THE COMMUTATOR SUBGROUP AND *CLT*(*NCLT*) GROUPS

By Fran Barry*
Department of Mathematics, National University of Ireland, Cork

[Received 31 July 2002. Read 19 June 2003. Published November 2004.]

Abstract

The commutator subgroup G' can indicate if a finite group G is a *CLT* (Converse Lagrange's Theorem) group or an *NCLT* (Non-Converse Lagrange's Theorem) group. We give general results and some examples of their application to groups of small order.

1. Introduction

Throughout the paper G is a finite group, p and q are primes and P_q is a Sylow-q subgroup of G.

Lagrange's Theorem states that, if H is a subgroup of a finite group G, then $|H|$ divides $|G|$. However, if d divides $|G|$, it is not always true that G has a subgroup of order d.

Definition 1. *If a finite group G has a subgroup H of order d for every divisor d of $|G|$, then G is a CLT (Converse Lagrange's Theorem) group; otherwise G is an NCLT (Non-Converse Lagrange's Theorem) group.*

Some well-known *NCLT* groups are the symmetric group S_n, $n \geq 5$, the alternating group A_n, $n \geq 5$, and some relative holomorphs of elementary abelian p-groups, e.g. A_4. The following hierarchy of classes of finite groups is given in [3, p. 156]:

$$C \subset A \subset N \subset SS \subset S,$$

where C, A, N, SS, S are the classes of cyclic, abelian, nilpotent, supersoluble and soluble groups, respectively. The definition of supersolubility shows immediately that all finite supersoluble groups are *CLT* groups. Furthermore, the characteristic property of finite soluble groups given in [1] shows that all *CLT* groups are soluble, but not all soluble groups are *CLT* groups, e.g. A_4. Moreover, while all subgroups and factor groups of supersoluble groups are supersoluble, [3, p. 155], all subgroups and factor groups of *CLT*(*NCLT*) groups are not *CLT*(*NCLT*) groups, e.g. $A_4 \times C_2$ and A_4. However, the direct product of a finite number of *CLT* groups is a *CLT* group.

*E-mail: fran.barry@ucc.ie
This paper is based on part of a PhD thesis supervised by Professor Des MacHale, National University of Ireland, Cork.

Now, the commutator subgroup G' of a finite group G can indicate whether G is a *CLT* group or an *NCLT* group. In considering G', we use the following well-known results:
R1 [Sylow] If p^i divides $|G|$, where p is a prime, then G has a subgroup of order p^i.
R2 [Hall] If G is a finite, soluble group with $|G| = mn$, $(m, n) = 1$, then G has a subgroup of order m.
R3 If H is a subgroup of G, then $N_G(H)/C_G(H)$ can be embedded in $AutH$.
R4 [Schur] If K is a normal, Hall subgroup of a finite group G, then G splits over K.
R5 [Zassenhaus] If A is an abelian, normal, Hall subgroup of a finite group G, then

$$A = (A \cap G') \times (A \cap Z(G)).$$

R6 [Frattini] If H is a normal subgroup of G and P is a Sylow p-subgroup of H, then $G = N_G(P)H$.
R7 [Scott] If H is cyclic and G/H is supersoluble, then G is supersoluble [3, p. 158].
R8 [Scott] If $|G| = 2p^i$, then G is supersoluble [3, p. 158].
R9 [Hall] If $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ then G is soluble if and only if there exists a subgroup H_i of G such that $[G : H_i] = p_i^{\alpha_i}$ for each i, where $1 \leq i \leq r$ [1].

2. Preliminary results

Lemma 1. *If G' is cyclic, then G is a CLT group. Furthermore, if G' is cyclic of order p, where p is the smallest prime dividing $|G|$, then G is nilpotent.*

Proof. By $R7$ if G' is cyclic, then G is supersoluble and so is a *CLT* group. Furthermore, if p is the smallest prime dividing $|G|$, then $G' \subseteq Z(G)$ and G is nilpotent. ■

Lemma 2. *If G' is abelian, then $G/C_G(G')$ is abelian.*

Proof. If G' is abelian, then $G' \subseteq C_G(G')$ so that $G/C_G(G')$ is abelian. ■

Lemma 3. *G' is soluble if and only if G is soluble.*

Lemma 4. *If G' is a p-group, then P_p is normal in G. Furthermore, $G = P_p A$, where A is abelian and $(p, |A|) = 1$.*

Proof. Every subgroup containing G' is normal in G, so that P_p is normal in G. By $R4$, G splits over P_p and $G = P_p A$ for some A. Moreover, since $G' \subseteq P_p$ if and only if G/P_p is abelian, then A is abelian since

$$G/P_p = P_p A/P_p \cong A/A \cap P_p = A. \quad ■$$

Lemma 5. *If $|G'| = p^i$ and G' contains normal subgroups of G of order p^l for each $l < i$, then G is a CLT group.*

Proof. Let $|G| = p^n.m$, $(p, m) = 1$. By $R2$ and Lemma 3 G has a subgroup M of order m. Since $G'M/G'$ is abelian, then $G'M/G' \cong M/M \cap G' = M$ is abelian (and so is a *CLT* group) and there exist subgroups of order d for each divisor d of m. Furthermore, since $|P_p/G'| = p^{n-i}$, there exist normal subgroups of order p^k, $i \leq k \leq n$, so that there also exist subgroups of order $p^k.d$. Now, if there exist normal subgroups of order p^l for each $l < i$, then there exist subgroups of order $p^l.d$ and G is a *CLT* group. ■

Lemma 6. *If P_q is a subgroup of $C_G(G')$, then P_q is normal in G.*

PROOF. If P_q is a subgroup of $C_G(G')$, then $G' \subseteq C_G(P_q) \subseteq N_G(P_q)$ so that $N_G(P_q)$ is normal in G. Moreover, P_q is the Sylow q-subgroup of $N_G(P_q)$ so that by $R6$ $G = N_G(P_q).N_G(P_q)$ and $G = N_G(P_q)$. ■

Corollary 1. *$C_G(G')$ is nilpotent.*

Lemma 7. *If $|G| = mn$, $(m,n) = 1, |G'|$ divides m and $(n, |AutG'|) = 1$, then G is decomposable.*

PROOF. Let $|G'| = m'$. Since G/G' is abelian, then G/G' has a normal subgroup of order $\frac{m}{m'}$ so that G has a normal subgroup, M say, of order m. By Corollary 1, if $(n, |AutG'|) = 1$, then $C_G(G')$ has a characteristic subgroup, N say, of order n, so that N is normal in G. Now $M \lhd G$, $N \lhd G$, $M \cap N = 1$ and $G = MN$, so that $G = M \times N$, the direct product of M and N. ■

Lemma 8. *If a finite group G is the direct product of H and K, then $G' = H' \times K'$.*

Lemma 9. *If G' and $AutG'$ are both p-groups, then G is nilpotent.*

PROOF. By Lemma 4 P_p is normal in G. Furthermore, by $R3$ $G/C_G(G')$ is a p-group so that by Lemma 6 all Sylow subgroups are normal in G and so G is nilpotent. ■

Lemma 10. *If G' is an elementary abelian p-group of order p^2, where p is the smallest prime dividing $|G|$ and the order of G is odd, then G is nilpotent.*

PROOF. If G' is a p-group, then P_p is normal in G by Lemma 4. By $R3$ $|G/C_G(G')|$ divides $|AutG'| = (p^2 - p)(p^2 - 1) = p(p-1)^2(p+1)$. Since $|G|$ is odd and p is the smallest prime dividing $|G|$, then $|G/C_G(G')|$ is co-prime to $(p-1)^2(p+1)$ so that $|G/C_G(G')|$ divides p. Therefore, for $q > p$, P_q is normal in G by Lemma 6. Now all the Sylow subgroups of G are normal in G and G is nilpotent. ■

Lemma 11. *If p is the smallest prime dividing $|G|$ and P_p is a subgroup of G', then G is an NCLT group.*

PROOF. If there exists a subgroup H of index p, then H is normal in G and G/H is abelian. Then G' is a subgroup of H, contradiction. Therefore G has no subgroup of index p and G is an *NCLT* group. ■

Lemma 12. *G' is not isomorphic to*: (*i*) *D_{2n}, the dihedral group of order $2n$, $n \geq 2$*; (*ii*) *Q_{4n}, the dicyclic group of order $4n$, $n \geq 3$; or* (*iii*) *H, where H is the non-abelian group of order pq.*

Proof. This is an immediate consequence of [2, Lemma 7]. ■

3. Examples of application to groups of small order

If G is an *NCLT* group whose commutator subgroup has order less than 16, then by Lemmas 1, 9, 10 and 12 we need only consider the following cases, for all of which G is soluble by Lemma 3:

(*i*) $G' = C_2 \times C_2$
(*ii*) $G' = C_2 \times C_2 \times C_2$
(*iii*) $G' = Q_8$, the quaternion group of order 8
(*iv*) $G' = C_3 \times C_3$, $|G|$ even,
(*v*) $G' = C_6 \times C_2$
(*vi*) $G' = A_4$.

We show that under certain conditions G may be an *NCLT* group in the first five cases and that G is a *CLT* group in the sixth case. We give examples of each in Section 4.

Theorem 1. *Let G be a finite group with its commutator subgroup G' elementary abelian of order 2^2, then G is an NCLT group if and only if G' is a Sylow 2-subgroup of G.*

Proof. Since G' is a 2-group, then P_2 is normal in G by Lemma 4. Moreover, by $R3$ and Lemma 2 $G/C_G(G')$ is isomorphic to an abelian subgroup of S_3. If $G/C_G(G')$ is a 2-group, then G is nilpotent by Lemma 6, and is therefore a *CLT* group.

Suppose $G/C_G(G')$ has order 3 and let P_3 be a Sylow 3-subgroup of G. Then, by Lemmas 6 and 7, $G = P_2P_3 \times A$, where A is abelian and $(2, |A|) = (3, |A|) = 1$. We show that if $|P_2| = 2^i, i \geq 3$ then G is a *CLT* group. Let $|P_3| = 3^j$, say. If P_2P_3/G' contains a subgroup H/G' of order 2.3^j, then P_2P_3 contains a subgroup H of order $2^3.3^j$, which is non-abelian since $P_3 \nsubseteq C_G(G')$ and in which the number of Sylow 3-subgroups $n_3 = 4$ and $|N_H(P_3)| = 2.3^j$. By $R8$ $N_H(P_3)$ is a *CLT* group so that P_2P_3 has subgroups of order $2.3^k, 1 \leq k \leq j$. Furthermore, since P_2P_3/G' is abelian, P_2P_3 now contains subgroups of every order dividing $|P_2P_3|$, and so G, the direct product of *CLT* groups, is a *CLT* group.

Therefore, if G is an *NCLT* group, then $P_2 = G'$. Furthermore, if P_2 is a subgroup of G', then G is an *NCLT* group by Lemma 11. This yields Theorem 1. ■

Theorem 2. *Let G be a finite group with its commutator subgroup G' elementary abelian of order 2^3. Let P_2 be a Sylow 2-subgroup of G. Then G is an NCLT group if and only if P_2/G' has order $2^i, 0 \leq i \leq 1$.*

Proof. Suppose G' is elementary abelian of order 8. Then $G/C_G(G')$ is isomorphic to an abelian subgroup, S say, of $GL(3, 2)$. If S is a 2-group, then G is nilpotent and is therefore a *CLT* group. Suppose S is not a 2-group. Since an element of order 3 and an element of order 7 are both self-centralising in $GL(3, 2)$, it follows that S has order 3 or 7. Suppose S has order 3. Then an element z of S of order 3 has characteristic polynomial x^3-1 and thus has an invariant eigenspace corresponding to the eigenvalue 1 and an invariant two-dimensional subspace corresponding to the irreducible factor x^2+x+1. It follows that G' has subgroups of order 2 and order 4 which are normal in G and we conclude, therefore, that G is a CLT group.

Suppose S has order 7. Let v be an element of G such that $vC_G(G')$ has order 7 in $G/C_G(G')$, and let s be the corresponding element of S. Then s has characteristic polynomial x^3+x^2+1 or x^3+x+1 and acts irreducibly on G'. Let P_2 be a Sylow 2-subgroup of G. Now $[P_2, \langle v\rangle]$ is contained in G' and is $\langle v\rangle$-invariant and therefore equals G' since $[G', \langle v\rangle]=G'$. Since P_2 is normal in G, P_2' is $\langle v\rangle$-invariant and P_2' is contained in G'. Hence $P_2'=G'$ or $P_2'=\{1\}$. If $P_2'=G'$, then v acts trivially on the Frattini factor group $P_2/\Phi(P_2)$, and this implies that v acts trivially on P_2 [3 (7.3.12), p. 162], contrary to hypothesis. Hence $P_2'=\{1\}$ and thus P_2 is abelian and $P_2 = G'\times C_{P_2}(v)$. Since no non-trivial proper subgroup of G' is normal in G, it follows that G has subgroups of order $2m$ and $4m$, where m is the odd part of $|G|$, if and only if $|C_{P_2}(v)|$ is divisible by 4. So G is a CLT group if $|C_{P_2}(v)|>2$, while if $|C_{P_2}(v)|\leq 2$, then G is an $NCLT$ group. This yields Theorem 2. ∎

Theorem 3. *Let G be a finite group with its commutator subgroup G′ isomorphic to Q_8, the quaternion group of order* 8. *Then G is an NCLT group if and only if G′ is a Sylow* 2-*subgroup of G.*

PROOF. P_2 is normal in G by Lemma 4, and by $R3$ $G/C_G(G')$ is isomorphic to a subgroup of S_4. If $G/C_G(G')$ is a 2-group, then by Lemmas 4 and 6 G is nilpotent and therefore is a CLT group.

Let P_3 be a Sylow 3-subgroup of G. Then by Lemma 7 $G=P_2P_3\times A$, where A is abelian. We show that if P_2P_3/G' has a subgroup H/G' of order 2, then G is a CLT group.

Suppose P_2P_3/G' has a subgroup H/G' of order 2. Then H is a non-abelian subgroup of order 2^4. Both $Z(H)$ and H' are characteristic in H, which is normal in G, and it is easy to check using Representation Theory that in all non-abelian groups of order 2^4, one of $Z(H)$ and H' is of order 2 and the other is of order 2^2. P_2P_3 now has normal subgroups of order 2 and 2^2 and by Lemma 5 P_2P_3 is a CLT group so that G, the direct product of CLT groups, is a CLT group. Therefore, if G is an $NCLT$ group, then $P_2=G'$. In the other direction, $P_2=G'$ implies that G is an $NCLT$ group by Lemma 11. This yields Theorem 3. ∎

Theorem 4. *Let G be a finite group of even order with its commutator subgroup G′ elementary abelian of order* 9. *Let P_2 be a Sylow* 2-*subgroup of G. Then G is an NCLT group if and only if G′ is a Sylow* 3-*subgroup of G and $P_2/C_{P_2}(G')$ is cyclic of order* 4 *or* 8.

PROOF. If G' is a 3-group, then P_3 is normal in G by Lemma 4. Furthermore, by $R3$ and Lemma 2, $G/C_G(G')$ is isomorphic to an abelian subgroup of $GL(2, 3)$. Let P_2 be a Sylow 2-subgroup of G. Let P_3 be a Sylow 3-subgroup of G. By Lemma 7

$G = P_2P_3 \times A$, where A and P_2 are abelian. If $G/C_G(G')$ is a 3-group, then by Lemma 6 G is nilpotent and is therefore a CLT group.

Furthermore, if the order of P_3 is 3^j and $j > 2$, we show that G is a CLT group. Suppose the order of P_2 is 2^i. If P_2P_3/G' contains a subgroup H/G' of order $2^i.3$, then there exists a non-abelian subgroup H of order $2^i.3^3$ whose Sylow 3-subgroup K is normal in P_2P_3. If K is non-abelian, then K' is of order 3 and is characteristic in K, which is normal in P_2P_3, so that K' is normal in P_2P_3. Now, by Lemma 5, P_2P_3 is a CLT group so that G, the direct product of CLT groups, is a CLT group. If K is abelian, then by $R5$

$$K \cong (K \cap H') \times (K \cap Z(H)).$$

Since H is non-abelian, then either $|H'| = 3$ or $|H'| = 3^2$. Again, P_2P_3 is a CLT group so that G is a CLT group.

Therefore, if G is an $NCLT$ group, then G' is a Sylow 3-subgroup of G. Now, consider $G \cong G'P_2 \times A$. P_2 must act irreducibly on G'; otherwise G is a CLT group. By Schur's Lemma and the fact that the non-zero elements of the finite field $GF(3^2)$ form a cyclic multiplicative group, $G'P_2/C_{G'P_2}(G')$ must be cyclic of order 4 or 8. If u is an element of order 4 in $GL(2, 3)$, then u has characteristic polynomial $x^2 + 1$ and this is irreducible over $GF(3)$. Similarly the characteristic polynomial of an element w of order 8 is $x^2 + x - 1$ or $x^2 - x - 1$ and is therefore irreducible over $GF(3)$ also. Therefore, if G' is elementary abelian of order 3^2 and G is an $NCLT$ group, then $P_3 = G'$ and $P_2/C_{P_2}(G')$ is cyclic of order 4 or 8. In the other direction, if $P_3 = G'$ and either an element of order 4 or an element of order 8 acts irreducibly on G', then G is an $NCLT$ group. This yields Theorem 4. ■

Theorem 5. *Let G be a finite group with its commutator subgroup G' isomorphic to $C_6 \times C_2$, then G is an NCLT group if and only if the Sylow 2-subgroup of G is contained in G'.*

PROOF. By $R3$ $G/C_G(G')$ is isomorphic to an abelian subgroup of D_{12}, the dihedral group of order 12. Let P_2 be a Sylow 2-subgroup of G. Let P_3 be a Sylow 3-subgroup of G. Let the order of P_2 be 2^i and the order of P_3 be 3^j. By Lemma 7 $G = P_2P_3 \times A$, where A is abelian.

If P_3 is a subgroup of $C_G(G')$, then by Lemma 6 P_3 is normal in G and there exists a subgroup, say P_3N, of order 2.3^j which is a CLT group by $R8$. Therefore there exist subgroups of order 2.3^l, $0 \le l \le j$. There also exist subgroups of order $2^k.3^l$, $2 \le k \le i$, $1 \le l \le j$ since P_2P_3/G' is abelian (and so is a CLT group). Now G, the direct product of CLT groups, is a CLT group.

If P_2 is not a subgroup of G', then P_2P_3/G' contains a subgroup H/G' of order 2.3^{j-1}, where $|H| = 2^3.3^j$ and the following cases apply:

(a) if P_3 is normal in H, H/P_3 contains a subgroup K/P_3 of order 2 and again we have a CLT subgroup K of order 2.3^j;

(b) if P_3 is not normal in H, $|N_H(P_3)| = 2.3^j$ and is a CLT group.

In both cases G is a CLT group. Therefore, if G is an $NCLT$ group, then P_2 is contained in G'. In the other direction, by Lemma 11, if P_2 is contained in G', then G is an $NCLT$ group. This yields Theorem 5. ■

Theorem 6. *Let G be a finite group with its commutator subgroup G' isomorphic to A_4. Then G is a CLT group.*

PROOF. By $R2$, $G/C_G(G')$ can be embedded in a subgroup of S_4. Since $G' \cap C_G(G') = 1$, then

$$(G/C_G(G'))' \cong G'/G' \cap C_G(G') = G',$$

so that $G/C_G(G') = S_4$ and $|P_2| \geq 2^3$. By Lemma 7 $G = P_2P_3 \times A$, where A is abelian. Since $|P_2| > 2^3$, then P_2P_3/G' contains a subgroup H/G' of order 2, with $|H| = 2^3.3^j$. H is a non-abelian subgroup in which $|N_H(P_3)| = 2.3^j$, so that, by $R8$, H (and so P_2P_3) has subgroups of order 2.3^l, $0 \leq l \leq j$. P_2P_3/G' is abelian of order $2^{i-2}.3^{j-1}$ so that P_2P_3 also has subgroups of order $2^k.3^l$, $2 \leq k \leq i$, $1 \leq l \leq j$. Now P_2P_3 is a CLT group and G, the direct product of CLT groups, is a CLT group. This yields Theorem 6. ■

4. Examples of *NCLT* groups with $|G'| < 16$

$G' = C_2 \times C_2$,	$G = A_4$ and $G = A_4 \times B$,
$G' = C_2 \times C_2 \times C_2$,	$G = E$ and $G = E \times F$,
$G' = Q_8$,	$G = SL(2, 3)$ and $G = SL(2, 3) \times B$,
$G' = C_3 \times C_3$,	$G = H$ and $G = K$,
$G' = C_6 \times C_2$,	$G = A_4 \times L$,

where

B is an abelian group of odd order,

E is a relative holomorph of the elementary abelian group of order 2^3 by C_7, where $\{a, b, c\}$ generates P_2, $\{d\}$ generates C_7 and the action is given by

$$[a, b] = [b, c] = [a, c] = 1, \quad [a, d] = a^{-1}c^{-1}, \quad [b, d] = ba, \quad [c, d] = b,$$

F is an abelian group and 2^2 does not divide $|F|$,

H is a relative holomorph of the elementary abelian group of order 3^2 by C_4, where $\{a, b\}$ generates P_3, $\{c\}$ generates C_4 and the action is given by

$$[a, b] = 1, \quad [a, c] = ab, \quad [b, c] = a,$$

K is a relative holomorph of the elementary abelian group of order 3^2 by C_8, where $\{a, b\}$ generates P_3, $\{c\}$ generates C_8 and the action is given by

$$[a, b] = 1, \quad [a, c] = b, \quad [b, c] = a^{-1},$$

and L is the non-abelian group of order 3^3, exponent 3.

ACKNOWLEDGEMENT

I would like to thank the referee for many helpful suggestions.

References

[1] P. Hall, A characteristic property of soluble groups, *Journal of the London Mathematical Society*, **12** (1937), 199–201.
[2] D. MacHale, and P. Ó Murchú, Commutator subgroups of groups with small central factor groups. *Proceedings of the Royal Irish Academy* **93**A (1993), 123–9.
[3] W.R. Scott, *Group theory*, Dover Publications, New York, 1987.