

GROUPS WHOSE NON-NORMAL SUBGROUPS HAVE FINITE CONJUGACY CLASSES

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ABSTRACT

We consider the class of all groups whose non-normal subgroups are *FC*-groups. In particular, it will be proved that if such a group is soluble-by-finite, then it is a locally *FC*-group and its commutator subgroup is an *FC*-group. We consider also the case when such groups have Chernikov commutator subgroups.

1. Introduction

One of the most important trends in group theory is the study of the influence of given systems of subgroups of a group on the structure of the group itself. Examples of such systems are the family of all subgroups, that of all normal subgroups, that of all abelian subgroups, that of all nilpotent subgroups, that of all finite subgroups and many others. We review some of the most important contributions made in the past in this direction of research.

The influence of the family $\mathfrak{F}_{norm}(G)$ of all its normal subgroups on the structure of a group G was detected by R. Dedekind; in his classical paper [6] he described finite groups all of whose subgroups are normal, that is $\mathfrak{F}_{norm}(G)$ coincides with the family of all subgroups. Dedekind's description was extended to arbitrary groups by

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R. Baer [1], though in honour of Dedekind, groups with only normal subgroups are known as *Dedekind groups*.

Later, O.Y. Schmidt raised the general question of studying groups G in which the family $\mathfrak{F}_{\text{nonnorm}}(G)$ of all non-normal subgroups of G satisfies certain strong restrictions. In particular, he studied finite groups G in which $\mathfrak{F}_{\text{nonnorm}}(G)$ is a conjugacy class or the union of two conjugacy classes, respectively (see [15; 16]). On the other hand, S.N. Chernikov in the paper [4] considered groups whose non-normal subgroups are finite. The study of the structure of groups for which the family $\mathfrak{F}_{\text{nonnorm}}(G)$ satisfies other natural finiteness conditions has been continued by several authors.

G.M. Romalis and N.F. Sesekin began the study of those groups G in which every member of the family $\mathfrak{F}_{\text{nonnorm}}(G)$ satisfies a given condition. In particular, in the papers [12; 13; 14], they considered groups G in which the members of $\mathfrak{F}_{\text{nonnorm}}(G)$ are abelian, the so called *metahamiltonian groups*. The final description of metahamiltonian groups was obtained in a series of papers by N.F. Kuzenny and N.N. Semko.

The next natural step in this direction seems to be the consideration of groups in which the members of $\mathfrak{F}_{\text{nonnorm}}(G)$ lie in some class of groups \mathfrak{X} , which is a good extension both of the class of all finite groups and of the class of all abelian groups. The best candidate is the class of the *FC*-groups or groups with finite conjugacy classes. Nowadays, the theory of *FC*-groups forms a well-developed topic and represents one of the main branches of group theory with finiteness conditions. Many famous mathematicians have made relevant contributions to this area (see [7; 17; 18]).

The aim of this article is to investigate the class of all groups whose non-*FC*-subgroups are normal. Clearly, such a class contains both the class of metahamiltonian groups and the class of *minimal non-FC*-groups, i.e. groups that are not *FC*-groups for which all their proper subgroups have the property *FC*. These groups have been studied by several authors. We will prove, among other results, that a soluble-by-finite group whose non-*FC*-subgroups are normal is a locally *FC*-group and its commutator subgroup is an *FC*-group. We also consider the case when such a group has Chernikov commutator subgroup.

Most of our notation is standard and can be found in [10] and [11].

2. Some preliminary results

Clearly, the class of groups whose non-normal subgroups are *FC*-groups is closed with respect to subgroups and homomorphic images. Moreover, if G is such a group and S is a non-*FC*-subgroup of G , then S is normal and G/S is a Dedekind group.

Lemma 2.1. *Let G be a group whose non-normal subgroups are *FC*-groups, and let A be a torsion-free finitely generated abelian normal subgroup of G . Then A centralises every element of finite order of G .*

PROOF. Suppose by contradiction that there exists an element of finite order g of G that is not contained in $C_G(A)$, and let p be a prime number such that $\langle g, A^{p^n} \rangle$ is not

abelian, for every positive integer n . Since the commutator subgroup of an FC -group is periodic (see, for instance, [10, theorem 4.32]), then $\langle g, A^{p^n} \rangle$ is not an FC -subgroup, for each $n \in \mathbb{N}$, and hence $\langle g \rangle = \bigcap_{n \in \mathbb{N}} \langle g, A^{p^n} \rangle$ is normal. It follows that $g \in C_G(A)$. This contradiction proves the lemma. ■

Lemma 2.2. *Let G be a group whose non-normal subgroups are FC -groups. Then the commutator subgroup $[G, G]$ of G either is an FC -group or it contains a G -invariant subgroup H such that $|[G, G]:H| \leq 2$ and H is a minimal non- FC -group.*

PROOF. Suppose that $[G, G]$ contains a subgroup H that is not an FC -group. Clearly H is normal and G/H is a Dedekind group. It follows that $|[G/H, G/H]| \leq 2$, that is $|[G, G]:H| \leq 2$. In particular, every proper subgroup of H is an FC -group. ■

Recall that a minimal non- FC -group with a proper commutator subgroup is a Chernikov group (see [2, theorem 1] and [3, proposition 3 and proposition 4]), and that every perfect locally (soluble-by-finite) minimal non- FC -group is a countable locally finite p -group for some prime number p (see [9]).

Proposition 2.3. *Let G be a locally (soluble-by-finite) group whose non-normal subgroups are FC -groups. Then the set of all elements of finite order is a (characteristic) subgroup of G .*

PROOF. Clearly, we may suppose that G contains a subgroup which is not an FC -group. Then, by Lemma 2.2 the commutator subgroup $[G, G]$ either is an FC -group or it contains a G -invariant subgroup H such that $|[G, G]:H| \leq 2$ and H is a minimal non- FC -group. In the latter case, by the quoted remark, $[G, G]$ is periodic, and the statement is proved. Assume now that $[G, G]$ is an FC -group, and let x and y be elements of G of finite order. Moreover, put $Y = \langle x, y \rangle$ and $X = Y \cap [G, G]$. Then Y/X is finite. It follows that X is a finitely generated FC -group. This means that X is central-by-finite, so that it contains an Y -invariant torsion-free subgroup Z of finite index, and by Lemma 2.1 $x, y \in C_Y(Z)$. Therefore Y is central-by-finite, and the statement follows also in this case. ■

Lemma 2.4. *Let G be a locally (soluble-by-finite) group whose non-normal subgroups are FC -groups. If G is torsion-free, then G is abelian.*

PROOF. Since a torsion-free FC -group is abelian, then G is metahamiltonian. Thus the statement follows from theorem 2.1 of [5]. ■

Corollary 2.5. *Let G be a locally (soluble-by-finite) group whose non-normal subgroups are FC -groups. Then the commutator subgroup of G is periodic.*

PROOF. The statement follows easily from Proposition 2.3 and Lemma 2.4. ■

We note that the above result can be extended with similar arguments to the class of groups in which every non-trivial finitely generated subgroup contains a proper subgroup of finite index (such groups are said to be *locally graded*).

Corollary 2.6. *Let G be a locally (soluble-by-finite) group whose non-normal subgroups are FC-groups. Then G is a locally (central-by-finite) group.*

PROOF. Clearly we may assume that G is finitely generated. If G is not nilpotent, then G contains a subgroup H of finite index that is not subnormal (see [11, theorem 10.51]). Therefore H is an FC-group, so that the factor group $G/Z(\text{Core}_G(H))$ is finite. It follows that G is polycyclic, and Corollary 2.5 yields that G' is finite. In particular, $Z(G)$ has finite index in G . ■

Lemma 2.7. *Let G be a group whose non-normal subgroups are FC-groups. If G is not an FC-group, then the commutator subgroup of G is countable.*

PROOF. Let $K = [G, G]$. Since G is a non-FC-group, there is an element $g \in G$ such that the index $|G:C_G(g)|$ is infinite. Then G contains a countable subgroup H such that $g \in H$ and the index $|H:C_H(g)|$ is infinite. It follows that H is not an FC-group, and G/H is a Dedekind group. It turns out that $K/(K \cap H) (\simeq KH/H)$ has order at most 2. Thus K is countable. ■

The last result of this section is a technical lemma that will be useful in the following.

Lemma 2.8. *Let G be a group whose non-normal subgroups are FC-groups, and let g be an element of G and A and B subgroups of G satisfying the following conditions:*

- (1) A and B are $\langle g \rangle$ -invariant;
- (2) $\langle A, B \rangle \cap \langle g \rangle = \langle 1 \rangle$;
- (3) $A \cap B = \langle 1 \rangle$;
- (4) $C_A(g) \neq A$.

Then $\langle B, g \rangle$ is an FC-group, and in particular the index $|B:C_B(g)|$ is finite.

PROOF. Clearly, the subgroup A contains an element a such that $[a, g] = d \neq 1$. Then $a^{-1}ga = gd \neq g$. Suppose first that gd belongs to $B\langle g \rangle$, and let $gd = g^m b$ for some $m \in \mathbb{Z}$ and $b \in B$. Then $g^{m-1} = db^{-1}$, and, by hypothesis (2), $db^{-1} = 1$. It follows, by (3), that $d = b = 1$. This contradiction shows that $gd \notin B\langle g \rangle$, and in particular, $B\langle g \rangle$ is not normal. Hence $B\langle g \rangle$ is an FC-subgroup. ■

3. Groups whose non-normal subgroups are FC-groups with no central Chernikov subgroups

In the theory of metahamiltonian groups finite-by-abelian groups play a central role. For groups whose non-FC subgroups are normal, this role is played by groups with

Chernikov commutator subgroup. Lemma 2.2 shows that we have two basic situations:

- (1) the commutator subgroup of G is an FC -group;
- (2) the commutator subgroup of G contains a subgroup of finite index that is a minimal non- FC -group.

We have already noted that a soluble minimal non- FC -group is Chernikov; thus in the second case the commutator subgroup is likewise Chernikov. Groups with Chernikov commutator subgroups are very common.

Clearly, if G is a group whose commutator subgroup is a Prüfer group, then every non-normal subgroup of G is finite-by-abelian. Groups with Chernikov commutator subgroups whose non-normal subgroup are FC -groups appear in the following way.

Let A be an infinite abelian group, $H \leq \text{Aut}(A)$. We will say that H is an *infinitely irreducible* group (or H acts *infinitely irreducibly* on A), if A contains no proper infinite H -invariant subgroup.

Let D be a divisible Chernikov group and H an abelian subgroup of $\text{Aut}(D)$ such that every non-identity element of H acts on D infinitely irreducibly. Put $G = H \rtimes D$, and let L be a subgroup of G . If $D \leq L$ then L is normal in G . Suppose that L does not contain D , and let g be any element of $L \setminus D$. Then $D \cap L$ is $\langle g \rangle$ -invariant, so that $D \cap L$ must be finite. This means that L is finite-by-abelian.

We will show soon that the above situation is more or less typical.

Let D be a non-trivial divisible Chernikov group and $g \in \text{Aut}(D)$. Since D satisfies the minimal condition on subgroups, it has a $\langle g \rangle$ -invariant series

$$\{1\} = D_0 \leq D_1 \leq \dots \leq D_t = D$$

such that g acts infinitely irreducibly on every factor of this series.

We say that g acts *unitriangularly* on D if $[D_{j+1}, g] \leq D_j$ for each $j \in \{0, 1, \dots, t-1\}$. Note that in this case every factor D_{j+1}/D_j is a Prüfer group.

Lemma 3.1. *Let D be a divisible Chernikov group and $g \in \text{Aut}(D)$. Suppose that D contains a $\langle g \rangle$ -invariant subgroup C satisfying the following conditions:*

- (1) $[C, g] = \{1\}$;
- (2) g induces on D/C a non-identity infinitely irreducible automorphism.

Then D contains a divisible $\langle g \rangle$ -invariant subgroup E such that g acts non-trivially, infinitely and irreducibly on E .

PROOF. Clearly the map $\phi: d \in D \rightarrow [g, d] \in D$ is an endomorphism of D . In particular, $\text{Im}\phi$ and $\text{Ker}\phi$ are $\langle g \rangle$ -invariant subgroups of D . By (1) $C \leq \text{Ker}\phi$, and (2) yields that $\text{Ker}\phi \neq D$. It follows that g acts on the divisible subgroup $\text{Im}\phi$ non-trivially, infinitely and irreducibly. ■

Corollary 3.2. *Let D be a non-trivial divisible Chernikov group and $y \in \text{Aut}(D) \setminus \{1\}$. Then either y acts unitriangularly on D or D contains a divisible $\langle y \rangle$ -invariant subgroup E such that y acts on E non-trivially, infinitely and irreducibly.*

Lemma 3.3. *Let G be a group whose non-normal subgroups are FC -groups, and let D be a divisible Chernikov subgroup of G . If $C_G(D)$ is properly contained in $N_G(D)$ and g is*

an element of $N_G(D) \setminus C_G(D)$ that does not act unitriangularly on D , then the commutator subgroup $K = [G, G]$ of G is a Chernikov group.

PROOF. By Corollary 3.2 D contains a divisible $\langle g \rangle$ -invariant subgroup A such that $g \notin C_G(A)$ and g acts on A infinitely irreducibly. Since the centre of an FC -group contains every \mathfrak{F} -perfect subgroup (see, for example [17, theorem 1.9]), the subgroup $\langle g, A \rangle$ is not an FC -group. It follows that $\langle g, A \rangle$ is a normal subgroup of G and the factor group $G/\langle g, A \rangle$ is a Dedekind group, so that $K\langle g, A \rangle/\langle g, A \rangle$ is finite.

Suppose first that the element g has finite order. Clearly, the index $|\langle g, A \rangle : A|$ is finite, so that the factor group KA/A is finite. If $K \cap A = A$, then $A \leq K$, and hence K is a Chernikov group. Thus we may suppose that the $\langle g \rangle$ -invariant subgroup $K \cap A$ is properly contained in A , and so it is finite. It follows that also K is finite, a contradiction.

Assume now that g has infinite order. We have that

$$|KA : KA \cap \langle g, A \rangle| = |K\langle g, A \rangle : \langle g, A \rangle|$$

is finite and $KA \cap \langle g, A \rangle = A(KA \cap \langle g \rangle)$. On the other hand, KD is periodic by Corollary 2.5. Therefore the factor group KA/A is finite, and so K is a Chernikov group. This completes the proof. ■

Lemma 3.4. *Let D be a divisible Chernikov group and g a non-identity automorphism of D . If g acts unitriangularly on D , then D contains a $\langle g \rangle$ -invariant subgroup C satisfying the following conditions:*

- (1) $[C, g] = E$ and C/E are Prüfer groups;
- (2) $[E, g] = \{1\}$.

PROOF. By hypothesis, the subgroup D has a series of $\langle g \rangle$ -invariant subgroups

$$\{1\} = D_0 \leq D_1 \leq \dots \leq D_t = D$$

such that g acts infinitely irreducibly on every factor of this series. Since g acts unitriangularly on D then, for each $j \in \{0, 1, \dots, t-1\}$ we have that $[D_{j+1}, g] \leq D_j$. This means that every factor of this series is a Prüfer group. Consider a positive integer k such that $[D_k, g] = \{1\}$ but $[D_{k+1}, g] \neq \{1\}$. Clearly, the map $\phi: d \rightarrow [d, g], d \in D_{k+1}$, is an endomorphism of D_{k+1} . In particular, $Im\phi$ and $Ker\phi$ are $\langle g \rangle$ -invariant subgroups of D . Put $E = Im\phi$. Then $E \leq D_k$, so that $[E, g] = \{1\}$. On the other hand, $E \simeq D_{k+1}/Ker\phi$ and $D_k \leq Ker\phi$. Therefore E is a Prüfer group, and so

$$E = \langle e_n \mid e_1^p = 1, e_{n+1}^p = e_n, n \in \mathbb{N} \rangle.$$

From $[D_{k+1}, g] = E$ it follows that there is an element b_1 such that $[b_1, g] = e_1$. Since D is divisible, there is an element b_2 such that $b_2^p = b_1$. Moreover $[b_2, g]^p = [b_2^p, g] = [b_1, g] = e_1$. It follows that $\langle [b_2, g] \rangle = \langle e_2 \rangle$. Using similar arguments we can find a family $\{b_n \mid n \in \mathbb{N}\}$ of elements satisfying the following conditions:

$$b_1^p = 1, b_{n+1}^p = b_n, \langle [b_n, g] \rangle = \langle e_n \rangle, n \in \mathbb{N}.$$

Put $C = \langle E, b_n \mid n \in \mathbb{N} \rangle$. Then C/E is a Prüfer group and $[C, g] = E$. ■

Lemma 3.5. *Let G be a group whose non-normal subgroups are FC-groups. Suppose that G contains a divisible Chernikov subgroup D . If g is an element of $N_G(D) \setminus C_G(D)$ that acts unitriangularly on D , then the commutator subgroup $K = [G, G]$ of G is abelian, and either it contains a Prüfer subgroup of finite index or G is nilpotent and K is a direct product of two Prüfer groups.*

PROOF. Note that the element g has infinite order (see, for example [10, lemma 3.13]). Put $x = g^8$. By Lemma 3.4 D contains an $\langle x \rangle$ -invariant subgroup C such that $x \notin C_G(C)$, $[C, x] = E$, and C/E are Prüfer groups and $[E, x] = \{1\}$. Since $\langle x, C \rangle$ is not an FC-group, it is a normal subgroup of G such that $G/\langle x, C \rangle$ is a Dedekind group. This factor group contains the element $g\langle x, C \rangle$ of order 8. It follows that $G/\langle x, C \rangle$ is abelian. On the other hand, since by Corollary 2.5 K is periodic, we have that G/C is abelian, so that K is also abelian and $E = [C, x]$ is a normal subgroup of G . If there is an element y such that $y \notin C_G(E)$, then, arguing as above, we may assume that $K \leq \langle E, y \rangle$. If $E \leq Z(G)$ but there is an element z such that $z \notin C_G(C/E)$, then Lemma 3.1 yields that C contains a $\langle g \rangle$ -invariant Prüfer subgroup U such that $z \notin C_G(U)$, and again $K \leq \langle U, z \rangle$. In other words, either K includes a Prüfer subgroup of finite index or $E \leq Z(G)$ and $C/E \leq Z(G/E)$. Since G/C is abelian, in the last case G is nilpotent. ■

Corollary 3.6. *Let G be a non-nilpotent group whose non-normal subgroups are FC-groups, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and g is an element of G such that $g \notin C_G(D)$, then g acts infinitely irreducibly on D . In particular, D is a p -subgroup for some prime p .*

PROOF. By Lemma 3.5 we may assume that g does not act unitriangularly on D . Then the statement follows easily, arguing as in the proof of Lemma 3.3. ■

Corollary 3.7. *Let G be a non-nilpotent group whose non-normal subgroups are FC-groups, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and g is an element of G such that $g \notin C_G(D)$, then $C_D(g)$ is finite and $D = [g, D]$.*

PROOF. Clearly the map $\phi: d \in D \rightarrow [g, d] \in D$, is a $\mathbb{Z}\langle g \rangle$ -endomorphism of D . Therefore, $Im\phi = [g, D]$ and $Ker\phi = C_D(g)$ are $\langle g \rangle$ -invariant subgroups of D . If we assume that $C_D(g)$ is infinite, then Corollary 3.6 yields that $C_D(g) = D$, a contradiction. It follows that $C_D(g)$ is finite. On the other hand, since $[g, D]$ is isomorphic to $D/C_G(g)$, then $[g, D]$ is infinite, and again by Corollary 3.6, $D = [g, D]$. ■

Corollary 3.8. *Let G be a non-nilpotent group whose non-normal subgroups are FC-groups, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K and $P/C_G(D)$ is a periodic subgroup of $G/C_G(D)$, then $P/C_G(D)$ is a finite cyclic group. Moreover, $\pi(D) = \{p\}$ and either $|P/C_G(D)| = p$, or $P/C_G(D)$ is a cyclic p' -subgroup.*

PROOF. Clearly, we may assume that G is not an FC -group. Since all periodic subgroups of the automorphism group of a divisible Chernikov group are finite (see, for example [10, theorem 3.29.2]), the factor group $P/C_G(D)$ is finite. On the other hand, $P/C_G(D)$ is an infinitely irreducible automorphism group of D by Corollary 3.6. Thus the statement follows easily from theorem 6.1 of [8]. ■

Corollary 3.9. *Let G be a non-nilpotent group whose non-normal subgroups are FC -groups, and suppose that the commutator subgroup K of G is a Chernikov group. If D is the divisible part of K , then $G/C_G(D)$ is abelian. In particular, K is central-by-finite.*

PROOF. By Corollary 3.8 we may suppose that the group $G/C_G(D)$ is not periodic. Clearly, every subgroup of G properly containing $C_G(D)$ is not an FC -group. It follows that $G/C_G(D)$ is abelian. ■

Corollary 3.10. *Let G be a soluble-by-finite group whose non-normal subgroups are FC -groups. Then the commutator subgroup of G is an FC -group.*

PROOF. If G is nilpotent, then by Lemma 3.5 K is abelian. So we may suppose that G is not nilpotent. Assume by contradiction that $K = [G, G]$ is not an FC -group. It follows from Lemma 2.2 that K contains a G -invariant subgroup H such that $|K:H| \leq 2$ and H is a minimal non- FC -group. Therefore H , and hence also K , is a Chernikov group, contradicting Corollary 3.9. ■

Theorem 3.11. *Let G be a non-nilpotent group with Chernikov commutator subgroup K , and let D be the divisible part of K . If every non- FC -subgroup of G is normal and $C_G(D) \neq G$, then $G = DL$, where L is a finite-by-abelian subgroup of G and $D \cap L$ is a finite G -invariant subgroup. Moreover, every non-normal subgroup of G has finite commutator subgroup.*

PROOF. Let g be an element of G such that $g \notin C_G(D)$. Then the subgroup $\langle g, D \rangle$ is not an FC -subgroup, so that by hypothesis, $\langle gD \rangle$ is a normal subgroup of the factor group G/D . If x is an arbitrary element of G , then $(gD)^{xD} = g^k D$ for some positive integer k . It follows that $g^x = g^k d$ for some $d \in D$. If $d = 1$, then $x \in N_G(\langle g \rangle)$. Let $d \neq 1$. Since $g^k \notin C_G(D)$, Corollary 3.7 yields that $D = [g^k, D]$, and in particular, $d = [g^k, d_1]$, for some element $d_1 \in D$. It follows

$$g^x = g^k [g^k, d_1] = d_1^{-1} g^k d_1,$$

so that $xd_1^{-1} \in N_G(\langle g \rangle) = L$. Thus, in any case $G = DL$.

Since D is abelian, $D \cap L$ is a normal subgroup of G . In particular, $D \cap L$ is $\langle g \rangle$ -invariant. If we suppose that $D \cap L$ is infinite, Corollary 3.6 yields that $D \cap L = D$, that is $D \leq N_G(\langle g \rangle)$. If g has finite order, then $\langle g, D \rangle$ has finite commutator subgroup, so that $g \in C_G(D)$. If g has infinite order, then $D \cap \langle g \rangle = \{1\}$ and again $g \in C_G(D)$. This contradiction shows that $D \cap L$ is finite.

Clearly, the factor group G/D is finite-by-abelian. Thus $L/(L \cap D)$ has finite commutator subgroup, so that, since $L \cap D$ is finite, also $[L, L]$ is finite.

Finally, let H be an arbitrary subgroup of G . Assume first that $H \leq C_G(D)$. The subgroup $C_G(D)$ is the product of its central subgroup D and the subgroup $L \cap C_G(D)$, which has finite commutator subgroup. It follows that also H has finite commutator subgroup.

Suppose now that $H \not\leq C_G(D)$. If the intersection $H \cap D$ is infinite, then $H \cap D = D$, because every element $y \in H \setminus C_G(D)$ acts infinitely irreducibly on D by Corollary 3.6. Moreover, $\langle y, D \rangle$ is not an FC -group, so that it is normal in G and $G/\langle y, D \rangle$ is a Dedekind group. It follows that H is normal in G . So we may suppose that $H \cap D$ is finite. Since $H/(H \cap D)$ is isomorphic to a subgroup of G/D , then $H/(H \cap D)$ has finite commutator subgroup. The theorem is proved. ■

4. Groups whose non- FC -subgroups are normal with an FC -subgroup of finite index

A group G is said to be an *almost FC -group* if G contains a subgroup that is an FC -group and has finite index in G . Plenty of examples of almost FC -groups can be found in many different works devoted to groups with restrictions on various natural systems of subgroups. A specific case of almost FC -groups are almost abelian groups. In this section we will deal with almost FC groups whose non-normal subgroups are FC -groups. It will turn out that such groups either are FC -groups or have Chernikov commutator subgroup.

Proposition 4.1. *Let G be an abelian-by-finite group. If G is not an FC -group and every non-normal subgroup of G is an FC -group, then the commutator subgroup of G is a Chernikov group.*

PROOF. Suppose first that G is periodic. Denote by A a normal abelian subgroup having finite index in G . By hypothesis G is not an FC -group, so it contains an element g that does not centralise A such that $|G : C_G(g)|$ is infinite. In particular, $|A : C_A(g)|$ is infinite, and the subgroup $H = \langle g, A \rangle$ is not an FC -group.

Clearly the intersection $B = \langle g \rangle \cap A$ is finite and it is also contained in $Z(H)$. If we suppose that the index $|A/B : C_{A/B}(gB)|$ is finite, then H/B is central-by-finite. In particular, by Schur’s theorem (see, for example, [10, theorem 4.12]) H/B and hence H have finite commutator subgroup. This contradiction shows that the index $|A/B : C_{A/B}(gB)|$ is infinite. In other words, we can suppose that $B = \{1\}$. Let a be an element of A such that $[g, a] = a_1 \neq 1$. Put $C = \langle a \rangle^{(g)}$. Then C is a finite $\langle g \rangle$ -invariant subgroup of A . Clearly, a_1 belongs to C and there exists a subgroup E of A such that $E \cap C = \{1\}$ and E is maximal with respect to this condition. Then A/E is a Chernikov group. Since $\langle g \rangle$ is finite, the set

$$\{E^x \mid x \in \langle g \rangle\} = \{E_1, \dots, E_m\}$$

is finite, and the factor group $A/Core_{\langle g \rangle}(E)$ is isomorphic with a subgroup of $A/E_1 \times \dots \times A/E_m$, and so it is also a Chernikov group. Put $Q = Core_{\langle g \rangle}(E)$. Then Q is a $\langle g \rangle$ -invariant subgroup and $C \cap Q = \{1\}$. By Lemma 2.8 $Q\langle g \rangle$ is an FC -subgroup, and in particular $C_Q(g)$ has finite index in Q . Since A/Q is a Chernikov group, $A/C_A(g)$ is also a Chernikov group.

The mapping $\phi: a \rightarrow [g, a]$, $a \in A$, is a $\mathbb{Z}\langle g \rangle$ -endomorphism of A . Therefore $\text{Im}\phi = [g, A]$ and $\text{Ker}\phi = C_A(g)$ are $\langle g \rangle$ -invariant subgroups and $[g, A] \cong A/C_A(g)$. It follows that $R = [g, A]$ is a Chernikov subgroup. Let S be a finite G -invariant subgroup of R such that R/S is divisible. It is enough to show that the commutator subgroup of G/S is a Chernikov group, and hence we may suppose that $S = \{1\}$.

Assume that $g \in C_G(R)$. Thus H is a nilpotent subgroup of nilpotency class at most equal to 2. Put $|g| = k$, so that $[g, h]^k = [g^k, h] = 1$ for each element $h \in H$. It follows that $[g, H] = [g, A]$ is a bounded Chernikov subgroup, and hence it is finite. In particular H is an FC -group. This contradiction shows that $g \notin C_G(R)$. Since g has finite order, then g does not act on R unitriangularly (see, for example [10, lemma 3.13]), and Lemma 3.3 yields that $[G, G]$ is a Chernikov subgroup.

In the general case, we can consider a maximal torsion-free subgroup U of A . Since G/A is finite, the set $\{U_1, \dots, U_m\}$ of all conjugates of U in G is finite. Put $V = \text{Core}_G(U)$. Clearly A/U is periodic, and the natural embedding of $A/\text{Core}_G(U)$ in $A/U_1 \times \dots \times A/U_m$ yields that also A/V is periodic. The first part of the proof shows that $G'V/V \cong G'/(G' \cap V)$ is a Chernikov group. On the other hand, G' is periodic by Corollary 2.5, so that $G' \cap V = \{1\}$ and G' is a Chernikov group. ■

Lemma 4.2. *Let G be a periodic group, and let H be a normal residually finite subgroup of G such that G/H is cyclic. If H and every non-normal subgroup of G are FC -groups, then G is likewise an FC -group.*

PROOF. Put $G/H = \langle gH \rangle$. Since H is residually finite and the element g has finite order, then H contains a G -invariant subgroup R such that the factor group G/R is finite and $R \cap \langle g \rangle = \{1\}$.

Assume first that $[g, R] = \{1\}$. Therefore $C_H(g)$ has finite index in H . On the other hand, G/H is finite, so that g is an FC -element, and G is an FC -group.

Thus, we may suppose that R contains an element x that does not centralise g . Put $A = \langle g, x \rangle \rtimes B$, where $B = \langle x \rangle^G$. Clearly A is finite. It follows that H contains a G -invariant subgroup V of finite index such that $V \cap A = \{1\}$, and Lemma 2.8 yields that $V\langle g \rangle$ is an FC -group. In particular, $|V:C_V(g)|$ is finite, and G is an FC -group. ■

Lemma 4.3. *Let G be a periodic group, and let H be a normal FC -subgroup of G such that G/H is cyclic of prime order. If every non-normal subgroup of G is an FC -group, then G either is an FC -group or its commutator subgroup is a Chernikov group.*

PROOF. Let g be an element of G such that $G/H = \langle gH \rangle$, and assume that G is not an FC -group. If N is a normal finite subgroup of G , then G/N is not an FC -group. Therefore, in order to show that G' is a Chernikov group, we may replace G by G/N , and assume $N = \{1\}$. In particular, we may suppose that the normal closure of $\langle g \rangle \cap H$ in G is trivial by Dietzmann's Lemma.

Denote by Z the centre of H . If the factor group H/Z is finite, then G is abelian-by-finite, and Proposition 4.1 yields that $[G, G]$ is a Chernikov group. So we may suppose that H/Z is infinite. Moreover, G/Z is an FC -group by Lemma 4.2. Let x be

an element of G that is not contained in the FC -centre of G . Then every subgroup of finite index in G that contains x is not an FC -group, and hence it is normal. It follows that the cyclic subgroup $\langle xZ \rangle$ is a normal subgroup of the nilpotent and abelian-by-finite group G/Z .

Suppose first that $g \in C_G(Z)$. Then Z is contained in the centre of G . Put $C/Z = C_{G/Z}(\langle gZ \rangle)$. Clearly, C has finite index in G and $g \notin FC(C)$. Moreover, the map $\phi: x \in C \rightarrow [g, x] \in C$ is an endomorphism of C , so that $[g, C]$ is isomorphic with $C/C_C(g)$. Let $p = |G/H|$. Since $g^p = 1$, the subgroup $[g, C]$ is an infinite elementary abelian p -group.

Let now a_1 an element of $H \cap C$ such that $[g, a_1] = b_1 \neq 1$. Then $|C:C_C(a_1)|$ is finite. Since $g \notin FC(C)$, there is an element $a_2 \in C_C(a_1)$ such that $[g, a_2] = b_2 \neq b_1$. By induction we can define a sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of C such that $a_{n+1} \in C_C(a_1, \dots, a_n)$ and $b_{n+1} \notin \langle b_1, \dots, b_n \rangle$ where $b_n = [g, a_n], n \in \mathbb{N}$. Put $E = \langle b, a_n | n \in \mathbb{N} \rangle$ and $Z_1 = \langle a_n, b_n | n \in \mathbb{N} \rangle$. Then $E = \langle g \rangle Z_1$ is abelian-by-finite, and by Lemma 4.1 either E is an FC -group or $[E, E]$ is a Chernikov group. On the other hand, $[E, E] = \text{Dr}_{n \in \mathbb{N}} \langle b_n \rangle$, and this is a contradiction.

So we may suppose that $g \notin C_G(Z)$. We have already noted that G/Z is abelian-by-finite. Since it is also an FC -group, then G/Z has finite commutator subgroup. Therefore if we assume that Z is a Chernikov group, then $[G, G]$ is a Chernikov group.

Suppose now that Z is not a Chernikov group. Let z be an element of Z such that $[g, z] \neq \{1\}$, and put $Y = \langle z \rangle^{(g)}$. Moreover, let U be a subgroup of Z which is maximal with respect to the condition $U \cap Y = \{1\}$. Since Y is finite, then Z/U is a Chernikov group.

On the other hand, if $\{U_1, \dots, U_m\}$ is the conjugacy class of U in G , the natural embedding of $Z/\text{Core}_G(U)$ in $Z/U_1 \times \dots \times Z/U_m$ shows that there exists a $\langle g \rangle$ -invariant subgroup V of Z such that Z/V is a Chernikov group. Therefore, by Lemma 2.8 the index $|V:C_V(g)|$ is finite, and so $Z/C_Z(g)$ is likewise a Chernikov group. The endomorphism of Z defined by the rule $z \rightarrow [g, z]$ shows that also $W = [g, Z]$ is a Chernikov group. Clearly $[g, Z]$ is a G -invariant subgroup of G , and, as above, we may suppose that W is a non-trivial divisible group. In particular W has not finite exponent, and hence $g \notin C_G(W)$. Since g has finite order, then g does not act on W unitriangularly (see, for instance [10, lemma 3.13]), and Lemma 3.3 yields that $[G, G]$ is a Chernikov subgroup. ■

Theorem 4.4. *Let G be an almost FC -group. If every non-normal subgroup of G is an FC -group, then G either is an FC -group or its commutator subgroup is a Chernikov group.*

PROOF. Suppose first that G is periodic. Let H be a normal subgroup of G that is an FC -group and has finite index in G , and assume that G is a counterexample such that the order of G/H is minimal. Clearly, by Lemma 4.3 $|G/H|$ is not a prime number.

If X is a proper subgroup of G that contains H and is not an FC -group, then X is normal in G , $[X, X]$ is a Chernikov group and the factor group $G/[X, X]$ is abelian-by-finite. It follows by Proposition 4.1 that either $G/[X, X]$ is central-by-finite or its

commutator subgroup is a Chernikov group. Therefore $[G, G]$ is likewise a Chernikov group.

This contradiction shows that every proper subgroup of G that contains H is an FC -group. Put now $G = g_1H \cup \dots \cup g_tH$, where g_1, \dots, g_t are elements of G and $t \in \mathbb{N}$, and assume that every subgroup $\langle g_i \rangle H$ is properly contained in G . Then, by above G is an FC -group. Hence there exists an element $g \in G$ such that $G/H = \langle gH \rangle$.

If L/H is a proper subgroup of G/H , then L is a proper normal FC -subgroup of G , so that by the minimality of the counterexample, G either is an FC -group or its commutator subgroup is a Chernikov group. This last contradiction proves the theorem when G is periodic.

In the general case, we can consider a maximal torsion-free subgroup U of $Z(G)$. Since G/H is finite, then also the set

$$\{U^x \mid x \in G\} = \{U_1, \dots, U_m\}$$

is finite. Put $V = \text{Core}_G(U)$. Clearly H/U is periodic, and the natural embedding of H/V in $H/U_1 \times \dots \times H/U_m$ shows that also H/V is periodic. It follows from the first part of the proof that either G/V is an FC -group or $G'/(G' \cap V) (\simeq G'V/V)$ is a Chernikov group. In the latter case, since G' is periodic by Corollary 2.5, then $G' \cap V = \{1\}$, and G' is a Chernikov group. On the other hand, if G/V is an FC -group, then the embedding of G in $G/V \times G/G'$ yields that G is likewise an FC -group. The theorem is proved. ■

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