $\pi_0(A) = \pi_{00}(A) = E(A)$ and $\pi_{00}(A^*) = \pi_0(A^*) = E(A^*)$.*

PROOF. Let $\lambda \in E(A)$. Then λ being an isolated point of $\sigma(A)$ is a pole of $P_\lambda(A)$ of order 1, and hence $\lambda \in \pi_0(A) \subset \pi_{00}(A) \subset E(A)$.

If instead $\lambda \in E(A^*)$, then λ being isolated in $\sigma(A^*)$ is isolated in $\sigma(A)$. Hence λ is a pole of $P_\lambda(A)$ of order 1 and $(A - \lambda)P_\lambda(A) = 0$. Talking adjoint this implies that

$$P_\lambda^*(A)(A-\lambda)^* = 0 \Rightarrow P_\lambda(A^*)(A^*-\lambda) = 0,$$

and hence $\lambda \in \pi_0(A^*) \subset \pi_{00}(A^*) \subset (A^*)$, which implies that $\lambda \in \pi_0(A^*) = \pi_{00}(A^*) = E(A^*)$ ■

For the next lemma we need some terminology. Let F be a closed subset of $\mathbb{C}$. Then the (generally non-closed) subspace $\chi_A(F)$ is defined to be the set of $x \in X$ such that $(A-\lambda)f(\lambda) = x$ has an analytic solution $f : \mathbb{C}\backslash F \to X$. The *local spectrum* $\sigma_A(x)$ of A at $x \in X$ is defined by $\sigma_A(x) = \mathbb{C}\backslash \cup \{U : (A-\lambda)f(\lambda) = x$ has an analytic solution $f : U \to X$ on the open $U \subset \mathbb{C}\}$ and we define the space $\mathcal{K}_A(\{\lambda\})$ by $\mathcal{K}_A(\{\lambda\}) = \{x \in X : \lambda \in \mathbb{C}\{\sigma(x)\}$. It is known (see [17]) that $\chi_A(\{\lambda\})(= R(P_\lambda(A))) = \{x \in X : \lim_{n\to\infty} \|(A-\lambda)^n x\|^{1/n} = 0\}$ and $\mathcal{K}_A(\{\lambda\}) = \cup_{n=1}^{\infty} \chi_A(\mathbb{C}\backslash B(\lambda, 1/n))$.

Lemma 2.4. *If $A \in \mathcal{B}(X)$ satisfies the following conditions*:

(i) *$A_{|M}$ satisfies Weyl's theorem for each invariant subspace M of A;*

(ii) *$\chi_A(\{\lambda\})$ is finite dimensional for each $\lambda \in E(A)$,*

then A satisfies the generalised Weyl's theorem.

Proof. We start by proving that if $\lambda \in \sigma(A)\backslash\sigma_{Bw}(A)$, then $\lambda \in E(A)$.

Suppose that $\lambda \in \sigma(A)\backslash\sigma_{Bw}(A)$. Then $A-\lambda$ is a B-Fredholm operator of index zero and there exists a direct sum decomposition $X = X_1 \oplus X_2$ such that $A_1 = (A-\lambda)_{|X_1}$ is a Fredholm operator of index 0, $A_2 = (A-\lambda)_{|X_2}$ is nilpotent and $A - \lambda = A_1 \oplus A_2$ (see [2, lemma 4.1]). We have two possibilities: either $\lambda \in \sigma(A_{|X_1})$ or $\lambda \notin \sigma(A_{|X_1})$.

Suppose $\lambda \in \sigma(A_{|X_1})$. By hypothesis $A_{|X1}$ satisfies Weyl's theorem, and so if $\lambda \in \sigma(A_{|X_1})$, then $\lambda \in \pi_{00}(A_{|X_1})$. In particular $\lambda \in \mathrm{iso}\ \sigma(A_{|X_1})$. Since $A - \lambda = (A_{|X_1} - \lambda I_{|X_1}) \oplus A_2$, and A_2 is nilpotent, $\sigma(A_1)\backslash\{0\} = \sigma(A-\lambda)\backslash\{0\}$ and $\lambda \in \mathrm{iso}\ \sigma(A)$. This implies that $\lambda \in \pi_{00}(A) \subset E(T)$.

If instead $\lambda \notin \sigma(A_{|X_1})$, then λ is a pole of A which implies that $\lambda \in E(A)$.

Conversely, let $\lambda \in E(A)$. Then, since $\chi_A(\{\lambda\})$ is finite dimensional, $X = \chi_A(\{\lambda\}) \oplus \mathcal{K}_A(\{\lambda\})$, where $\mathcal{K}_A(\{\lambda\})$ is closed ([17, lemma 1]. Since $0 \neq N(A-\lambda) \subset \chi_A(\{\lambda\})$, λ is a pole of the resolvent $R_\lambda(A)$ (which implies the existence of a $q > 0$ such that the space $(A-\lambda I)^{-q}(0)$ is non-zero and complemented by the closed A-invariant subspace $(A-\lambda I)^q X \subset (A-\lambda I)X$ [15, theorem 3.4]. Hence $A-\lambda$ is B-Fredholm of index zero, i.e. $\lambda \notin \sigma_{Bw}(A)$. ■

Lemma 2.5. *If $A \in \mathrm{HN}$, then A and A^* satisfy Weyl's theorem.*

Proof. Let

$$\Delta_A = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A \text{ and } A-\lambda \text{ is Fredholm of index } 0\}.$$

Then by [10, theorem 1] a necessary and sufficient condition for A to satisfy Weyl's theorem is that: (i) *asc*$(A-\lambda) < \infty$ at every $\lambda \in \Delta_A$ and (ii) *dsc*$(A-\lambda) < \infty$ at every $\lambda \in \pi_{00}(A)$, where *asc* and *dsc* denote ascent and descent of an operator.

If $\lambda \in \Delta_A$ then there exists an algebraic direct sum decomposition $X = \mathcal{M} \oplus \mathcal{N}$, where $\mathcal{M} = N(A-\lambda)$ is finite dimensional and the closed subspace $\mathcal{N}$ satisfies $\mathcal{N} \subset (A-\lambda)X$. This by [15, theorem 3.2] implies that $\lambda \in \mathrm{iso}\ \sigma(A)$, and so λ is a pole of $R_\lambda(A)$ of order 1, both $asc(A-\lambda)$ and $dsc(A-\lambda)$ are finite [20, theorem V.10.5]. Hence, A satisfies Weyl's theorem.

Recall that $asc(A^*)=dsc(A)$ and $dsc(A^*)=asc(A)$ for a Fredholm operator. Hence, if $\lambda\in\Delta_{A^*}$, then $asc(A^*)$ and $dsc(A^*)$ are both finite and A^* satisfies Weyl's theorem. ■

Theorem 2.1. *If* $A\in\mathrm{HN}$, *then* $\sigma_{Bw}(A)=\sigma(A)\backslash E(A)$ *and* $\sigma_{Bw}(A^*)=\sigma(A^*)\backslash E(A^*)$.

PROOF. Combine Lemmas 2.3, 2.4 and 2.5 to prove that $\sigma_{Bw}(A)=\sigma(A)\backslash E(A)$. For the case of the operator A^*, if $\lambda\in E(A^*)$, then $\lambda\in\mathrm{iso}\,\sigma(A)$, which implies that $\lambda\notin\sigma_{Bw}(A)$. Hence $A-\lambda$ is a B-Fredholm operator of index 0. But then $A^*-\lambda$ is a B-Fredholm operator of index 0 [2, remark B(2)], and so $\lambda\notin\sigma_{Bw}(A^*)$. If instead $\lambda\notin\sigma_{Bw}(A^*)$, then $A^*-\lambda$ is a B-Fredholm operator of index zero which implies that λ is an isolated eigenvalue of A and hence $\lambda\in E(A^*)$. ■

3. Paranormal operators

A particularly important subclass of *HN* is the class of operators that satisfy the norm condition $\|Ax\|^2\le\|A^2x\|$ for each unit vector $x\in X$. Such operators have been called *paranormal* (see [1; 12]). (The class of paranormal operators coincides with the class of operators considered in the Remark, following the proof of Lemma 2.2, in the case in which $k=2$.) Let $\mathcal{P}$ denote the class of paranormal operators $A\in\mathcal{B}(H)$, where H is a Hilbert space, such that non-zero eigenvalues of A are normal. Then the class $\mathcal{P}$ properly contains a number of the commonly considered classes of (Hilbert space) operators, amongst them the class of hyponormal operators ($AA^*\le A^*A$), p-hyponormal operators ($\|A^*\|^{2p}\le\|A\|^{2p}$, $0<p<1$), quasihyponormal operators ($(A^*A)^2\le(A^{*2}A^2)$) and the class of operators satisfying the absolute value condition ($\|A\|^2\le\|A^2\|$) (see [7; 13] for further references). Our Theorem 2.1 implies that if $A\in\mathcal{B}(H)$ is in any of these classes, then A satisfies the generalised Weyl's theorem. (That operators $A\in\mathcal{P}$ satisfy Weyl's theorem was first proved by Chourasia and Ramanujan [5].) More is true, as we now proceed to show.

Let $Q(k)$ denote the class of k-quasihyponormal operators (i.e. the class of $A\in\mathcal{B}(H)$ for which $A^{*k}(A^*A-AA^*)A^k\ge0$ for some integer $k\ge1$). Neither of the classes $Q(k)$ and $\mathcal{P}$ is contained in the other; for example, operators in $Q(k)$ need not even be normaloid. We say that $A\in\mathcal{B}(X)$ has the *single valued extension property* (SVEP) if for every open set U of $\mathbb{C}$ the only analytic function $f\colon U\to X$ that satisfies the equation $(A-\lambda)f(\lambda)=0$ is the constant function $f\equiv0$ on U.

Lemma 3.1.

(i) *If* $A\in THN$ *and the Banach space* X *is separable, then* A *has SVEP.*
(ii) *If* A *is in either* $\mathcal{P}$ *or* $Q(k)$, *then* A *has SVEP.*

PROOF. Part (i) of the lemma is proved in [5] for the case in which $A\in\mathcal{B}(X)$ is paranormal. A similar argument works for operators $A\in THN$: we include the argument to be self contained. An operator with countable point spectrum $\sigma_p(A)$ has

SVEP [6], and we claim that if $A \in THN$ has uncountable point spectrum $\{\lambda_j\}$ then the space X can not be separable. This follows from Lemma 2.2: if x_j is a unit vector for λ_j then $i \neq j$ implies $||x_i - x_j|| \geq ||x_i|| = 1$.

(ii) If $A \in \mathcal{P}$, then $||Ax||^2 \leq ||A^2x||$ for each unit vector $x \in H$ implies that $N(A) = N(A^2)$. Let $\lambda \neq 0$ be an eigenvalue of A. Then λ is a normal eigenvalue of A, i.e. $N(A-\lambda) \subset N((A-\lambda)^*)$. Let $x \in N(A-\lambda)^2$; then $(A-\lambda)x \in N(A-\lambda) \subset N(A-\lambda)^*$ and so

$$((A-\lambda)^2x, x) = 0 = ((A-\lambda)^*(A-\lambda)x, x) = ||(A-\lambda)x||^2.$$

Hence if $A \in \mathcal{P}$, then $asc(A-\lambda) = 1$.

If $A \in Q(k)$ and $x \in N(A^{k+1})$, then

$$||A^kx||^2 = (A^*A^kx, A^{k-1}x) \leq ||A^*A^kx|| \; ||A^{k-1}x|| \leq ||A^{k+1}x|| \; ||A^{k-1}x|| = 0$$

and $x \in N(A^k)$. Since the non-zero eigenvalues of an operator $A \in Q(k)$ are normal eigenvalues of A [4], if $0 \neq \lambda \in \sigma_p(A)$ and $(A-\lambda)^{k+1}x = 0$, then $(A-\lambda)(A-\lambda)^kx = 0 = (A-\lambda)^*(A-\lambda)^kx$ and

$$||(A-\lambda)^kx|| = ((A-\lambda)^*(A-\lambda)^kx, (A-\lambda)^{k-1}x) = 0.$$

Hence, if $A \in Q(k)$, then $asc(A-\lambda) = k$.

Since operators with finite ascent have SVEP [16], the proof is complete. ■

It is known that SVEP is stable under the functional calculus, i.e. if $A \in \mathcal{B}(X)$ has SVEP, then so does $f(A)$ for each f analytic on a neighbourhood of $\sigma(A)$ [6]. Operators with SVEP satisfy Browder's theorem [8]. Hence if the Banach space X and the operator A are as in Lemma 3.1, then

$$f(\sigma(A)\backslash\pi_0(A)) = f(\sigma_b(A)) = \sigma_b(f(A)) = \sigma(f(A))\backslash\pi_0(f(A)) = f(\sigma(A))\backslash\pi_0(f(A))$$

and

$$f(\sigma_{Bw}(A)) = \sigma_{Bw}(f(A)).$$

Theorem 3.1. *If A is in either $\mathcal{P}$ or $Q(k)$, then $f(A)$ satisfies the generalised Weyl's theorem for every function f analytic on a neighbourhood of $\sigma(A)$.*

Proof. As seen above, $\sigma_{Bw}(f(A)) = f(\sigma_{Bw}(A)) = f(\sigma(A)\backslash E(A))$, and so to prove the theorem it is sufficient to prove that $f(\sigma(A)\backslash E(A)) = \sigma(f(A))\backslash E(f(A))$. But this follows because the operator A is isoloid in the sense that the isolated points of its spectrum are eigenvalues (see Lemma 2.1 and [3, lemma 2.9]). ■

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