A GENERAL WOLFF THEOREM FOR ARBITRARY BANACH SPACES

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Abstract

The Kobayashi distance is used to generalise the classical theorem of Wolff to compact holomorphic fixed-point-free mappings on the open unit ball of an arbitrary complex Banach space E and more generally on bounded convex domains in E, thereby extending results of Abate for $\mathbf{C}^n$. This is compared to earlier results on bounded symmetric domains. The boundary behaviour of the Kobayashi distance κ on bounded symmetric domains is also discussed, with estimates given for $\kappa(z, w)$ as one or both of z, w tend to the boundary.

Introduction

Wolff's classical theorem on the unit disc Δ in $\mathbf{C}$ [28] shows that to every holomorphic fixed-point-free self map of Δ there corresponds a point $\xi \in \partial\Delta$ with the property that each disc in Δ that is internally tangent at ξ is f-invariant. This was extended to the finite dimensional Hilbert ball by Hérve [15] in 1963 and the infinite dimensional Hilbert ball by Goebel [11] in 1982. More recently, the author extended this result to all finite-dimensional bounded symmetric domains and for classes of holomorphic functions in the infinite-dimensional case [24]. In another vein, Abate in the 1980s tackled similar problems for certain classes of bounded domains D in $\mathbf{C}^n$ [1; 3]. While in general the results of Abate fall short of finding domains that are actually invariant under the (fixed-point-free) holomorphic mapping, he does prove a more general result for bounded convex domains in $\mathbf{C}^n$ [1], namely, he shows that there exists a point ξ on ∂D such that for each $R > 0$ and base point $z_0 \in D$ there exists a small and large horosphere $E_{z_0}(\xi, R)$ and $F_{z_0}(\xi, R)$ with $E_{z_0}(\xi, R) \subset F_{z_0}(\xi, R)$ satisfying $f^m(E_{z_0}(\xi, R)) \subset F_{z_0}(x, R)$ for all $m \in \mathrm{IN}$.

In this paper we extend this result of Abate to compact holomorphic maps on the open unit ball of a complex Banach space E, and more generally to such maps on any bounded convex domain in E. This includes, of course, the bounded symmetric domains, being the open unit balls of the Banach spaces known as the JB^*-triples [17]. The bounded symmetric domains provide us with a large class of interesting examples within which the constraints of the above theorem can be examined. Section 4 of this paper is, in fact, motivated by the desire to reconcile for bounded symmetric domains the above approach to a Wolff theorem using the Kobayashi distance and the approach, and the results obtained in [24], using the Jordan triple structure. We note that the bounded symmetric domains include the

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Hilbert ball for which the above theorem reduces to the well-known 'Wolff' result [11; 15], where small and large horospheres coincide and satisfy $\partial E_{z_0}(\xi, R) \cap \partial B = \partial F_{z_0}(\xi, R) \cap \partial B = \{\xi\}$. At the other extreme, the bounded symmetric domains also include the n-dimensional polydisc Δ^n, for all n, where the small and large horospheres never coincide and $\{\xi\} \subsetneq \partial F_{z_0}(\xi, R) \cap \partial B$ [2]. In the case of the bounded symmetric domains, we show in Section 4 that the f-invariant domains found in [24] lie sandwiched between the small and large horospheres.

In Section 5, we discuss the usefulness and some of the shortcomings of the Wolff type results achieved to the ultimate aim of understanding the behaviour of iterates of holomorphic functions on bounded symmetric domains. In particular, it follows from results in [24] that the closure of the horospheres $F_{z_0}(x, R)$ can intersect the boundary in quite a large set. In fact, $\partial F_{z_0}(\xi, R) \cap \partial B$ can properly contain the holomorphic boundary component (in the sense of [19]) of the point ξ. In general, this causes difficulties in studying the behaviour of the iterates of f and also in studying the boundary behaviour of the holomorphic function f.

Along the way, we obtain estimates for the boundary behaviour of the Kobayashi distance on bounded symmetric domains. We also show that for bounded symmetric domains, the horospheres can be approximated in some sense by a sequence of Kobayashi balls. For a thorough overview of the literature in the finite-dimensional case we refer to [2]. For additional background material see [12] and for a recent survey of related results on iteration see [26]. A recent survey of Denjoy-Wolff theorems in Banach spaces can be found in [22, pp 483–96].

1. Notation and background

Throughout we let $\Delta = \{z \in \mathbf{C}: |z| < 1\}$. We let E and F denote arbitrary complex Banach spaces and let D and $\tilde{D}$ be domains in E and F respectively. We denote the set of all holomorphic maps from D to $\tilde{D}$ as $H(D, \tilde{D})$ and use $H(D)$ for $H(D, D)$. The Kobayashi pseudo-distance can be defined on any complex manifold [7], although we restrict our attention here to the case of a bounded domain D in a complex Banach space E. For domains D there are different possible ways to introduce the definition, and we choose to integrate the infinitesimal Kobayashi pseudo-metric and refer to [7] for other possibilities and details. The infinitesimal Kobayashi pseudo-metric is defined on the tangent bundle $D \times E$ of D using holomorphic maps from Δ to D.

Definition 1.1. The infinitesimal Kobayashi pseudo-metric $k(p, v)$ at a point (p, v) of the tangent bundle $D \times E$ is

$$k(p, v) := \inf\{\lambda > 0: \exists f \in H(\Delta, D), f(0) = p, f'(0)(\lambda) = v\}.$$

The pseudo-metric is a means of defining lengths of tangent vectors that can then be used, by integrating, to define lengths of curves and thereby distances on a manifold. An admissible curve on D is a parametrised curve $\gamma: [a, b] \to D$ having piecewise continuous derivative. We define the length of such a curve, $L_k(\gamma)$, with respect to the infinitesimal Kobayashi pseudo-metric as

$$L_k(\gamma) := \int_a^b k(\gamma(t), \gamma'(t))dt.$$

This then allows us to measure distances on D.

Definition 1.2. The Kobayashi pseudo-distance κ on D is given by

$$\kappa(z,w) := \inf\{L_k(\gamma) \colon \gamma \text{ is an admissible curve joining } z \text{ and } w\}$$

for all $z, w \in D$.

In this case, cf. [7, chapters 4 and 5], it turns out that κ is continuous and generates the original topology, thus ensuring that it is actually a distance on D.

We now introduce the JB^*-triples as our primary class of Banach space examples. We use H and K to denote arbitrary complex Hilbert spaces and $\mathcal{L}(X, Y)$ to denote the space of all continuous linear operators from a Banach space X to a Banach space Y. We let $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\mathsf{GL}(X)$ all be invertible elements in $\mathcal{L}(X)$.

Definition 1.3. A JB^*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} \colon Z \times Z \times Z \to Z$ satisfying:

(i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.

(ii) The map $z \mapsto \{x, x, z\}$, denoted $x \square x$, is Hermitian, $\sigma(x \square x) \geq 0$ and $\|x \square x\| = \|x^2\|$ for all $x \in Z$, where σ denotes the spectrum.

(iii) The product satisfies the following 'triple identity'

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

Example 1.4. (i) $\mathcal{L}(H, K)$ is a JB^*-triple for the product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ where y^* denotes the usual adjoint of y.

(ii) $C(X)$, the continuous C-valued functions on a compact Hausdorff space X, is a JB^*-triple for the product $\{x, y, z\} = x\bar{y}z$.

In the 1970s Kaup [17] classified the bounded symmetric domains by showing that every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a JB^*-triple and vice versa.

Let Z be a JB^*-triple. Several types of linear operators on Z can be defined naturally in terms of the triple product:

$$x \square y \in \mathcal{L}(Z) \colon z \mapsto \{x, y, z\};$$

$$Q_x \in \mathcal{L}_{\mathbb{R}}(Z) \colon z \mapsto \{x, y, x\};$$

and the fundamentally important Bergman operators

$$B(x, y) = I - 2x \square y + Q_x Q_y \in \mathcal{L}(Z).$$

As Banach spaces the JB^*-triples are characterised by the fact that their open unit balls are homogeneous. In fact, if we let $\mathsf{Aut}(B)$ denote all biholomorphic maps from B to B, then for all z in B we have $g_z \in \mathrm{Aut}(B)$ defined by

$$g_z(w) = z + B(z,z)^{\frac{1}{2}}(I + w \Box z)^{-1} w$$

(cf. [17]), which satisfies $g_z(0) = z$, $g_z^{-1} = g_{-z}$ and $g_z'(0) = B(z,z)^{\frac{1}{2}}$ (defined in terms of a functional calculus). For a recent survey of JB^*-triples and bounded symmetric domains we refer to [5].

2. Horospheres

In this section we define the small and large horospheres at a boundary point of D in terms of the Kobayashi distance. While the definitions make sense in the context of hyperbolic manifolds with boundary, we are only interested here in the case of a bounded domain D in a complex Banach space E. We recall that every holomorphic mapping from the domain to itself contracts the Kobayashi distance. It is not surprising therefore that this distance plays a key role in the study of fixed points, iterates and invariant domains of holomorphic functions. The definitions in this section were first introduced by Abate in [1] and [2], to which we refer for related details and more information. We use $B_\kappa(z, r)$ to denote the Kobayashi ball about x of radius r, namely, the set $\{w \in D: \kappa(z,w) < r\}, r > 0$.

Definition 2.1. For $x \in \partial D$, $z_0 \in D$ and $R > 0$, we define, respectively, the small and large horosphere about x of radius R and base point z_0 as follows:

$$E_{z_0}(x,R) = \{w \in D: \limsup_{z \to x} [\kappa(w,z) - \kappa(z_0,z)] < \frac{1}{2}\log R\},$$

$$F_{z_0}(x,R) = \{w \in D: \liminf_{z \to x} [\kappa(w,z) - \kappa(z_0,z)] < \frac{1}{2}\log R\}.$$

An application of the triangle inequality gives $|\kappa(w,z) - \kappa(z_0,z)| \leq \kappa(z_0,w)$ for all z, $w \in B$ and hence the lim inf and lim sup above are both actually finite.

The following is lemma 1.1 in [1] for bounded domains in $\mathbf{C}^n$. The proof there follows more or less immediately from the definition and works equally well in the Banach space setting.

Lemma 2.2. *Let* $x \in \partial D$, $z_0 \in D$. *Then*

(*i*) $E_{z_0}(x,R) \subset F_{z_0}(x,R)$ *for all* $R > 0$;

(*ii*) $E_{z_0}(x,R_1) \subset E_{z_0}(x,R_2)$ *and* $F_{z_0}(x,R_1) \subset F_{z_0}(x,R_2)$ *for all* $0 < R_1 \leq R_2$;

(*iii*) *For all* $R > 1$ *we have that* $B_\kappa(0, \frac{1}{2}\log R) \subset E_{z_0}(x,R)$;

(*iv*) *For all* $R < 1$ *we have* $F_{z_0}(x,R) \cap B_\kappa(z_0, -\frac{1}{2}\log R) = \emptyset$;

(*v*) $\bigcup_{R>0} E_{z_0}(x,R) = \bigcup_{R>0} F_{z_0}(x,R) = D$

and

$$\bigcap_{R>0} E_{z_0}(x,R) = \bigcap_{R>0} F_{z_0}(x,R) = \emptyset.$$

(*vi*) *For all* $g \in \mathrm{Aut}(D) \cap C^0(\bar{D})$ *and* $R > 0$

$$g(E_{z_0}(x,R)) = E_{g(z_0)}(g(x),R) \text{ and } g(F_{z_0}(x,R)) = F_{g(z_0)}(g(x),R).$$

(*vii*) *Let* $z_1 \in D$ *and let* $L := \exp(2\limsup_{z\to x}[\kappa(z_1,z) - \kappa(z_0,z)])$. *Then*

$$E_{z_1}(x,R) \subset E_{z_0}(x,LR) \text{ and } F_{z_1}(x,R) \subset F_{z_0}(x,LR)$$

for all $R > 0$.

We note from Lemma 2.2 (vii) that the base point z_0 plays no significant role.

In the Euclidean ball of $\mathbf{C}^n$ $\lim_{z\to x}[\kappa(w,z) - \kappa(z_0,z)]$ exists [29] so that the small and large horospheres coincide. Although this is also true for the infinite dimensional Hilbert ball (see Example 4.4 below) and also for all strongly convex domains with C^3 boundary [2, corollary 2.6.48], it is not true in general even for finite dimensional bounded symmetric domains, cf. proposition 2.4.12 of [2] for the unit polydisc in $\mathbf{C}^n$. Moreover, the closure of the horospheres generally intersect the boundary in more than one point. For example, [2, corollary 2.4.13], if B is the unit polydisc Δ^n in $\mathbf{C}^n$ then $\partial F_{z_0}(x,R) \cap \partial\Delta^n$ always contains $\{x\}$ properly. This phenomenon is also not special to the polydisc since any rank n bounded symmetric domain contains a copy of the polydisc Δ^n preserved with regard to the Kobayashi distance, and therefore we can find this 'bad' behaviour of the horospheres in any bounded symmetric domain apart from the Hilbert space (that is, rank 1) case. As this phenomenon is crucial to the study of the iterates of f we will return to it again in Section 5.

3. Main results

We first present a Julia type lemma for the horospheres. This is proved in [3] for bounded domains in $\mathbf{C}^n$, and once again the proof extends immediately to the Banach space case.

Lemma 3.1. *Let* D *be a bounded domain in* E *and* $f: D \to D$ *be holomorphic. Fix* z_0 *in* D. *If there exists a sequence* $(z_k)_k$ *in* B *such that* $\lim_k z_k = x \in \partial B$ *and* $\lim_k f(z_k) = y \in \partial B$ *and*

$$\limsup_k[\kappa(z_0,z_k) - \kappa(z_0,f(z_k))] < \infty$$

then

$$f(E_{z_0}(x,R)) \subset F_{z_0}(y,\alpha R)$$

for all $R > 0$ *where* α *satisfies* $\frac{1}{2}\log\alpha = \limsup_k[\kappa(z_0,z_k) - \kappa((z_0,f(z_k))]$.

Note 3.2. If Z is a JB^*-triple then this lemma has a more familiar statement. As we will see from equation 4.7 below, the condition

$$\limsup_k [\kappa(z_0, z_k) - \kappa(z_0, f(z_k))] < \infty \text{ becomes } \limsup_k \frac{1 - \|g_{-f(z_k)}(z_0)\|}{1 - \|g_{-z_k}(z_0)\|} < \infty.$$

In particular, since the characterising property of a JB^*-triple as a Banach space is the homogeneity of its open unit ball B, we may in the light of Lemma 2.2 (vii) above take $z_0 = 0$, in which case we get the following neater statement.

Lemma 3.3. *Let Z be a JB^*-triple with open unit ball B and let f: $B \to B$ be holomorphic. If there exists a sequence $(z_k)_k$ in B such that $\lim_k z_k = x \in \partial B$ and $\lim_k f(z_k) = y \in B$ and*

$$\alpha := \limsup_k \frac{1 - \|f(z_k)\|}{1 - \|z_k\|} < \infty$$

then

$$f(E_0(x, R)) \subset F_0(y, \alpha R)$$

for all $R > 0$.

Our first main result is the following Wolff type theorem for the open unit ball B of a complex Banach space E. It generalises to the Banach space setting a result in [1] for bounded convex domains in $\mathbf{C}^n$. The proof is a fairly straightforward adaptation of that in [1] to the infinite dimensional case. The following notion of a compact holomorphic mapping is well known in the literature, namely, a holomorphic map f: $B \to B$ is said to be compact if $f(B)$ is relatively compact in E (but is not necessarily relatively compact in B). In particular, if E is finite dimensional then every f: $B \to B$ is compact. For an arbitrary domain D in E, f: $D \mapsto D$ is, likewise, said to be compact if $f(D)$ is relatively compact in E (but not necessarily in D). We recall that a set K in a domain D is said to be contained strictly inside D, written $K \subset\subset D$ if $dist(K, \partial D) > 0$ where

$$dist(K, \partial D) := \inf\{\|w - y\| : w \in K, y \in E \backslash D\}.$$

Note that we write $B(x, r)$ for the set $\{y \in E : \|x - y\| < r\}$.

Theorem 3.4. *Let E be a complex Banach space with open unit ball B and let f: $B \to B$ be a compact holomorphic map with no fixed points in B. Then, there exists ξ in ∂B such that*

$$f^n(E_{z_0}(\xi, R)) \subset F_{z_0}(\xi, R)$$

for all $R > 0$, $z_0 \in B$ and all $n \in IN$, where $f^1 = f$ and $f^n = f \circ f^{n-1}$ for all $n > 1$.

Proof. Choose a sequence $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$. Let $f_k := \alpha_k f$ for all k. Since f_k maps B strictly inside B, the Earle–Hamilton theorem [9] implies that f_k has a fixed point z_k in $\alpha_k B$. Then $\frac{1}{\alpha_k} z_k = f(z_k) \in f(B)$, and the relative compactness of $f(B)$ then implies that $(\frac{1}{\alpha_k} z_k)_k$, and hence $(z_k)_k$, has a convergent subsequence. Without loss of generality, we assume that $z_k \to \xi \in \bar{B}$. It is easy to see that if $\xi \in B$ then $f(\xi) = \xi$, which is impossible, so $\xi \in \partial B$. We also have $\lim_k f(z_k) = \xi$. Fix w and z_0 in B and $n \in$ IN arbitrary. For all $k \in$ IN

$$\begin{aligned}|[\kappa(f^n(w), z_k) - \kappa(z_0, z_k)] - [\kappa(f_k^n(w), z_k) - \kappa(z_0, z_k)]| \\ = |\kappa(f^n(w), z_k) - \kappa(f_k^n(w), z_k)| \\ \le \kappa(f^n(w), f_k^n(w)) \text{ by the triangle inequality.}\end{aligned}$$

Since n is fixed and $\lim_k f_k = f$ we have $\lim_k f_k^n(w) = f^n(w)$ for all $w \in B$ and as κ is a continuous distance on B we have $\lim_k \kappa(f^n(w), f_k^n(w)) = 0$. Therefore $\lim_k |[\kappa(f^n(w), z_k) - \kappa(z_0, z_k)] - [\kappa(f_k^n(w), z_k) - \kappa(z_0, z_k)]| = 0$. This then gives that

$$\liminf_k [\kappa(f^n(w), z_k) - \kappa(z_0, z_k)] \le \limsup_k [\kappa(f_k^n(w), z_k) - \kappa(z_0, z_k)]. \tag{3.1}$$

Then

$$\liminf_{z\to\xi} [\kappa(f^n(w), z) - \kappa(z_0, z)] \le \liminf_k [\kappa(f^n(w), z_k) - \kappa(z_0, z_k)] \tag{3.2}$$

$$\le \limsup_k [\kappa(f_k^n(w), z_k) - \kappa(z_0, z_k)] \text{ by (3.1)} \tag{3.3}$$

$$= \limsup_k [\kappa(f_k^n(w), f_k^n(z_k)) - \kappa(z_0, z_k)] \tag{3.4}$$

$$\le \limsup_k [\kappa(w, z_k) - \kappa(z_0, z_k)] \tag{3.5}$$

$$\le \liminf_{z\to\xi} [\kappa(w, z) - \kappa(z_0, z)]. \tag{3.6}$$

In particular, if $w \in E_{z_0}(\xi, R)$ the above then gives that $f^n(w) \in F_{z_0}(\xi, R)$, that is,

$$f^n E_{z_0}(\xi, R) \subset F_{z_0}(\xi, R)$$

for all $R > 0$ and all $n \in \mathrm{IN}$. ■

The following result is lemma 4.1 of [27]. We provide the proof for clarity.

Lemma 3.5. *Let be a bounded convex domain in E and let* $0 < \alpha < 1$. *Then for all* $x \in D$ *the set*

$$\alpha D + (1-\alpha)x \textit{ is contained strictly inside } D.$$

PROOF. Fix $0 < \alpha < 1$ and fix $x \in D$. Let $r > 0$ be such that $B(x, r) \subset D$. We use proof by contradiction. Suppose $\alpha D + (1-\alpha)x$ is not contained strictly inside D. Then for all $\epsilon > 0$, there exists $w \in D$ and $y \in E \setminus D$ such that $\|\alpha w + (1-\alpha)x - y\| < \epsilon$. In particular, there are sequences $(w_n)_n \subset D$ and $(y_n)_n \subset E \setminus D$ such that $z_n := \alpha w_n + (1-\alpha)x - y_n$ satisfies $\|z_n\| \mapsto 0$. In particular, $\|z_n\| < (1-\alpha)r$ for all n large. As $B(x, r) \subset D$ and

$$\left\|\frac{z_n}{1-\alpha}\right\| = \left\|x - \left(\frac{1}{1-\alpha} y_n - \frac{\alpha}{1-\alpha} w_n\right)\right\| < r$$

it follows that $v_n := \frac{1}{1-\alpha} y_n - \frac{\alpha}{1-\alpha} w_n \in D$ for all n large. Then $y_n = (1-\alpha)v_n + \alpha w_n$. Since v_n and w_n are in D for all n large and D is convex, this implies that $y_n \in D$ for n large. This contradiction means that $\alpha D + (1-\alpha)x$ is contained strictly inside D. ■

The following proposition generalises the above mentioned result of Abate [1] to the infinite dimensional case.

Proposition 3.6. *Let D be a bounded convex domain in E. Let f: $D \to D$ be a compact holomorphic map with no fixed points in D. Then the conclusion of Theorem* 3.4 *holds.*

PROOF. Fix x arbitrary in D. Choose a sequence $(\alpha_k)_k$, $0 < \alpha_k < 1$, $\alpha_k \uparrow 1$. Let

$$f_k := \alpha_k f + (1 - \alpha_k)x.$$

Then $f_k(D) = \alpha_k f(D) + (1 - \alpha_k)x \subset \alpha_k D + (1 - \alpha_k)x$ and hence Lemma 3.5 implies that $f_k(D)$ is contained strictly inside D for all k. Therefore, by the Earle-Hamilton theorem [9], each map f_k has a fixed point z_k in D. Since f is a compact map, $f(D)$ is relatively compact in E, so $(f(z_k))_k$ has a convergent subsequence in E. As $z_k = f_k(z_k) = \alpha_k f(z_k) + (1 - \alpha_k)x$, it follows that $(z_k)_k$ also has a convergent subsequence. We assume, without loss of generality, that $(z_k)_k$ converges to $\xi \in \bar{D}$. Moreover $\xi = \lim_k z_k = \lim_k f(z_k)$. If $\xi \in D$ this would mean that $f(\xi) = \xi$, and since f has no fixed point in D it follows that $\xi \in \partial D$. We note that since D is a bounded convex domain, the Kobayashi distance κ_D on D is (still) a continuous distance on D. The rest of the proof now continues exactly as in Theorem 3.4 with the open unit ball B replaced by D. ■

Note 3.7.
(i) Given a point y in D, we seek a 'minimal' value of R such that $y \in \partial E_{z_0}(\xi, R)$; namely, if $y \in E_{z_0}(\xi, S)$ for any $S > 0$ then $S > R$. We will call such a value R_y It is not difficult to see that for $y \in D$, R_y is given by

$$R_y = \exp(2 \limsup_{z \to \xi} [\kappa(y, z) - \kappa(z_0, z)]).$$

Then Theorem 3.4 implies that $f^n(y) \in \bar{F}_{z_0}(\xi, R_y)$ for all n, and therefore this set $\bar{F}_{z_0}(\xi, R_y)$ is the 'best' one to use to study the iterates of the point y under f.

(ii) One might note that the only properties of the Kobaysahi distance used in the above results are that it is a continuous distance on D that is contracted by holomorphic mappings. Therefore the Kobayashi distance in the definition of the horospheres and in all of the above results could be replaced by any Schwarz-Pick distance [7, chapter 4] and analagous results would be obtained. We recall that the Kobayashi distance is the largest Schwarz-Pick distance and for convex domains all Schwarz-Pick distances coincide [8; 23].

4. Bounded symmetric domains

We return now to the example of the JB^*-triples Z whose open unit balls B are exactly the bounded symmetric domains. We compare Theorem 3.4 above to results of [24] that produce invariant domains for compact holomorphic fixed-point-free self maps of B. In order to do this we need to describe the Kobayashi distance and thereby the horospheres in terms of the Jordan triple product. In particular, we need to study the boundary behaviour of $\kappa(z, w)$ as one or both of z tend to the boundary of B. This is shown by the following result to be described in terms of Bergman operators. We recall [17] that for all $z \in B$ the operator $B_z := B(z,z)^{\frac{1}{2}}$ and B_z^{-1} exist. We also know from Section 1 that for $z \in B$, there exists $g_z \in \mathsf{Aut}(B)$ satisfying $g_z(0) = z$, $g_z^{-1} = g_{-z}$ and $g_z'(0) = B_z$.

Proposition 4.1. *For all z and w in B, we have*

$$\kappa(z,w) = \frac{1}{2}\log(\|B_w^{-1}B(w,z)B_z^{-1}\|(1 + \|g_{-z}(w)\|)^2).$$

In particular,

$$\frac{1}{2}\log\|B_w^{-1}B(w,z)B_z^{-1}\| \le \kappa(z,w) \le \frac{1}{2}\log\|B_w^{-1}B(w,z)B_z^{-1}\| + \log 2.$$

PROOF. We have $\kappa(z,w) = \kappa(0, g_z(w))$ and therefore

$$\kappa(z,w) = \tanh^{-1}\|g_{-z}(w)\| \tag{4.1}$$

$$= \frac{1}{2}\log\frac{1 + \|g_{-z}(w)\|}{1 - \|g_{-z}(w)\|} \tag{4.2}$$

$$= \frac{1}{2}\log\left(\frac{1}{1 - \|g_{-z}(w)\|^2}(1 + \|g_{-z}(w)\|^2)\right). \tag{4.3}$$

The first statement follows since, from [24, proposition 3.1],

$$\frac{1}{1 - \|g_{-z}(w)\|^2} = \|B_w^{-1}B(w,z)B_z^{-1}\|.$$

Since $\|g_{-z}(w)\| \le 1$ for all $z, w \in B$ we get the second statement. ■

Note 4.2. One can see, cf. [19], that the operator B_z is never invertible for $z \in \partial B$ and in particular, if z is an extreme point of $\bar{B}$ then $B_z = 0$ [20].

We now use Proposition 4.1 to obtain further estimates on $\kappa(z, w)$ as z tends to the boundary of B and w is fixed. Compare for instance [1, theorem 1.4; 13] and [25, propositions 2.3 and 2.4]. We write $d(z, \partial B)$ for $dist(\{z\}, \partial B) = 1 - \|z\|$. We find that $\kappa(z, w)$ behaves essentially like $-\frac{1}{2}\log d(z, \partial B)$ for w fixed and $z \to \partial B$.

Corollary 4.3. *Fix w_0 in B. Then there exists a constant c_1 and a function c_2: $B \to \mathbb{R}$ such that for all z in B*

$$-\frac{1}{2}\log d(z,\partial B)+c_2(z)\leq \kappa(z,w_0)\leq -\frac{1}{2}\log d(z,\partial B)+c_1$$

where $\lim_{z\to x} c_2(z)<\infty$ *for all* $x\in\partial B$.

PROOF. From Proposition 4.1 we have that for all $z, w\in B$

$$\kappa(z,w)\leq \frac{1}{2}\log\|B_w^{-1}B(w,z)B_z^{-1}\| + \log 2.$$

Since [10] $\|\{x,y,z\}\|\leq \|x\|\|y\|\|z\|$ one gets immediately that $\|B(z,w)\|\leq (1+\|z\|\|w\|)^2$. Together with the fundamental equality [18]

$$\|B_z^{-1}\|=\frac{1}{1-\|z\|^2}$$

this then gives

$$\begin{aligned}\kappa(z,w)&\leq \log\frac{1+\|w\|\|z\|}{\sqrt{(1-\|z\|^2)(1-\|w\|^2)}}+\log 2\\ &=-\frac{1}{2}\log(1-\|z\|)+\log\frac{1+\|w\|\|z\|}{\sqrt{(1+\|z\|)(1-\|w\|^2)}}+\log 2\\ &\leq -\frac{1}{2}\log(1-\|z\|)+\frac{1}{2}\log\left(\frac{1+\|w\|}{1-\|w\|}\right)+\log 2\\ &=-\frac{1}{2}\log d(z,\partial B)+\tanh^{-1}\|w\|+\log 2.\end{aligned}$$

For w_0 fixed, we therefore take $c_1:=\tanh^{-1}\|w_0\|+\log 2$. In the other direction, we have

$$\|B(w,z)B_w^{-1}\|\leq \|B_w\|\|B_w^{-1}B(w,z)B_z^{-1}\|\leq \|B_w^{-1}B(w,z)B_z^{-1}\| \tag{4.4}$$

since [18, corollary 3.6] $\|B_w\|\leq 1$ for all $w\in B$ It is well known [17, corollary 3.4] that $B(x, y)$ is invertible whenever $\|x\|\|y\|<1$ and therefore for z, $w\in B$

$$\frac{1}{1-\|z\|^2}=\|B_z^{-1}\|\leq \|B(w,z)^{-1}\|\|B(w,z)B_z^{-1}\|$$

and hence from (4.4) we have that $\frac{1}{1-\|z\|^2}\,\frac{1}{\|B(w,z)^{-1}\|}\leq \|B_w^{-1}B(w,z)B_z^{-1}\|$. Proposition 4.1 then gives that

$$-\frac{1}{2}\log(1-\|z\|)-\frac{1}{2}\log(1+\|z\|)\|B(w,z)^{-1}\|\leq \kappa(z,w)$$

for all $z,w\in B$. For w_0 fixed in B, we let $c_2(z):=-\frac{1}{2}\log(2\|B(w_0,z)^{-1}\|)$. Then

$$-\frac{1}{2}\log d(z,\partial B)+c_2(z)\leq \kappa(z,w_0)$$

and $\lim_{z\to x} c_2(z) = -\frac{1}{2}\log(2\|B(w_0,x)^{-1}\|) < \infty$ for all $x \in \partial B$ (recall from above that $\|w_0\|\|x\| < 1$ implies that $B(w_0,x)^{-1}$ exists). ■

Example 4.4. Let Z be a complex Hilbert space H. We show that the small and large horospheres coincide and are exactly the ellipsoids of the already known 'Wolff' result on the Hilbert ball [11; 15]. One can calculate (cf. [7, section 11.2]) that for all z, w in a Hilbert ball,

$$\frac{1}{1-\|g_{-z}(w)\|^2} = \frac{|1-\langle w,z\rangle|^2}{(1-\|z\|^2)(1-\|w\|^2)}$$

and therefore

$$\frac{1-\|g_{-z}(z_0)\|^2}{1-\|g_{-z}(w)\|^2} = \frac{|1-\langle w,z\rangle|^2}{(1-\|w\|^2)}\,\frac{1-\|z_0\|^2}{|1-\langle z_0,z\rangle|^2}\ . \tag{4.5}$$

From (4.3) one gets that for all $z, w \in B$

$$\kappa(w,z) - \kappa(z_0,z) = \frac{1}{2}\log\left(\frac{1-\|g_{-z}(z_0)\|^2}{1-\|g_{-z}(w)\|^2}\right) + \log\left(\frac{1-\|g_{-z}(w)\|^2}{1-\|g_{-z}(z_0)\|^2}\right)\ . \tag{4.6}$$

By the continuity of the triple product one sees fairly easily [19], that in any bounded symmetric domain $\lim_{z\to x}\|g_{-z}(w)\| = 1$ for all $w \in B$ and $x \in \partial B$ and it follows therefore from (4.5) and (4.6) above that for all x, w in the Hilbert ball

$$\lim_{z\to x}[\kappa(w,z) - \kappa(z_0,z)] = \frac{1}{2}\log\lim_{z\to x}\frac{1-\|g_{-z}(z_0)\|^2}{1-\|g_{-z}(w)\|^2} = \frac{1}{2}\log\frac{|1-\langle w,x\rangle|^2}{1-\|w\|^2}\,\frac{1-\|z_0\|^2}{|1-\langle z_0,x\rangle|^2}\ .$$

Therefore the small and large horospheres about x of radius R (with base point z_0) coincide and

$$E_{z_0}(x,R) = F_{z_0}(x,R) = \left\{ w \in B\colon \frac{|1-\langle w,x\rangle|^2}{1-\|w\|^2} < R\,\frac{|1-\langle z_0,x\rangle|^2}{1-\|z_0\|^2} \right\}.$$

For the remainder of this paper we take $f\colon B \to B$ to be a compact holomorphic map with no fixed points in B and we let ξ be the point in ∂B found in Theorem 3.4. We now compare the horospheres $E_{z_0}(\xi, R)$ and $F_{z_0}(\xi, R)$ to the f-invariant domains found in [24]. Fix z_0 in B. Since $\lim_{z\to\xi}\|g_{-z}(w)\| = \lim_{z\to\xi}\|g_{-z}(z_0)\| = 1$, it follows from (4.6) that

$$\limsup_{z\to\xi}[\kappa(w,z) - \kappa(z_0,z)] = \frac{1}{2}\log\limsup_{z\to\xi}\left(\frac{1-\|g_{-z}(z_0)\|^2}{1-\|g_{-z}(w)\|^2}\right) \tag{4.7}$$

and a similar expression holds for the lim inf.

We now restrict ourselves to the case where Z is finite dimensional (although the following arguments also hold for arbitrary Z if, cf. [24, section 3], f is either of boundary finite rank type or f has property I). It is shown in [24, proposition 3.1] that we can choose the sequence $(z_k)_k$ in Theorem 3.4 above so that the following limit exists

$$\lim_k \frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(w)\|^2} = \lim_k (1 - \|z_k\|^2)\|B_w^{-1}B(w, z_k)B_{z_k}^{-1}\|$$
$$= \|B_w^{-1}B(w, \xi)R\| > 0$$

for a suitable $R \in \mathcal{L}(Z)$. In other words, if we write

$$\lambda_{\varpi} := \lim_k \left(\frac{1 - \|z_k\|^2}{1 - \|g_{-z_k}(w)\|^2} \right)$$

then λ_w exists and $\lambda_w > 0$ for all $w \in B$. It follows then from (4.6) that

$$\lim_k [\kappa(w, z_k) - \kappa(z_0, z_k)] = \frac{1}{2} \log \lim_k \left(\frac{1 - \|g_{-z_k}(z_0)\|^2}{1 - \|g_{-z_k}(w)\|^2} \right) \tag{4.8}$$

exists and equals $\frac{1}{2} \log \frac{\lambda_w}{\lambda_{z_0}}$ for all $w, z_0 \in B$. It then follows immediately from equations (3.3), (3.4) and (3.5) in Theorem 3 above that the following domain $E_{\xi,R}^{z_0}$ is an f-invariant domain in B satisfying

$$E_{z_0}(\xi, R) \subset E_{\xi,R}^{z_0} \subset F_{z_0}(\xi, R) \tag{4.9}$$

where

$$E_{\xi,R}^{z_0} := \left\{ w \in B : \lim_k [\kappa(w, z_k) - \kappa(z_0, z_k)] < \frac{1}{2} \log R \right\} \tag{4.10}$$

$$= \left\{ w \in B : \lim_k \left(\frac{1 - \|g_{-z_k}(z_0)\|^2}{1 - \|g_{-z_k}(w)\|^2} \right) < R \right\} \tag{4.11}$$

$$= \{ w \in B : \lambda_w < R\lambda_{z_0} \} \tag{4.12}$$

$$= E_{\xi, R\lambda_{z_0}}^{0}. \tag{4.13}$$

Since $E_{\xi,R}^{z_0} = E_{\xi,R\lambda_{z_0}}^{0}$ we can always take the base point z_0 to be 0 and we write

$$E_{\xi,R} := E_{\xi,R}^{0}.$$

Therefore the f-invariant domains $E_{\xi,R}$ studied in detail in [24] lie sandwiched between the small and large horospheres. Although the domains $E_{\xi,R}$ may initially look more natural here described in terms of the Kobayashi distance (which is not surprising since they are shown in [24, theorem 3.8] to be the limit in a certain sense of a sequence of Kobayashi balls) it is the explicit description of these domains in terms of the Jordan triple product (cf. [24], proposition 3.1)

$$E_{\xi,R} = \left\{ w \in B : \lim_k (1 - \|z_k\|^2)\|B_w^{-1}B(w, z_k)B_{z_k}^{-1}\| < R \right\}$$

that allows one to show using JB^*-triple techniques [24, theorem 3.10] that the invariant domains have the very simple form

$$E_{\xi,R} = c_R + T_R(B)$$

for some $c_R \in B$ and T_R an invertible linear operator on Z ($c_r + T_R(B)$ is referred to as an operator ball). In addition, this immediately shows that the invariant domains $E_{\xi,R}$ are always non-empty convex domains in B for all $R > 0$. Consequently we know that even if $R < 1$ the horospheres are non-empty (this is not known in general). We also note that the horospheres are not convex in general although strongly convex C^3 domains do have convex horospheres [3].

One sees easily from Example 4.4 above that if Z is a complex Hilbert space then

$$E_{z_0}(\xi, R) = E^{z_0}_{\xi,R} = F_{z_0}(\xi, R).$$

In general, however, the three domains are different.

Example 4.5. Let $B = \Delta^n$ and $z_0 = 0$. Then [2, proposition 2.4.12]

$$E_0(\xi, R) = \left\{ w \in \Delta^n : \max_j \left\{ \left(\frac{|\xi_j - w_j|^2}{1 - |w_j|^2} \right) : |\xi_j| = 1 \right\} < R \right\}$$

and

$$F_0(\xi, R) = \left\{ w \in \Delta^n : \min_j \left\{ \left(\frac{|\xi_j - w_j|^2}{1 - |w_j|^2} \right) : |\xi_j| = 1 \right\} < R \right\} .$$

On the other hand, it is shown in [24, theorem 3.10 and example 3.17] that there exists real numbers $a_1, \ldots, a_n \in [0,1]$ with $|a_i| = 1$ for at least one $i \in \{1, \ldots, n\}$ so that

$$E_{\xi,R} = \left\{ w \in \Delta^n : \max_j \left\{ \left(\frac{|\xi_j - w_j|^2}{1 - |w_j|^2} \right) a_j : |\xi_j| = 1 \right\} < R \right\} .$$

Since it is possible that $a_i \neq 1$ even if $|\xi_i| = 1$ (cf. [24, theorem 3.10]) these three domains are in general distinct.

The small and large horospheres about the point $\xi \in \partial B$ can in a crude sense be approximated by a sequence of Kobayashi balls as follows. For $(z_k)_k$ as above, one sees directly from the definition of $E^{z_0}_{\xi,R}$ (or alternatively from [24, theorem 3.8]) that the following hold:

(i) if $w \in E^{z_0}_{\xi,R}$ then $w \in B_\kappa(z_k, r_k)$ for all k large and

(ii) if $w \in B_\kappa(z_k, r_k)$ for all k large then $w \in \bar{E}^{z_0}_{\xi,R}$

where $r_k = \kappa(z_0, z_k) + \frac{1}{2}\log R = \frac{1}{2}\log(R(1 + \|g_{-z_k}(z_0)\|)^2 \|B^{-1}_{z_0} B(z_0, z_k) B^{-1}_{z_k}\|)$ from Proposition 4.1.

In particular then we have that

(a) if $w \in E_{z_0}(\xi, R)$ then $w \in B_k(z_k, r_k)$ for all k large and

(b) if $w \in B_\kappa(z_k, r_k)$ for all k large then $w \in \bar{F}_{z_0}(\xi, R)$ with r_k as above.

5. Horospheres and the boundary of *D*

Our final comments concern the use of Theorem 3.4 and its corollaries to study the iterates of a fixed-point-free compact holomorphic function f. Given a point y in [Trial mode], we know that $y \in E_0(\xi, R)$ for some $R > 0$ and therefore from Theorem

3.4 all iterates of y under f must lie in $F_0(\xi, R)$. Now, if $(f^n(y))_n$ converges it must converge to a point of ∂D (otherwise the limit would be a fixed point of f) and therefore the limit would have to lie in $\partial F_0(\xi, R) \cap \partial D$. If this sequence does not converge then we study the subsequential limits of $(f^n(y))_n$ and again if such limits exist on ∂D, Theorem 3.4 locates them to within $\partial F_0(\xi, R) \cap \partial D$. The problem, however, is that $\partial F_0(\xi, R) \cap \partial D$ can be quite large even if ξ is on the Shilov boundary of D. Most of the Denjoy-Wolff type iteration results available in the literature [6; 28; 11; 15; 4; 1; 3; 16; 21; 25] are achieved in cases where the domain has sufficient convexity to control the size of $\partial F_0(\xi, R) \cap \partial D$ and generally to force $\partial F_0(\xi, R) \cap \partial D = \{\xi\}$. Since this is far from happening in the case of bounded symmetric domains (for example Δ^n) we dwell now on the possible size of $\partial F_0(\xi, R) \cap \partial D$. In order to do this, we restrict ourselves once again to the bounded symmetric domains B, for which there is a well-developed theory of holomorphic boundary components [19]. We recall the basic definitions.

Definition 5.1. Let $a, b \in \bar{B}$. We say a and b are affinely equivalent in $\bar{B}$ if there is a finite number of complex affine functions $q_k\colon \Delta \to \bar{B}$, $0 \le k \le n$, satisfying $a \in q_0(\Delta)$, $b \in q_n(\Delta)$ and $q_k(\Delta) \cap q_{k+1}(\Delta) \neq \emptyset$ for $0 \le k \le n-1$.

Definition 5.2. We say a, b in $\bar{B}$ are holomorphically equivalent if instead of complex affine functions in Definition 5.3 we have holomorphic functions.

Definition 5.3. The affine (resp. holomorphic) boundary components of B are the affine (resp. holomorphic) equivalence classes of $\bar{B}$.

It is not too difficult to see [19] that the affine and holomorphic boundary components of B coincide and we henceforth simply refer to boundary components. We denote the boundary component of a point $x \in \partial B$ by K_x.

As it was shown in [24, proposition 4.4] that $K_\xi \subset \partial \in_{\xi,R} \cap \partial B$ we can apply (4.9) to immediately give the following result.

Corollary 5.4.

$$K_\xi \subset \partial Fz_0(\xi, R) \cap \partial B \quad \textit{for all } R > 0.$$

In the case of the polydisc Δ^n in $\mathbf{C}^n$ this containment is even strict (cf. [2, corollary 2.4.13] or [24, example 4.6]). We see then from the comments at the end of Section 2 that, in general, this makes the study of the iterates of f, and also the study of the angular limits of f, a difficult problem. It is already known that no Denjoy-Wolff type convergence result can hold for the polydisc Δ^n and even the pointwise behaviour of sequences of iterates on Δ^n is bad [4, example 1]. In fact, since any rank n bounded symmetric domain contains a copy of Δ^n preserved with respect to the Kobayashi distance, one can reasonably expect the difficulties arising in the case of Δ^n to be representative of the difficulties arising in general bounded symmetric domains. We note that the case of the bidisc Δ^2 was effectively dealt with by Hervé in 1954 [14].

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