GENERALISED P-SYMMETRIC OPERATORS

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Abstract

Let H denote a complex Hilbert space, $L(H)$ the algebra of all bounded linear operators on H and $C_1(H)$, the trace class operators. We study the pairs of operators $A, B \in L(H)$ with the property that $AT = TB$ and $T \in C_1(H)$ implies $B^*T = TA^*$. The main result is that this property is equivalent to the self-adjointness of the ultraweak closure of the range of a generalised derivation.

1. Introduction

For $A, B \in L(H)$, $\delta_{A,B}$ denotes the operator on $L(H)$ defined by $\delta_{A,B}(X) = AX - XB$ (for $X \in L(H)$). When $A = B$, $(\delta_{A,A} = \delta_A)$ is called the inner derivation induced by $A \in L(H)$. In [1] J. Anderson *et al*. show that if A is D-symmetric (i.e. $\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}$, where $\overline{R(\delta_A)}$ denotes the norm closure of the range of δ_A) then for $T \in C_1(H)$, $AT = TA$ implies $A^*T = TA^*$.

S. Bouali and J. Charles [2] introduced P-symmetric operators (operators $A \in L(H)$ with the property that $T \in C_1(H)$, $AT = TA$ implies $A^*T = TA^*$ and gave some properties of P-symmetric operators. We generalise these results to what we call P-symmetric pairs (A, B) of operators.

The ideal $C_1(H)$ of $L(H)$ admits a trace function $tr(T)$, given by $tr(T) = \sum_n (Te_n, e_n)$ for any complete orthonormal system (e_n) in H. As a Banach space, $C_1(H)$ can be identified with the dual of the ideal K of compact operators by means of the linear isometry $T \mapsto f_T$, where $f_T = tr(XT)$. Moreover $L(H)$ is the dual of $C_1(H)$. The ultraweakly continuous linear functionals on $L(H)$ are those of the form f_T for $T \in C_1(H)$ and the weakly continuous ones are those of the form f_T where T is of finite rank.

Theorem 1.1. [1] *If $A \in L(H)$, then the following two statements are equivalent*

(i) *A is D-symmetric*

(ii) (a) *$[A]$, the corresponding element of the Calkin algebra, is D-symmetric and*

(b) *$T \in C_1(H)$, $AT = TA$ implies $A^*T = TA^*$.*

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Theorem 1.2. [2] *For* $A \in L(H)$,

(i) *A is P-symmetric if and only if* $\overline{R(\delta_A)}^{w*}$ (*the ultraweak closure of the range*) *is self-adjoint*.

(ii) $F_0(H)$ (*the set of P-symmetric operators*) *is self-adjoint.*

Lemma 1. [2] *Let* $A \in L(H)$ *have the property that there is some* $\lambda \in C$ *so that*

(i) *There exists* $x \in H$, $x \neq 0$ *with* $Ax = \lambda x$ *and* $A^*x \neq \bar{\lambda}x$

(ii) *There exists* $y \in H$, $y \neq 0$ *with* $A^*y = \bar{\lambda}y$.

Then $\overline{R(\delta_A)}^{w*}$ *fails to be self-adjoint*.

2. Genaralised *P*-symmetric and *D*-symmetric operators

In this section we use techniques different from those of S. Bouali and J. Charles [2] to generalise Theorem 1.2 and Lemma 1.1.

Definition 2.1. *For* $A, B \in L(H)$, *the pair* (A, B) *is called D-symmetric if* $\overline{R(\delta_{A,B})} = \overline{R(\delta_{B^*,A^*})}$ (*norm closures of the ranges*). *The set of such pairs is denoted* $GS(H)$.

Definition 2.2. *For* $A, B \in L(H)$, *the pair* (A, B) *is called P-symmetric if* $T \in C_1(H)$, $BT = TA$ *implies* $A^*T = TB^*$. *The set of such pairs is denoted* $GF_0(H)$.

Theorem 2.1. *Let* $A, B \in L(H)$ *have the property that there is some* $\lambda \in C$ *so that*

(i) *There exists* $x \in H$, $x \neq 0$ *with* $Ax = \lambda x$, $A^*x \neq \bar{\lambda}x$ *and* $Bx = \lambda x$

(ii) *There exists* $y \in H$, $y \neq 0$ *with* $A^*y - \bar{\lambda}y$.

Then $\overline{R(\delta_{A,B})}^{w*}$ *fails to be self-adjoint*.

Proof. If $\overline{R(\delta_{A,B})}^{w*}$ is self-adjoint then for each $T \in C_1(H)$ such that f_T vanishes on $R(\delta_{A,B}) = \{AX - XB\}$ we must have f_T vanishing on

$$\{(AX - XB)^*\colon X \in L(H)\} = R(\delta_{B^*,A^*}).$$

We exhibit a $T \in C_1(H)$ contradicting this.

Let $T = x \otimes y$ (meaning $T_Z = (z \mid y)x$). Then

$$tr\{(AX - XB)T\} = tr[\{(A - \lambda)X - X(B - \lambda)\}T] = tr\{XT(A - \lambda)\} - tr\{X(B - \lambda)T\}$$
$$= tr(x \otimes \{(A^* - \bar{\lambda})y\}) - tr\{X[(B - \lambda)x] \otimes y\} = 0 \quad for\ all \quad X.$$

If we take $Y = y \otimes u$ with $u = (A - \lambda)^*x$, then

$$Yx = (x \mid u)y = ((A - \lambda)x \mid x)y = 0,$$

$$tr\{[Y(A - \lambda)^*x] \otimes y\} = ||u||^2||y||^2 \neq 0$$

and

$$tr\{(B^*Y - YA^*)T\} = tr\{(B-\lambda)^*YT - Y(A-\lambda)^*T\}$$
$$= tr\{(B-\lambda)^*[Yx]\otimes y\} - tr\{[Y(A-\lambda)^*x]\otimes y\} \neq 0. \quad \blacksquare$$

Theorem 2.2. *For* $A, B \in L(H)$,

(i) $(A,B) \in GF_0(H)$ *if and only if* $\overline{R(\delta_{A,B})}^{w^*}$ *is self-adjoint*

(ii) $GF_0(H)$ *is self-adjoint* (*that is* $(B^*, A^*) \in GF_0(H)$ *if* $(A,B) \in GF_0(H)$).

PROOF. (i) The *weak**-topology is generated by all f_T with $T \in C_1(H)$ and so $\overline{R(\delta_{A,B})}^{w^*}$ is the intersection

$$\cap\{\ker f_T : f_T(AX - XB) = 0 \,\forall X \in L(H)\}.$$

Since

$$f_T(AX - XB) = tr(T(AX - XB)) = tr((TA - BT)X)$$

this intersection is $\ker\delta_{B,A} \cap C_1(H)$.

If $(A,B) \in GF_0(H)$, then

$$\ker\delta_{B,A} \cap C_1(H) = \ker\delta_{A^*,B^*} \cap C_1(H)$$

and so the weak*-closure of $(R(\delta_{A,B}))^* = R(\delta_{B^*,A^*})$ is the same as the weak*-closure of $R(\delta_{A,B})$.

Conversely, if $\overline{R(\delta_{A,B})}^{w^*}$ is self-adjoint then the set of $T \in C_1(H)$ for which f_T vanishes on $R(\delta_{A,B})$ must be self-adjoint ($Y \in R(\delta_{A,B})$ implies $0 = f_T(Y^*) = tr(TY^*) = \overline{tr(T^*Y)}$. Hence

$$\ker\delta_{B,A} \cap C_1(H) = \ker\delta_{A^*,B^*} \cap C_1(H)$$

and $(A,B) \in GF_0(H)$.

(ii) Let $(A,B) \in GF_0(H)$. If $S \in C_1(H)$ with $B^*S = SA^*$, then $S^*B = AS^*$ and so $A^*S^* = B^*S^*$ or $SA = BS$. Hence $(B^*, A^*) \in GF_0(H)$. $\blacksquare$

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