LINE INSERTIONS IN TOTALLY POSITIVE MATRIX FUNCTIONS

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ABSTRACT

Between any two rows (columns) of an m-by-n totally nonnegative matrix polynomial, it is not difficult to show that a new polynomial row (column) may be inserted to form a larger, totally nonnegative matrix polynomial. The analogous question, in which 'totally nonnegative' is replaced by 'totally positive' arises in completion problems and the extension of collocation matrices, and its answer is far less clear. Here the totally positive question is answered affirmatively, and by similar techniques affirmative answers may also be given for several other classes of matrix functions.

1. Introduction

An m-by-n matrix A is called *totally positive (nonnegative)* if every minor of A is positive (nonnegative). See [1; 5; 6; 7; 10] for background and ample motivation. Such matrices have arisen in a surprising variety of ways throughout the twentieth century and have received increasing attention of late. The following applications from interpolation and computer-aided geometric design are discussed more thoroughly in [7].

(1) Let $y = (y_0, y_1, \ldots, y_n)^T$ be a system of functions defined on an interval. The *collocation matrix* of y at $t_0 < t_1 < \ldots < t_m$ is the matrix $M = (y_j(t_i))$, $i = 0, \ldots, m$; $j = 0, \ldots, n$. We say y is *totally positive* if all its collocation matrices are totally positive. Collocation matrices are important in interpolation problems.

(2) In a finite dimensional space that has a totally positive basis, there exist special bases called *B-bases*. Every totally positive basis $y = (y_1, y_2, \ldots, y_n)$ can be obtained from a B-basis $b = (b_1, b_1, \ldots, b_n)$ via the totally positive change of basis matrix A, i.e. $y = bA$. The Bernstein basis of polynomials of

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degree n or less on a compact interval and the B-spline basis in the space of polynomial splines are examples of B-bases. In computer-aided geometric design, totally positive matrices provide good shape properties and there is a correspondence between B-bases and optimal shape-preserving properties. For instance, the Bernstein basis has been shown to have optimal shape-preserving properties [3].

A *matrix polynomial (function)* is simply a matrix whose entries are polynomials (functions) in a single real variable, e.g. $A(x)=(a_{ij}(x))$, in which each $a_{ij}(x)$ is an independent polynomial (function). Modifiers of the term 'matrix function' apply to each entry function individually, so that a *continuous matrix function* is one in which each function is continuous, for example. After [8], we call a matrix polynomial or matrix function *totally positive* (*TP*) or *totally nonnegative* (*TN*) if it is so, point-wise, i.e. if $A(x)$ is *TP* (or *TN*) when evaluated at each value of the real variable x. Note that in the *TP* case each entry must be a positive polynomial (function), i.e. the value of the polynomial (function) at each value of x is positive. Examples of any size are easily constructed.

It is straightforward to show that between the ith and $(i+1)$st rows (jth and $(j+1)$st columns), $i=0,\ldots, m$ ($j=0,\ldots, n$) of an m-by-n *TN* matrix A, a row (column) may be inserted to form an $(m+1)$-by-n (m-by-$(n+1)$) *TN* matrix $\hat{A}$.

For example, repetition of an adjacent line (i.e. row or column) will do. However, if *TN* is replaced by *TP*, solvability of the line insertion problem is much less clear; but, it was recently shown [9] that line insertion is always possible in the *TP* case. Our purpose here is to discuss generalisations of this result to *TP* matrix functions, with the functions possibly of restricted type. Again, the *TN* case is straightforward via repetition of an adjacent line.

If $A(x)$ is a general *TP* matrix function, then it is not difficult to show, using the result of [9], that line insertion to produce another *TP* matrix function $\hat{A}(x)$ is always possible. For example, one may apply the result of [9] point-wise to obtain a particular line to insert for each value of x, and then let that give the function value at each entry of the line. If, however, the *TP* matrix function $A(x)$ is restricted in some entry-wise way and $\hat{A}(x)$ is required to meet the same entry-wise restrictions, then the above prescription may not work for every combination of inserted lines, and the existence of an $\hat{A}(x)$ is much less clear. It is our purpose here to show that the line insertion problem always has a solution for *TP* matrix polynomials, *TP* matrix rational functions, *TP* matrix continuous functions and *TP* matrix differentiable functions.

The following observation about line insertions in matrix polynomials (respectively, matrix rational functions, matrix continuous functions and matrix differentiable functions) may be easily made.

Observation 1.1. *In either the TP or TN case, the set of possible insertions (in a specific place) is convex, because of the linearity of the determinant as a function of a line.*

Observation 1.2. *In the TN case, the set of possible insertions is always nonempty (i.e. line insertion is always possible in every position in every TN matrix polynomial, TN matrix rational function, TN matrix continuous function and TN matrix differentiable* *function) because either repetition of an adjacent line or use of the zero subline suffices. However, neither of these insertions is possible in the general TP case, and the existence of a solution to the line insertion problem in the TP case, though plausible, is not obvious.*

We shall need the following lemmas to establish our third observation. A function g *dominates* a function f if $g(x) > f(x)$ for all x.

Lemma 1.3. *If f is a polynomial (respectively, continuous function, differentiable function), then there is a positive polynomial (respectively, positive continuous function, positive differentiable function) that dominates f.*

PROOF. If f is a polynomial (respectively, continuous function, differentiable function), then $h(x) = (f(x))^2 + 1$ is a positive polynomial (positive continuous function, positive differentiable function) that dominates f. ■

Lemma 1.4. *If $R(x) = \frac{f(x)}{g(x)}$ in which $f(x)$ is a polynomial (respectively, continuous function, differentiable function) and $g(x)$ is a positive continuous function, then there is a positive polynomial (respectively, positive continuous function, positive differentiable function) that dominates $R(x)$.*

PROOF. We prove the result for the case in which $f(x)$ is a polynomial (the other cases are similar). Since $g(x)$ is a positive continuous function, it attains its minimum value m at some critical value of x. By Lemma 1.3, there is a positive polynomial $\hat{f}$ that dominates f and hence the positive polynomial $p(x) = \frac{\hat{f}(x)}{m} + 1$ dominates $R(x)$. ■

Remark. Note that if $R(x) = \frac{f(x)}{g(x)}$ is *TP*, then f and g may be assumed to be positive functions.

Observation 1.5. *When the line is exterior (top/bottom, right/left), it is obvious that there is a solution in the TP matrix polynomial (respectively, TP matrix rational function, TP matrix continuous function, TP matrix differentiable function) case by adding sufficiently large positive polynomial (respectively, positive polynomial, positive continuous function, positive differentiable function) entries one at a time in the correct order. For example, to add a new last column, start from the top and move down the column; each successive entry enters only positively into every minor it completes, so that, by Lemma 1.4, a sufficiently large positive polynomial (respectively, positive polynomial, positive continuous function, positive differentiable function) can be chosen each time. (Note that, in view of the preceding remark, the rational function case reduces to the polynomial case upon simplification.)*

We show here that the line insertion problem always has a solution in the *TP* case and the mechanism may be of independent interest (as we have found it to be in other contexts). We give a careful analysis to 'totally positive linear systems'.

2. Main Results

The following lemma is fundamental to the insertion strategy and may be of independent interest.

Lemma 2.1. *If* $A(x) = (a_{ij}(x)) = [a_1(x), a_2(x), \ldots, a_n(x)]$ *is an* $(n-1)$*-by-*n *TP matrix polynomial (respectively, TP matrix rational function, TP matrix continuous function, TP matrix differentiable function), then, for* $k = 1, 2, \ldots, n$,

$$a_k(x) = \sum_{\substack{i=1 \\ i \neq k}}^{n} z_i(x) a_i(x) \tag{2.1}$$

in which each $z_i(x) = c_i \frac{f_i(x)}{g_i(x)}$ *and each* $f_i(x)(g_i(x))$ *is a positive polynomial (respectively, positive polynomial, positive continuous function, positive differentiable function) and* c_i *equals* $(-1)^i$ *if* k *is odd and* $(-1)^{i-1}$ *if* k *is even.*

Proof. If $k = 1$, (2.1) has solution

$$\begin{aligned} z(x) &= [a_2(x), a_3(x), \ldots, a_n(x)]^{-1} a_1(x) \\ &= \frac{1}{\det[a_2(x), a_3(x), \ldots, a_n(x)]} \begin{bmatrix} \det[a_1(x), a_3(x), a_4(x), \ldots, a_n(x)] \\ \det[a_2(x), a_1(x), a_4(x), \ldots, a_n(x)] \\ \vdots \\ \det[a_2(x), a_3(x), \ldots, a_{n-1}(x), a_1(x)] \end{bmatrix} \end{aligned}$$

and, for $i = 1, \ldots,$ n, $z_i(x) = (-1)^i \frac{f_i(x)}{g_i(x)}$ in which each $f_i(x)$ $(g_i(x))$ is a positive polynomial (respectively, positive poynomial, positive continuous function, positive differentiable function).

If $k > 1$, then (2.1) has solution

$$\begin{aligned} z(x) &= [a_1(x), \ldots, a_{k-1}(x), a_{k+1}(x), \ldots, a_n(x)]^{-1} a_k(x) \\ &= \frac{1}{\det[a_1(x), \ldots, a_{k-1}(x), a_{k+1}(x), \ldots, a_n(x)]} \\ &\quad \times \begin{bmatrix} \det[a_k(x), a_2(x), a_3(x), \ldots, a_{k-1}(x), a_{k+1}(x), \ldots, a_n(x)] \\ \det[a_1(x), a_k(x), a_3(x), \ldots, a_{k-1}(x), a_{k+1}(x), \ldots, a_n(x)] \\ \vdots \\ \det[a_1(x), \ldots, a_{k-1}(x), a_{k+1}(x), \ldots, a_{n-1}(x), a_k(x)] \end{bmatrix}. \end{aligned}$$

We see that, for $i = 1, \ldots, n$, there exist each positive polynomials (respectively, positive polynomials, positive continuous functions, positive differentiable functions) $f_i(x)$ and $g_i(x)$ such if k is odd, $z_i(x) = (-1)^i \frac{f_i(x)}{g_i(x)}$ while if k is even, $z_i(x) = (-1)^{i-1} \frac{f_i(x)}{g_i(x)}$. The result follows. ■

In other words, when expressing any column $a_k(x)$ of an $(n-1)$-by-n *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function, *TP* differentiable function), $A(x)$ as a linear combination of the remaining columns, say $a_k(x) = \sum_{\substack{i=1 \\ i \neq k}}^{n} z_i(x) a_i(x)$, all coefficients $z_i(x)$ are signed ratios of positive polynomials (respectively, positive polynomials, positive continuous functions, positive differentiable functions) and, for $i = 1, \ldots, k-1$ and $i = k+1, \ldots, n$, the signs of the coefficients alternate with the signs of $z_{k-1}(x)$ and $z_{k+1}(x)$ being positive. In terms of column insertion, this means that, if we insert a column $y(x)$ into a square *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function, *TP* matrix differentiable function) $A(x)$ and remain *TP*, then, when $y(x)$ is expressed as a linear combination of the columns of $A(x)$, the coefficients of the columns of $A(x)$ are 'appropriately signed' in the following sense: the signs of the coefficients of the columns on each side of $y(x)$ alternate with the signs of the coefficients of the columns adjacent to $y(x)$ being positive. It is easy to construct examples to show that just inserting a column that is an appropriately signed linear combination of the columns of a square *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function, *TP* matrix differentiable function) $A(x)$ does not necessarily result in a *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function, *TP* matrix differentiable function), even if the column entries of $A(x)$ are positive polynomials (respectively, positive rational functions, positive continuous function, positive differentiable functions).

Example 2.2. Consider the constant *TP* matrix polynomial

$$A(x) = \begin{bmatrix} x^2+1 & x^4+1 & x^4+1 \\ x^2+1 & x^4+2 & 2x^4+3 \\ x^2+1 & x^4+3 & 3x^4+6 \end{bmatrix} = [a_1(x), a_2(x), a_3(x)].$$

If we insert the constant column $y(x) = a_1(x) + a_2(x) - \frac{1}{3}a_3(x)$ between the first and second columns of A, we obtain $B(x) = [a_1(x), y(x), a_2(x), a_3(x)]$ in which

$$y(x) = \begin{bmatrix} \frac{2}{3}x^4 + x^2 + \frac{5}{3} \\ \frac{1}{3}x^4 + x^2 + 2 \\ x^2 + 2 \end{bmatrix}$$

Since $\det B[\{1,2\}] = -\frac{1}{3}(x^2+1)(x^4-1) \leq 0$ *if* $|x| \geq 1$, B is not a *TP* matrix polynomial.

Thus, Lemma 2.1 gives only a necessary condition for inserting a row (column) in a *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function), but not a sufficient condition. In addition, care must be taken in choosing the coefficients $z_i(x)$ so that a new *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function) is created. The proof of our main result shows how to select these coefficients so that we obtain a new *TP* matrix polynomial (respectively, *TP* matrix rational function, *TP* matrix continuous function) upon the insertion of the resulting column. Implicit in the proof is the use of Fekete's criterion [4] stated as follows: a matrix is totally positive if and only if the determinant of every square submatrix based on contiguous (e.g., i, $i+1,\ldots,i+k$) row *and* column index sets is positive.

Theorem 2.3. *Let $A(x)=(a_{ij}(x))$ be a TP matrix polynomial (respectively, TP matrix rational function, TP matrix continuous function, TP matrix differentiable function). Then, a line can be inserted between any pair of adjacent lines in $A(x)$ so that the resulting matrix is a TP matrix polynomial (respectively, TP matrix rational function, TP matrix continuous function, TP matrix differentiable function).*

Proof. We will prove the theorem for the case in which $A(x)$ is a TP matrix polynomial; the other cases follow in a similar fashion by applying Lemma 1.4. By transposition and/or external addition of rows/columns (see Observation 1.5), we may assume, without loss of generality, that $A(x)$ is square and of even order, say A is n-by-n in which $n=2k$, and that we wish to insert a column in the middle. Specifically, let $A(x)=[a_1(x),\ldots, a_k(x), a_{k+1}(x),\ldots, a_n(x)]$ in which $n=2k$ and let $\tilde{A}(x)=[a_1(x),\ldots,a_k(x),y(x),a_{k+1}(x),\ldots,a_n(x)]$ in which

$$y(x)=\sum_{i=1}^{k}(-1)^{i+1}z_i(x)a_{k-i+1}(x)+\sum_{i=1}^{k}(-1)^{i+1}z_i(x)a_{k+i}(x) \tag{2.2}$$

for some choice of positive polynomials $z_1(x),\ldots, z_k(x)$. Thus, the coefficients of $y(x)$ are appropriately signed and it remains to show that positive polynomials $z_1(x),\ldots, z_k(x)$ can be chosen so that $\tilde{A}(x)$ is a TP matrix polynomial.

For each square contiguous submatrix $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$, let $\hat{a}_i(x)$ denote the subcolumn of $a_i(x)$ having the same row indices as $\hat{y}(x)$ $(i=1,\ldots, n)$, let $l(\hat{A}(x))$ (respectively, $r(\hat{A}(x))$) denote the number of columns of $\hat{A}(x)$ which lie to the left (respectively, right) of $\hat{y}(x)$, and let $m=m(\hat{A}(x))=\min\{l(\hat{A}(x)),\ r(\hat{A}(x))\}$. Thus,

$$\hat{A}(x)=[\hat{a}_{k-l(\tilde{A}(x))+1}(x),\ldots,\hat{a}_k(x),\hat{y}(x),\hat{a}_{k+1}(x),\ldots,\hat{a}_{k+r(\tilde{A}(x))}(x)] \tag{2.3}$$

in which

$$\hat{y}(x)=\sum_{i=1}^{k}(-1)^{i+1}z_i(x)\hat{a}_{k-i+1}(x)+\sum_{i=1}^{k}(-1)^{i+1}z_i(x)\hat{a}_{k+i}(x). \tag{2.4}$$

First, set $z_k(x)=1$; we will show that this ensures that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x))=k-1$ have positive determinant for all x or, equivalently, that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x))=k-1$ have positive determinant for all x. This statement follows since no square contiguous submatrix $\hat{A}(x)$ of $\hat{A}(x)$ contains a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfies $m(\hat{A}(x))=k$. We need to consider the following cases since $\hat{A}(x)$

contains a subcolumn $\hat{y}(x)$ of $y(x)$ and $m(\hat{A}(x)) = k-1$ together imply that the order of $\hat{A}(x)$ must be either n (Cases I or II) or $n-1$ (Case III).

Case I. $\det \hat{A}(x) = \det[a_1(x), \ldots, a_k(x), y(x), a_{k+1}(x), \ldots, a_{n-1}(x)]$

$$= \det[a_1(x), \ldots, a_k(x), (-1)^{k+1} z_k(x) a_{2k}(x), a_{k+1}(x), \ldots, a_{2k-1}(x)]$$
$$= \det[a_1(x), \ldots, a_k(x), a_{k+1}(x), \ldots, a_{2k-1}(x), (-1)^{(k+1)+(k-1)} z_k(x) a_{2k}(x)]$$
$$= z_k(x) \det[a_1(x), \ldots, a_k(x), a_{k+1}(x), \ldots, a_{2k}(x)] = \det A(x) > 0$$

for all x.

Case II. $\det \hat{A}(x) = \det[a_2(x), \ldots, a_k(x), y(x), a_{k+1}(x), \ldots, a_n(x)] > 0$ for all x is similar.

Case III. $\det \hat{A}(x) = \det[\hat{a}_2(x), \ldots, \hat{a}_k(x), \hat{y}(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{n-1}(x)]$

$$= \det[\hat{a}_2(x), \ldots, \hat{a}_k(x), (-1)^{k+1} z_k(x) \hat{a}_1(x) + (-1)^{k+1} z_k(x) \hat{a}_n(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{n-1}(x)]$$
$$= \det[(-1)^{(k+1)+(k-1)} z_k(x) \hat{a}_1(x), \hat{a}_2(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{n-1}(x)]$$
$$+ \det[\hat{a}_2(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{n-1}(x), (-1)^{(k+1)+(k-1)} z_k(x) \hat{a}_n(x)]$$
$$= \det[\hat{a}_1(x), \ldots, \hat{a}_{n-1}(x)] + \det[\hat{a}_2(x), \ldots, \hat{a}_n(x)] > 0$$

for all x.

We will now show that positive polynomials $z_{k-1}(x), \ldots, z_1(x)$ can be sequentially chosen so that $\tilde{A}(x)$ is a *TP* polynomial matrix. We will need the following observation.

Observation 2.4. *Let $\hat{A}(x)$ be given by (3) in which $\hat{y}(x)$ is given by (4) and $m(\hat{A}(x)) \geq j-1$. Then the terms of (4) involving any of $z_1(x), \ldots, z_{j-1}(x)$ can be ignored when computing det $\hat{A}(x)$. [For, the terms of (3) involving any of $z_1(x), \ldots, z_{j-1}(x)$ correspond to the columns $\hat{a}_{k-j+2}(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+j-1}(x)$. Since $\hat{A}(x)$),* $r(\hat{A}(x)) \geq j-1$, all of these are columns of $\hat{A}(x)$ distinct from $\hat{y}(x)$ and therefore can be ignored when computing det $\hat{A}(x)$.]

For $j = k, k-1, \ldots, 2$, assume (inductively) that $z_k(x), z_{k-1}(x), \ldots, z_j(x)$ have been chosen so that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) \geq j-1$ have positive determinant for all x. Then we just need to show that there is a positive polynomial $z_{j-1}(x)$ such that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) = j-2$ have positive determinant for all x. [For, by Observation 2.4, we will then have chosen positive polynomials $z_k(x), z_{k-1}(x), \ldots, z_j(x), z_{j-1}(x)$ such that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) \geq j-2$ have positive determinant for all x.]

To simplify subscripting, let $m = j-2$ so that positive polynomials $z_k(x), \ldots, z_{m+2}(x)$ have been chosen such that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) \geq m+1$ have positive determinant for all x; a positive polynomial $z_{m+1}(x)$ needs to be chosen so that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $\mathrm{m}(\hat{A}(x)) = m$ have positive determinant for all x. We consider the various possibilities for $m = m(\hat{A}(x))$ where $\hat{A}(x)$ is given by (3) and $\hat{y}(x)$ is given by (4). In each case (4) has a splitting $\hat{y}(x) = s(x) + t(x) + u(x)$, in which $s(x)$ is the sum of the terms in (4) each of whose vector part is a column of $\hat{A}(x)$ distinct from $\hat{y}(x)$ (and hence $s(x)$ can be ignored in computing $\det(\hat{A}(x))$, $t(x)$ is the sum of the terms of $\hat{y}(x) - s(x)$ that involve $z_{m+1}(x)$ and $u(x) = \hat{y}(x) - s(x) - t(x)$. Thus, the terms that sum to $u(x)$ involve the previously chosen coefficients $z_k(x), \ldots, z_{m+2}(x)$ only.

Case I. $m = r = r(\hat{A}(x)) < l(\hat{A}(x)) = l$. Then $\hat{y}(x) = s(x) + t(x) + u(x)$ in which

$$s(x) = \sum_{i=1}^{l} (-1)^{i+1} z_i(x) \hat{a}_{k-i+1}(x) + \sum_{i=1}^{m} (-1)^{i+1} z_i(x) \hat{a}_{k+i}(x),$$
$$t(x) = (-1)^{m+2} z_{m+1}(x) \hat{a}_{k+m+1}(x),$$

and

$$u(x) = \sum_{i=l+1}^{k} (-1)^{i+1} z_i(x) \hat{a}_{k-i+1}(x) + \sum_{i=m+2}^{k} (-1)^{i+1} z_i(x) \hat{a}_{k+i}(x).$$

Thus, we must have

$$\begin{aligned} \det \hat{A}(x) &= \det[\hat{a}_{k-l+1}(x), \ldots, \hat{a}_k(x), t(x) + u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)] \\ &= \det[\hat{a}_{k-l+1}(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x), (-1)^{2m+2} z_{m+1}(x) \hat{a}_{k+m+1}(x)] \\ &\quad - \det[\hat{a}_{k-l+1}(x), \ldots, \hat{a}_k(x), -u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)] > 0 \end{aligned}$$

for all x, or equivalently,

$$z_{m+1}(x) > b(\hat{A}(x)) = \frac{\det[\hat{a}_{k-l+1}(x), \ldots, \hat{a}_k(x), -u(x)\, \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)]}{\det[\hat{a}_{k-l+1}(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x), \hat{a}_{k+m+1}(x)]}$$

for all x. Notice that $b(\hat{A}(x))$ is a rational function whose denominator is a positive polynomial. Hence, by Lemma 1.4, we can find a positive polynomial $z_{m+1}(x)$ that dominates $b(\hat{A}(x))$.

Case II. $m = r = r(\hat{A}(x)) = l(\hat{A}(x)) = l$. Then $\hat{y}(x) = s(x) + t(x) + u(x)$ in which

$$s(x) = \sum_{i=1}^{m} (-1)^{i+1} z_i(x) \hat{a}_{k-i+1}(x) + \sum_{i=1}^{m} (-1)^{i+1} z_i(x) \hat{a}_{k+i}(x),$$

$$t(x) = (-1)^{m+2} z_{m+1}(x) \hat{a}_{k-m}(x) + (-1)^{m+2} z_{m+1}(x) \hat{a}_{k+m+1}(x),$$

and

$$u(x) = \sum_{i=m+2}^{k} (-1)^{i+1} z_i(x)\hat{a}_{k-i+1}(x) + \sum_{i=m+2}^{k} (-1)^{i+1} z_i(x)\hat{a}_{k+i}(x).$$

Thus, we must have

$$\begin{aligned}\det \hat{A}(x) &= \det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_k(x), t(x) + u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)] \\ &= \det[(-1)^{2m+2} z_{m+1}(x)\hat{a}_{k-m}(x), \hat{a}_{k-m+1}, \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)] \\ &\quad + \det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x), (-1)^{2m+2} z_{m+1}(x) \\ &\quad \hat{a}_{k+m+1}(x)] - \det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_k(x), -u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)] > 0\end{aligned}$$

for all x, or equivalently,

$$z_{m+1}(x) > b(\hat{A}(x)) = \frac{\det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_k(x), -u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+m}(x)]}{\det[\hat{a}_{k-m}(x), \ldots, \hat{a}_{k+m}(x)] + \det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_{k+m+1}(x)]}$$

for all x. As before, since $b(\hat{A}(x))$ is a rational polynomial whose denominator is a positive polynomial, there is a positive polynomial $z_{m+1}(x)$ that dominates $b(\hat{A}(x))$.

Case III. $m = l(\hat{A}(x)) < r(\hat{A}(x)) = r$. Analogously to Case I, one has that $\det\hat{A}(x) > 0$ for all x is equivalent to

$$z_{m+1}(x) > b(\hat{A}(x)) = \frac{\det[\hat{a}_{k-m+1}(x), \ldots, \hat{a}_k(x), -u(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+r}(x)]}{\det[\hat{a}_{k-m}(x), \ldots, \hat{a}_k(x), \hat{a}_{k+1}(x), \ldots, \hat{a}_{k+r(\hat{A}(x))}(x)]}$$

for all x. Again, there is a positive polynomial $z_{m+1}(x)$that dominates $b(\hat{A}(x))$ by Lemma 1.4.

Notice that in each of the Cases I, II, and III, a positive polynomial $z_{m+1}(x)$ can be determined from the positive polynomials $z_k(x)$,..., $z_{m+2}(x)$ so that det $\hat{A}(x) > 0$ for all x. Since there are a finite number of square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) = m$, we can choose the positive polynomial $z_{m+1}(x)$ large enough so that for *all* square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) = m$, det $\hat{A}(x) > 0$ for all x. [As we can just find a positive polynomial that works for each such $\hat{A}(x)$, given that $z_{m+1}(x)$ equal the sum of all these polynomials.] This completes the proof of the theorem. ■

To clarify the proof, we reconsider Example 2.2. In this example, to insert a column between, say, the first and second columns, we first augment the matrix with exterior lines so that the resulting 4-by-4 matrix is TP and the column to be added is in the middle. The matrix resulting from one such augmentation is

$$\begin{bmatrix} 8 & x^2+1 & x^4+1 & x^4+1 \\ 4 & x^2+1 & x^4+2 & 2x^4+3 \\ 2 & x^2+1 & x^4+3 & 3x^4+6 \\ 1 & x^2+1 & x^4+4 & 4x^4+10 \end{bmatrix}$$

So we just need to add a column $y(x)$ in the middle (forming the matrix $\tilde{A}(x)$) while retaining total positivity. According to the proof of the theorem, we let

$$y(x) = -z_2(x)a_1(x) + z_1(x)a_2(x) + z_1(x)a_3(x) - z_2(x)a_4(x)$$

in which $z_1(x)$, $z_2(x)$ are positive polynomials chosen so as to ensure total positivity. In particular, choosing $z_2(x) = 1$, all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) = 2 - 1 = 1$ have positive determinant. The proof also shows that we can choose a positive polynomial $z_1(x)$ such that all square contiguous submatrices $\hat{A}(x)$ of $\tilde{A}(x)$ containing a subcolumn $\hat{y}(x)$ of $y(x)$ and satisfying $m(\hat{A}(x)) = 2 - 2 = 0$ (i.e. those submatrices such that a subcolumn of $y(x)$ is either the first column or the last column) have positive determinant. Together, these statements imply that $\tilde{A}(x)$ is *TP*. By deleting the first column and last row of $\tilde{A}(x)$, we obtain the desired insertion for our original *TP* matrix function.

As a potential application of our main result, consider an nth order system of linear first order initial value problems

$$\frac{dX}{dt} = AX' + F \tag{2.5}$$

for $a \leq t \leq b$ in which $X = (x_1(t), \ldots, x_n(t))^T$, $A = (a_{ij}(t))$, $i, j = 1, \ldots, n$, is known to be a *TP* matrix function, $F = (f_1(t), \ldots, f_n(t))^T$, and with initial conditions $X(a) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Runge Kutta methods can be used to solve such a system (see [2] for details). In mathematical modeling, one might be interested in such a system, but, in practice, it might be the case that either a) one of the equations is missing or unknown (a row of A is missing) or b) the effect of one of the functions x_i cannot be determined (a column of A is missing). To have any hope of determining a solution, it would then be necessary to be able to complete A so as to maintain total positivity (this, of course, would not be likely to be the *TP* completion we are looking for). Then, for each such completion, one could derive the corresponding solution to the first order system via Runge Kutta methods. From the proof of our theorem, we see that a grid of possible solutions can be obtained in this manner, one for each *TP* completion of A. Practical considerations might then allow us to determine which of these solutions best approximates the actual solution.

Another potential application of our result lies in the study of small oscillations of n masses concentrated at n movable points $x_1 < x_2 < \ldots < x_n$ of a segmentary elastic continuum stretched along a segment $0 \leq x \leq l$ of the x-axis [5, p. 110].

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