

FINITE 2-GROUPS WITH LARGE CENTRALISERS OF ABELIAN SUBGROUPS

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ABSTRACT

The finite 2-groups G satisfying $|N_G(A) : C_G(A)| \leq 2$ for all abelian subgroups A of G are determined by the defining relation, and G possesses the class $c(G) = 1 + \log_2(|G'|)$.

1. Introduction

For any prime p and any p -group G of finite order, we denote by $c(G)$ the class of G , and denote by G' the derived subgroup of G . Then the following inequality holds:

$$c(G) \leq 1 + \log_p(|G'|).$$

In fact, if $G' = 1$ then G is abelian, a trivial case. Suppose that $G' > 1$. Then $Z(G) \cap G' > 1$. Let Z be a subgroup of $Z(G) \cap G'$ of order p . By induction we have $c(G/Z) \leq 1 + \log_p(|(G/Z)'|)$. Noting that $(G/Z)' = G'/Z$ and $c(G) - 1 \leq c(G/Z)$, we obtain $c(G) - 1 \leq c(G/Z) \leq 1 + \log_p(|G'|/p) = \log_p(|G'|)$, which yields the inequality.

An interesting problem is to investigate the finite p -groups G satisfying $c(G) = 1 + \log_p(|G'|)$. Recall that a maximal class p -group G is a p -group whose order is $p^{1+c(G)}$. All such G have the class $1 + \log_p(|G'|)$. This fact and the inequality suggest the following

Definition 1.1. *A finite p -group G is said to be a generalised maximal class p -group if $c(G) = 1 + \log_p(|G'|)$.*

Taussky [3] determined all maximal class 2-groups. However, for the case when p is an odd prime, the classification of the maximal class p -groups has not been completed yet. Therefore it is also a difficult problem to determine all generalised maximal class p -groups. In this note, we consider the case when $p = 2$. Our aim is to give an important class of generalised maximal class 2-groups. For this, we define the following:

Definition 1.2. *A finite 2-group G is called an AC-group if $|N_G(A) : C_G(A)| \leq 2$ for all abelian subgroups A of G .*

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In this note, AC -groups are determined by means of the defining relation. These groups are generalised maximal class 2-group. In particular, an AC -2-group of class 2 is a non-abelian 2-group all of whose proper subgroups are abelian, and an AC -2-group with $|Z(G)| = 2$ is a maximal class 2-group.

An open problem about finite 2-groups is to find the complete classification of finite 2-groups generated by two elements with a cyclic derived subgroup (same problem for p -groups of odd order had been solved by Miech [4]). Our result shows that AC -2-groups form a subclass of the above groups. Therefore the note is related to the open problem.

Remark. The finite groups G satisfying $C_G(A) = N_G(A)$ or $C_G(A) = A$ for all abelian subgroups A of G was classified completely in [2]. In particular, when G is a p -group, G is either abelian or a non-abelian of order p^3 . This is a special case of AC -2-groups.

2. A description of AC -2-groups

First of all, we define the group $A(m, n, s) = \langle a, b, c \rangle$ with the defining relation:

$$a^{2^n} = b^{2^m} = c^{2^s} = 1, a^b = a^{-1}c, [a, c] = [b, c] = 1, \\ n \geq s \geq 0, m \geq s \geq 0.$$

Lemma 2.1. *The class of AC -2-groups is subgroup-closed.*

PROOF. Let G be an AC -2-group and let H be a subgroup of G . Any abelian subgroup A of H satisfies $|N_G(A) : C_G(A)| \leq 2$. Noting that $C_H(A)$ is normal in $N_H(A)$ and $N_H(A)/C_H(A) = N_H(A)/C_G(A) \cap C_H(A) \cong N_H(A)C_G(A)/C_G(A) \leq N_G(A)/C_G(A)$, we see that $|N_H(A) : C_H(A)|$ is a factor dividing $|N_G(A) : C_G(A)|$, so $|N_G(A) : C_G(A)| \leq 2$. Thus the result follows. ■

Lemma 2.2. *A finite 2-group G is an AC -group if and only if there exists an AC -subgroup H in G such that $Z(H) \subseteq \Phi(H)$ and $G = Z(G)H$.*

PROOF. Let G be an AC -2-group. Since G is finite, we may choose a subgroup H such that $G = Z(G)H$ and H is of minimal order. Then $Z(H) = Z(G) \cap H \leq \Phi(H)$ and, by Lemma 2.1, H is an AC -group as desired.

Conversely, for any abelian subgroup A of G , $Z(G)$ is contained in both $N_G(A)$ and $C_G(A)$, so $N_G(A) = Z(G) \cdot N_G(A) \cap H = Z(G)N_H(A)$ and $C_G(A) = Z(G)C_H(A)$ and $Z(G) \cap N_H(A) = Z(G) \cap C_H(A)$. Consequently $|N_G(A) : C_G(A)| = |N_H(A) : C_H(A)|$. Now, by hypothesis, H is an AC -2-group, so G is also an AC -2-group. ■

Lemma 2.3. *Suppose that G is a group with a normal abelian subgroup A such that G/A is cyclic. Write $G/A = \langle xA \rangle$. Then there is an epimorphism ϕ from A onto G' that is defined by*

$$\phi(a) = [a, x] \quad \forall a \in A.$$

PROOF. This is well-known (see [1, III, p. 259]). ■

Lemma 2.4. *Assume that $G = \langle a, x, c \rangle$ satisfies the following condition (not necessarily full defining relation):*

$$a^{2^n} = x^{2^m} = c^{2^s} = 1, \quad x^{-1}ax = a^{-1}c, \quad [a, c] = [x, c] = 1.$$

Then G , in particular the group $A(m, n, s)$, is an AC-group.

PROOF. Since any abelian 2-group is an AC-group, we need only to consider the case when G is non-abelian. Let $A = \langle a, c, x^2 \rangle$. Then A is an abelian subgroup of index 2 and $x^2 \in Z(G) \subseteq A$. Let B be any abelian subgroup of G . We must show that $|N_G(B) : C_G(B)| \leq 2$. If $B \leq A$, then $A \leq C_G(B)$ and so $|N_G(B) : C_G(B)| \leq |G : A| = 2$ as desired. Assume that B is not contained in A . We have $G = AB$. Since $G = A\langle x \rangle$ and $x^2 \in A$, there exists an element $u \in A$ such that $ux \in B$. Then $a^{ux} = (a^u)^x = a^x = a^{-1}c$. So we can replace x by ux in the relation of G . Without loss of generality, we assume $x \in B$.

First, we observe $C_G(B)$. Obviously, any element v in A belongs to $C_G(B)$ if and only if $[v, x] = 1$, and hence if and only if $v \in Z(G)$. Thus $C_G(B) = C_G(B) \cap G = C_G(B) \cap (AB) = (C_G(B) \cap A)B = Z(G)B$.

Secondly, we consider $N_G(B)$. Since G equals the group $A\langle x \rangle$, we have $N_G(B) = N_G(B) \cap A \cdot \langle x \rangle$. We claim that

$$|N_G(B) \cap A : Z(G)B_1| = 2 \quad \text{where } B_1 = B \cap A$$

In fact, for any $g \in N_G(B) \cap A$, we have $g \in A$ and $x^g \in B$. Because $B \leq C_G(B) = Z(G)B = (Z(G)B_1)\langle x \rangle$ and $x^2 \in Z(G)$, we have $x^g = xb$ for some $b \in Z(G)B_1$, i.e., $[x, g] \in Z(G)B_1$. Write $g = a^i z$ where $z \in Z(G)$. Then $[g, x] = [a^i z, x] = [a^i, x] = [a, x]^i = a^{-i}(a^x)^i = a^{-2i}c^i$, so $a^{2i} \in Z(G)B_1$ since $[g, x]$ and c both are in $Z(G)B_1$. It follows that $g^2 = a^{2i}z^2 \in Z(G)B_1$ and hence $N_G(B) \cap A / Z(G)B_1$ is elementary abelian. Also, by the definition of A , we see that $A/Z(G) = \langle aZ(G) \rangle$ is cyclic. So $N_G(B) \cap A / Z(G)B_1$, as a section of $A/Z(G)$, is cyclic as well. We thus deduce that $N_G(B) \cap A / Z(G)B_1$ is of order 2, as desired.

Now, as $x^2 \in Z(G) \subseteq N_G(B) \cap A$, we have $|N_G(B)| = 2 \cdot |N_G(B) \cap A|$. Consequently $|N_G(B) : C_G(B)| = 2|N_G(B) \cap A| / |Z(G)B| = |N_G(B) \cap A : Z(G)B_1| = 2$. By definition, G is an AC-group, completing the proof. ■

Theorem 2.1. *Let G be a finite 2-group with $Z(G) \leq \Phi(G)$. Then G is an AC-group if and only if $G \cong A(m, n, s)$ for suitable m, n and s .*

PROOF. By Lemma 2.4, $A(m, n, s)$ is an AC -group. Conversely, let G be an AC -2-group with $Z(G) \subseteq \Phi(G)$. Then G is non-abelian. We must show that $G \cong A(m, n, s)$ for suitable m, n and s . Our proof is divided into six steps.

(1) G contains an abelian subgroup A of index 2, so $G/A = \langle xA \rangle$ with $x^2 \in Z(G) \leq A$.

Since G is finite, we can choose a normal abelian subgroup A of G such that A is of order as large as possible. By a well-known elementary theorem on finite p -groups, we have $C_G(A) = A$. By hypothesis, $G/A = N_G(A)/C_G(A)$ has order 2, as desired.

(2) $x^{-1}gx = g^{-1}$ for each $g \in G'$.

By Lemma 2.3, the map ϕ defined by $\phi(a) = [a, x] \forall a \in A$ is an epimorphism from A onto G' . So, for every element g of G' , there exists an element of A , say a , such that $g = [a, x]$. Thus we have $a^x = ag$. It follows from $x^2 \in Z(G)$ that $a = a^{x^2} = (ag)^x = a^x g^x = agg^x$, which yields $g^x = g^{-1}$ as desired.

(3) $\Omega_1(G') = Z(G) \cap G'$.

Because G is a finite 2-group, $Z(G) \cap G' \neq 1$. For any $g \in Z(G) \cap G'$, we have $g^x = g$ and, by above conclusion (2), $g^x = g^{-1}$. So $g = g^{-1}$, i.e. $g^2 = 1$ and hence $g \in \Omega_1(G')$. Conversely, for $h \in \Omega_1(G')$ we have $h^2 = 1$ because $\Omega_1(G')$ is abelian. By conclusion (2), $h^x = h^{-1} = h$, so $h \in Z(G)$. Thus conclusion (3) holds.

(4) Set $B = Z(G)\langle x \rangle$. Then B is a maximal abelian subgroup of G .

In fact, $C_G(B) = C_G(B) \cap G = C_G(B) \cap A \cdot \langle x \rangle$. For any $g \in A \cap C_G(B)$, we have $[g, x] = 1$ and so $g \in Z(G)$, thus (4) follows.

(5) $A/Z(G)$ is cyclic.

By conclusion (1), $Z(G) \subseteq A$. Suppose that $A/Z(G)$ is non-cyclic, so that there exists an elementary abelian subgroup $V/Z(G)$ of $A/Z(G)$ of order 4. Then

$$V/Z(G) = \langle aZ(G) \rangle \times \langle bZ(G) \rangle$$

where $a \notin Z(G)$, $b \notin Z(G)$, but both a^2 and b^2 are in $Z(G)$. Now $[a, x]^2 = [a^2, x] = 1$, so $[a, x] \in \Omega_1(G') \leq Z(G)$ by (3), which implies that $\langle a \rangle$ normalises the subgroup B and $\langle x \rangle$ normalises V . Similarly $\langle b \rangle$ normalises B . Consequently V normalises B and $V\langle x \rangle$ is a subgroup of G . Let N denote the group $V\langle x \rangle$. Then $B \subseteq N \subseteq N_G(B)$, so $|N_G(B) : C_G(B)| \geq |N : B|$. Also, it is easy to see that $|N| = 2 \cdot 2^2 \cdot |Z(G)|$ and $|B| = 2 \cdot |Z(G)|$. Consequently $|N : B| \geq 4$ and hence $|N_G(B) : C_G(B)| \geq 4$, contrary to hypothesis. We thus deduce that $A/Z(G)$ is cyclic.

(6) G is isomorphic to some $A(m, n, s)$.

By conclusion (5), we can let $A = \langle a, Z(G) \rangle$. By hypothesis, $Z(G) \subseteq \Phi(G)$, and as we have $G = A\langle x \rangle$, it follows that G is generated by a and x . Also, $aa^x \in A$ and $(aa^x)^x = a^x a^{x^2} = a^x a = aa^x$, so aa^x lies in $Z(G)$. Choosing a and x having order as large as possible, we have

$$G = \langle a, x, c \rangle$$

satisfying the following condition:

$$a^{2^n} = x^{2^m} = c^{2^s} = 1, \quad a^x = a^{-1}c, \quad [a, c] = [x, c] = 1.$$

$$n \geq s \geq 0, \quad m \geq s \geq 0.$$

By comparison with the group $A(m, n, s)$, we know that G is isomorphic to a factor group of $A(m, n, s)$. Because $A(m, n, s)$ is an AC -2-group by Lemma 2.4, we thus conclude $G = A(m, n, s)$ for suitable m, n and s . The proof of the theorem is now complete. ■

Corollary 2.2. *For any non-abelian AC -2-group G , G' is cyclic and $G' \cap Z(G)$ is of order 2.*

PROOF. By Lemma 2.2, $G = Z(G)H$ where H is an AC -group with $Z(H) \subseteq \Phi(H)$. Obviously $G' = H'$. By Theorem 2.1, H is a group of the form $A(m, n, s)$. Checking the defining relation of $A(m, n, s)$, we know that $H' = \langle [a, x] \rangle$, so G' is generated by $[a, x]$ and hence cyclic. We claim $|G' \cap Z(G)| = 2$. Indeed, this follows from $G' = \langle [a, x] \rangle$ and $G' \cap Z(G) = \Omega_1(G')$ (see conclusion (3) of the proof of Theorem 2.1). ■

3. Sufficient and necessary conditions

Theorem 3.1. *For a finite non-abelian 2-group G , the following statements are equivalent to each other:*

- (1) G is an AC -group.
- (2) $G/Z(G)$ is a dihedral 2-group.
- (3) G contains an abelian subgroup of index 2 and G' is cyclic.

PROOF. (1) \Rightarrow (2): Since G is non-abelian, by Lemma 2.2, we can write $G = Z(G)H$ where H is an AC -subgroup of G with $Z(H) \subseteq \Phi(H)$. By Theorem 2.1, H is a group of the form $A(m, n, s)$ and hence $H/Z(H) = \langle \bar{a}, \bar{b} \rangle$ defined by the following relation: $\bar{a}^{2^n} = \bar{b}^{2^s} = 1, \bar{b}^{-1}\bar{a}\bar{b} = \bar{a}^{-1}$ for suitable n . So $H/Z(H)$ is a dihedral 2-group. Thus $G/Z(G)$, which is isomorphic to $H/Z(H)$, is a dihedral 2-group.

(2) \Rightarrow (3): By hypothesis, $G/Z(G) \cong D_{2^f}$, the dihedral group of order 2^f . Choose a subgroup H of G having minimal order satisfying $G = Z(G)H$. Then $Z(H) \subseteq \Phi(H)$ and $H/Z(H) \cong D_{2^f}$. Because D_{2^f} is generated by two elements and contains a cyclic subgroup of index 2, it follows that H is generated by two elements and contains an abelian subgroup A of index 2. Let $H = \langle a, b \rangle$ with $a \in A$. Then $b^2 \in Z(H)$, and H' is generated by $[a, b]$. Thus, as $G' = H'$, we deduce that G' is cyclic, as desired.

(3) \Rightarrow (1): Let A be the abelian subgroup of G of index 2 and let G' be cyclic. Write $G = Z(G)H$ such that $Z(H) \subseteq \Phi(H)$. It is clear that H inherits the condition on G . By Lemma 2.2, it suffices to show that H is an AC -group. Therefore, we can require, without loss of generality, $H = G$, i.e., $Z(G) \subseteq \Phi(G)$.

We now have $G = A\langle x \rangle$ with $Z(G) \subseteq \Phi(G)$. Since G' is cyclic, it follows by Lemma 2.3 that G' is generated by $[a, x]$ where some $a \in A$. We claim $G = \langle a, x \rangle$. For any

$b \in G$, $[b, x] \in G'$, so $[b, x]^{-1} = [a, x]^i$ for some i . Thus $[a^i b, x] = [a, x]^i [b, x] = 1$, and hence $a^i b \in Z(G)$ and we have $G = \langle a, x, Z(G) \rangle = \langle a, x \rangle$ since $Z(G) \subseteq \Phi(G)$, as desired.

Because A is normal in G and $x^2 \in A$, we have $[a^{x^2}, x] = 1$, so $aa^{x^2} = c$ for some $c \in Z(G)$. Thus $G = \langle a, x, c \rangle$ satisfies the following relation: $a^{2^n} = x^{2^m} = c^{2^s} = 1$, $a^x = a^{-1}c$, $[a, c] = [x, c] = 1$ for suitable n, m and s . Applying Lemma 2.4 we can conclude that G is an AC -2-group. The proof of the theorem is now complete. ■

The following corollary is interesting, but not obvious from the definition of an AC -2-group.

Corollary 3.2. *Every homomorphic image of an AC -2-group is an AC -2-group again.*

PROOF. Let G be an AC -2-group and let M be an homomorphic image of G . We may assume that M is non-abelian. Then G contains an abelian subgroup of index 2 and G' cyclic by Theorem 3.1. This property is inherited by M obviously. Again applying Theorem 3.1, we have that M is an AC -2-group. ■

Corollary 3.3. *Let G be a finite 2-group. Then*

- (1) *G is an AC -group of class 2 with $Z(G) \subseteq \Phi(G)$ if and only if G is a minimal non-abelian 2-group.*
- (2) *G is an AC -group with $|Z(G)| = 2$ if and only if G is a 2-group with maximal class.*

PROOF. (1) It is obvious that any minimal non-abelian 2-group is an AC -group with class 2 and $Z(G) = \Phi(G)$. Conversely, let G be an AC -2-group of class 2 with $Z(G) \subseteq \Phi(G)$. By Theorem 3.1 and Corollary 2.2, G contains an abelian subgroup A of index 2 and $G' \cap Z(G)$ is of order 2. Since the class of G is 2, it follows that $G' \leq Z(G)$. Hence G' is of order 2. Also, by Lemma 2.3, we have that $A/Z(G) \cong G'$. Consequently $|A/Z(G)| = 4$. Noting that $Z(G) \subseteq \Phi(G)$, we get that $Z(G) = \Phi(G)$. Now, it is clear that the maximal subgroups of G are abelian, so G is a minimal non-abelian 2-group.

(2) If G is a 2-group of maximal class, then G contains a cyclic subgroup of index 2 and $|Z(G)| = 2$ by [1, III, 11.9 Satz]. Of course, G' is cyclic. It follows from Theorem 3.1 that G is an AC -2-group. Conversely, suppose G is an AC -2-group with $|Z(G)| = 2$. By Theorem 3.1, $G/Z(G)$ is a dihedral 2-group. In particular, $G/Z(G)$ is a 2-group of maximal class. Thus, as $|Z(G)| = 2$, G is a 2-group of maximal class, completing the proof. ■

Corollary 3.4. *Let G be an AC -2-group of class $c(G)$. Then $c(G) = 1 + \log_2 |G'|$.*

PROOF. If G is abelian, then $c(G) = 1$, a trivial case. Assume that G is non-abelian. By Theorem 3.1, $G/Z(G)$ is a maximal class 2-group, and G has an abelian subgroup

A of index 2. Obviously $Z(G) \subseteq A$, and $A/Z(G) \cong G'$ by Lemma 2.3. Hence $|G/Z(G)| = 2 \cdot |G'| = 2^{k+1}$. It follows that $c(G) = k + 1$, as desired. ■

Finally, we end this note by posing the following problem, which is a special case of the famous problem: classify up to isomorphism the finite 2-groups generated by two elements with a cyclic derived subgroup.

Problem 3.1. *Classify AC-2-groups G with $Z(G) \subseteq \Phi(G)$ up to isomorphism.*

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