

GENERALISED SURFACES IN \mathbb{R}^3

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ABSTRACT

The correspondence between 2-parameter families of oriented lines in \mathbb{R}^3 and surfaces in $T\mathbb{P}^1$ is studied, and the geometric properties of the lines are related to the complex geometry of the surface. Congruences generated by global sections of $T\mathbb{P}^1$ are investigated, and a number of theorems are proven that generalise results for closed convex surfaces in \mathbb{R}^3 .

1. Introduction

In this paper we study a generalisation of the concept of a surface in Euclidean \mathbb{R}^3 through the twistor construction. The fundamental aspect of this generalisation is that, through its normal, an oriented surface in \mathbb{R}^3 determines a surface in the space of all oriented lines: $\mathbb{T} = T\mathbb{P}^1$. This 4-manifold inherits a natural complex structure from the Euclidean metric in \mathbb{R}^3 and has been useful in connection with monopoles [4].

A key feature of the construction is that a point in \mathbb{R}^3 corresponds to a holomorphic sphere in \mathbb{T} . In this paper we consider (not necessarily holomorphic) 2-spheres in \mathbb{T} and the *line congruences* they generate in \mathbb{R}^3 [11]. Our main aim is to show that, in this setting, curvature and umbilics can all be sensibly defined and used to obtain results generalising those of closed convex surfaces.

The sense in which we are generalising is given by Frobenius' theorem: we are allowing consideration of *twisting* line congruences. This twist is encoded in the anti-symmetric part of a suitably defined second fundamental form of the line congruence. The curvature K of a congruence can then be defined as the determinant of the second fundamental form.

This has the following bundle interpretation:

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Theorem 1. *A line congruence Σ is locally the graph of a section of the bundle $\pi: \mathbb{T} \rightarrow \mathbb{P}^1$ if and only if the curvature is non-zero.*

We say a line congruence is *globally convex* if it is the graph of a global section of the bundle $\pi: \mathbb{T} \rightarrow \mathbb{P}^1$. Thus, a globally convex congruence $\Sigma \subset \mathbb{T}$ is a topological sphere and the Gauss map $\pi|_{\Sigma}$ yields natural global coordinates $(\xi, \bar{\xi})$ on Σ . We show that a generalised Gauss-Bonnet theorem holds for globally convex congruences:

Theorem 2. *Let Σ be a globally convex congruence with curvature K . Then*

$$\int_{\Sigma} K \, d\mu = 4\pi$$

where $d\mu$ at $\gamma \in \Sigma$ is the pull-back via π of the volume form induced on the plane orthogonal to γ by the Euclidean metric on \mathbb{R}^3 .

In general these globally convex congruences, aside from twisting, will be non-holomorphic. However, we can always perturb a congruence so that it has isolated complex points. The total number of complex points, counted with index, is a topological invariant of the congruence. We show that this index is the number of shear-free points on a globally convex congruence and that:

Theorem 3. *The total number of shear-free lines (counted with index) on a globally convex congruence with only isolated shear-free lines is 4.*

This generalises the well-known result that the number of isolated umbilics (counted with index) on a closed convex surface is 4.

This paper is organised as follows: in the next section we recall the relevant twistor construction; for further details see [4; 5]. To prove our main results we use a canonical co-ordinate system, as in [3] and null frame adapted to the congruence under consideration. After describing these in Section 3, we turn to the first order description of congruences. Here the method of spin-coefficients [8; 10], applied to \mathbb{R}^3 , yields a compact description of the geometric data. Finally in Section 5 we prove the results regarding globally convex congruences.

2. The minitwistor construction

We begin by recalling the minitwistor construction of straight lines in \mathbb{R}^3 (see Hitchin [4] for further details). Given a choice of origin in \mathbb{R}^3 , a straight line can be uniquely described by two vectors: the (oriented) direction of the line \vec{W} and its perpendicular displacement from the origin \vec{U} . A straight line γ is given by

$$\gamma = \{ \vec{U} + r\vec{W} \in \mathbb{R}^3 \mid \langle \vec{U}, \vec{W} \rangle = 0 \quad |\vec{W}| = 1 \quad r \in \mathbb{R} \},$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $|\cdot|$ the associated norm.

The space of all oriented straight lines in \mathbb{R}^3 is the minitwistor space

$$\mathbb{T} = \{ (\vec{U}, \vec{W}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle \vec{U}, \vec{W} \rangle = 0 \quad |\vec{W}| = 1 \} \cong TS^2.$$

Throughout this paper we utilise this bijection to identify an oriented line $\gamma \subset \mathbb{R}^3$ with the point $\gamma \in \mathbb{T}$. This 4-manifold has a natural almost complex structure \mathbb{J} defined by rotation in \mathbb{R}^3 through 90° about the direction of the line. In fact, the almost complex structure \mathbb{J} is integrable and so \mathbb{T} is a complex surface. Alternatively, $\mathbb{J} = j \oplus j$, where the splitting $T\mathbb{T} = TS^2 \oplus TS^2$ and the complex structure j on S^2 are induced by the Euclidean metric on \mathbb{R}^3 . In addition, there is an anti-holomorphic involution $\tau: \mathbb{T} \rightarrow \mathbb{T}$ given by reversing the orientation of the line.

A point p in \mathbb{R}^3 is uniquely determined by the 2-sphere S_p^2 of oriented lines passing through it. This sphere $S_p^2 \subset \mathbb{T}$ has the following properties:

- (1) S_p^2 is a complex line in \mathbb{T} i.e. the complex structure on \mathbb{T} leaves invariant the tangent space of S_p^2
- (2) $S_p^2 \cap S_q^2$ consists of two points in \mathbb{T} —the two oriented lines in \mathbb{R}^3 passing through p and q
- (3) S_p^2 is invariant under the involution τ

From the above it follows that S_p^2 can be given, in terms of a holomorphic coordinate ξ on \mathbb{P}^1 , as the graph of a section of $\pi: \mathbb{T} \rightarrow \mathbb{P}^1$

$$s(\xi) = \frac{1}{2}((x^1 + ix^2) - 2x^3\xi - (x^1 - ix^2)\xi^3) \frac{\partial}{\partial \xi}, \tag{2.1}$$

where the Euclidean coordinates of p are (x^1, x^2, x^3) . This follows from the fact that global holomorphic sections can be at most quadratic in ξ (self-intersection 2) and then the invariance of the section under the antipodal map $\tau(\xi) = -\bar{\xi}^{-1}$ restricts the coefficients to be of the above form.

In this paper we will investigate *line congruences*, that is, surfaces $\Sigma \subset \mathbb{T}$ or two parameter families of lines in \mathbb{R}^3 . Every surface $S \subset \mathbb{R}^3$ gives rise to a congruence $\Sigma \subset \mathbb{T}$ by way of its normal, but not every congruence arises in this way. It is in this sense that we are working with generalised surfaces.

In the next section we introduce new local coordinates and a null frame on \mathbb{R}^3 that fit nicely with the description of a congruence as a surface in $T\mathbb{P}^1$.

3. Coordinates and an adapted frame on \mathbb{R}^3

Let (x^1, x^2, x^3) be the standard coordinates on \mathbb{R}^3 and set $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ and $x^3 = t$.

Definition 1. Consider the following transformation $\Phi: (u, v, r) \rightarrow (z, \bar{z}, t)$ on an open subset of \mathbb{R}^3 given by

$$z = \frac{2(F - \bar{F}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2} \tag{3.1}$$

and

$$t = \frac{-2(F\bar{\xi} + \bar{F}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}, \tag{3.2}$$

where $F(u, v)$ and $\xi(u, v)$ are smooth complex-valued functions of two real parameters u and v . We call the coordinates (u, v, r) the *congruence coordinates*.

For each (u, v) , $\Phi(r)$ is a straight line in \mathbb{R}^3 . Moreover, these lines are parameterised by arc length r and $\frac{\partial}{\partial r}$ is the unit tangent to the lines. The shortest distance from the origin to each line is given by the point $r = 0$.

These coordinates come from the twistor construction in the following way: consider one of these lines with direction \vec{W} and perpendicular displacement from the origin \vec{U} . Translate \vec{W} along \vec{U} to the origin and then translate \vec{U} along \vec{W} . This vector tangent to S^2 gives us the line as a point in \mathbb{T} . Moreover, ξ is the standard holomorphic coordinate on \mathbb{P}^1 induced from stereographic projection from the south pole and in these coordinates the point in \mathbb{T} is given by

$$\left(\xi(u, v), F(u, v) \frac{\partial}{\partial \xi} \right) \in T_\xi \mathbb{P}^1.$$

Thus $F(u, v)$ determines the perpendicular distance of the line from the origin. This can be viewed as parametric equations for a line congruence in terms of coordinates (ξ, F) on \mathbb{T} , which are holomorphic with respect to \mathbb{J} . For further details see [3].

A change of origin leads to a quadratic holomorphic translation of the function F . In particular, if the origin is translated $(0, 0, 0) \rightarrow (x_0^1, x_0^2, x_0^3)$ then

$$F \rightarrow F + \frac{1}{2}(\alpha_0 - 2t_0\xi - \bar{\alpha}_0\xi^2), \tag{3.3}$$

where $\alpha_0 = x_0^1 + ix_0^2$ and $t_0 = x_0^3$. This can be seen from equation (2.1) since the lines through the origin (the zero section of the bundle) will change to the section

$$s(\xi) = \frac{1}{2}(\alpha_0 - 2t_0\xi - \bar{\alpha}_0\xi^2) \frac{\partial}{\partial \xi}.$$

Our coordinates will then transform by:

$$(u, v, r) \rightarrow \left(u, v, r + \frac{\bar{\alpha}_0\xi - \alpha_0\bar{\xi} + t_0(1 - \xi\bar{\xi})}{1 + \xi\bar{\xi}} \right) \tag{3.4}$$

$$(z, t) \rightarrow (z + \alpha_0, t + t_0).$$

The change in r is just $\langle \vec{T}, \vec{W} \rangle$, where \vec{T} is the translation vector determined by α_0 and t_0 .

The quantities that have geometric significance are invariant under this translation. In particular, we have the following translation invariant derivatives of the twistor function F :

Proposition 1. Let $v = u + iv$, $\bar{v} = u - iv$, and $\partial = \frac{\partial}{\partial v}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{v}}$.

Then

$$\begin{aligned} \partial^+ F &\equiv \partial F + r\partial\zeta - \frac{2F\bar{\zeta}\partial\zeta}{1 + \zeta\bar{\zeta}} \\ \partial^- F &\equiv \bar{\partial} F + r\bar{\partial}\bar{\zeta} - \frac{2F\zeta\bar{\partial}\bar{\zeta}}{1 + \zeta\bar{\zeta}}, \end{aligned}$$

are invariant under the translations (3.3) and (3.4).

PROOF. This is a straight-forward calculation. ■

Note 1. The Jacobian of the transformation Φ is

$$\Delta = \frac{4}{(1 + \zeta\bar{\zeta})^2} (\partial^- F \bar{\partial}^+ \bar{F} - \partial^- F \bar{\partial}^- \bar{F})$$

Thus the transformation is a diffeomorphism wherever $\Delta \neq 0$.

A null frame in \mathbb{R}^3 is a trio $\{e_0, e_+, e_-\}$ of complex vector fields in $\mathbb{C} \otimes T\mathbb{R}^3$, where e_0 is real, e_+ is the complex conjugate of e_- and they satisfy the following orthogonality properties:

$$\langle e_0, e_0 \rangle = 1 \quad \langle e_0, e_+ \rangle = 0 \quad \langle e_+, e_+ \rangle = 0 \quad \langle e_+, e_- \rangle = 1$$

where we have extended the Euclidean inner product of \mathbb{R}^3 bilinearly over \mathbb{C} . Orthonormal frames $\{e_0, e_1, e_2\}$ on $T\mathbb{R}^3$ and null frames are related by

$$e_+ = \frac{1}{\sqrt{2}}(e_1 - ie_2) \quad e_- = \frac{1}{\sqrt{2}}(e_1 + ie_2). \tag{3.5}$$

Definition 2. A congruence null frame for $\Sigma \subset \mathbb{T}$ is a null frame $\{e_0, e_+, e_-\}$ if, for each $\gamma \in \Sigma$, we have e_0 tangent to γ in \mathbb{R}^3 , and the orientation of $\{e_0, e_1, e_2\}$ is the standard orientation on \mathbb{R}^3 .

Proposition 2. Let Σ be a line congruence and consider an open set $U \subset \Sigma$ with $\Delta \neq 0$. Suppose that F is the twistor function describing Σ on U . A null frame is a congruence null frame to Σ if and only if it has the following expression in terms of canonical coordinates

$$e_0 = \frac{\partial}{\partial r} \quad e_+ = \left(\alpha \frac{\partial}{\partial v} + \beta \frac{\partial}{\partial \bar{v}} + \Omega \frac{\partial}{\partial r} \right) e^{i\varphi} \quad e_- = \bar{e}_+,$$

where

$$\Omega = \frac{\sqrt{2}[\bar{\partial}^- \bar{F}(F\bar{\partial}\bar{\zeta} + \bar{F}\bar{\partial}\bar{\zeta}) - \bar{\partial}^+ \bar{F}(F\partial\bar{\zeta} + \bar{F}\partial\bar{\zeta})]}{(1 + \zeta\bar{\zeta})(\partial^- F \bar{\partial}^+ \bar{F} - \partial^+ F \bar{\partial}^- \bar{F})}$$

$$\alpha = \frac{\overline{\partial^+ F}(1 + \xi \bar{\xi})}{\sqrt{2}(\partial^- F \overline{\partial^- F} - \partial^+ F \overline{\partial^+ F})} \quad \beta = -\frac{\overline{\partial^- F}(1 + \xi \bar{\xi})}{\sqrt{2}(\partial^- F \overline{\partial^- F} - \partial^+ F \overline{\partial^+ F})}, \tag{3.6}$$

and ϕ is a function of v, \bar{v} and r .

PROOF. We can prove this by a change of coordinates from (v, \bar{v}, r) to (z, \bar{z}, t) using equations (3.1) to (3.2). This gives the following mixed form for the frame:

$$e_0 = \frac{2\xi}{1 + \xi \bar{\xi}} \frac{\partial}{\partial z} + \frac{2\bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial \bar{z}} + \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial t}$$

$$e_+ = \frac{\sqrt{2}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial z} - \frac{\sqrt{2}\xi^2}{1 + \xi \bar{\xi}} \frac{\partial}{\partial \bar{z}} - \frac{\sqrt{2}\xi}{1 + \xi \bar{\xi}} \frac{\partial}{\partial t}.$$

Now, since the Euclidean inner product on \mathbb{R}^3 in the coordinates z, \bar{z}, t is simply

$$g_{ij} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we can check that these vectors form a null frame.

Alternatively, we can work backwards as follows. Set

$$e_0 = \frac{\partial}{\partial r} \quad e_+ = \alpha \frac{\partial}{\partial v} + \beta \frac{\partial}{\partial \bar{v}} + \Omega \frac{\partial}{\partial r},$$

for some functions α, β and Ω to be determined. Now the first condition on these functions is $\langle e_0, e_+ \rangle = 0$, which gives us that

$$\Omega = -\alpha \left\langle e_0, \frac{\partial}{\partial v} \right\rangle - \beta \left\langle e_0, \frac{\partial}{\partial \bar{v}} \right\rangle.$$

Thus

$$e_+ = \alpha \left(\frac{\partial}{\partial v} - \left\langle e_0, \frac{\partial}{\partial v} \right\rangle \frac{\partial}{\partial r} \right) + \beta \left(\frac{\partial}{\partial \bar{v}} - \left\langle e_0, \frac{\partial}{\partial \bar{v}} \right\rangle \frac{\partial}{\partial r} \right) = \alpha Z_+ + \beta Z_-.$$

The second condition $\langle e_+, e_+ \rangle = 0$ now becomes

$$\langle Z_+, Z_+ \rangle \alpha^2 + 2 \langle Z_+, Z_- \rangle \alpha \beta + \langle Z_-, Z_- \rangle \beta^2 = 0. \tag{3.7}$$

A lengthy calculation, involving a change to Euclidean coordinates via Φ , shows that

$$\langle Z_+, Z_+ \rangle = \frac{4\overline{\partial^+ F}}{(1 + \xi \bar{\xi})} \quad \langle Z_+, Z_- \rangle = \frac{2(\partial^- F \overline{\partial^- F} + \partial^+ F \overline{\partial^+ F})}{(1 + \xi \bar{\xi})}. \tag{3.8}$$

With the aid of this, we can solve equation (3.7) for the ratio of α and β :

$$\frac{\alpha}{\beta} = -\frac{\overline{\partial^+ F}}{\partial^- F} \quad \text{or} \quad \frac{\alpha}{\beta} = -\frac{\partial^- F}{\partial^+ F}.$$

The fact that we get two answers simply represents the choice of orientation for the null frame. We choose the first in order to have our orientation agree with the standard orientation of \mathbb{R}^3 . This orientation coincides with the graph orientation Σ in $\mathbb{T} \rightarrow \mathbb{P}^1$.

The final equation we have to solve is $\langle e_+, e_- \rangle = 1$ and this gives us $\beta\bar{\beta}$, that is ϕ . This represents rotation of the frame about e_0 , which we can set to agree with the argument of $-\overline{\partial^- F}$ by parallel translation of the frame. ■

4. The local geometry of congruences

We now use the twistor coordinates and null frame of the last section to investigate the local geometry of congruences in \mathbb{R}^3 . To start we find expressions for the first order geometric properties:

Proposition 3. *The Lie derivative of e_+ in the e_0 direction is given by*

$$L_{e_0} e_+ = \bar{\rho} e_+ + \sigma e_-,$$

with

$$\rho = \frac{\partial^+ F \bar{\partial} \bar{\xi} - \partial^- F \partial \bar{\xi}}{\partial^- F \overline{\partial^- F} - \partial^+ F \overline{\partial^+ F}} \tag{4.1}$$

$$\sigma = \frac{\overline{\partial^+ F} \partial \bar{\xi} - \overline{\partial^- F} \bar{\partial} \bar{\xi}}{\partial^- F \overline{\partial^- F} - \partial^+ F \overline{\partial^+ F}}. \tag{4.2}$$

PROOF. This follows by using the computations in the previous proposition. For the twist and divergence,

$$\begin{aligned} \bar{\rho} &= \langle L_{e_0} e_+, e_- \rangle \\ &= \left\langle \frac{\partial \alpha}{\partial r} Z_+ + \frac{\partial \beta}{\partial r} Z_- , \bar{\alpha} Z_- + \bar{\beta} Z_+ \right\rangle. \end{aligned}$$

Now use the expressions for α and β and equation (3.8) to get the result.

Similarly

$$\begin{aligned} \sigma &= \langle L_{e_0} e_+, e_+ \rangle \\ &= \left\langle \frac{\partial \alpha}{\partial r} Z_+ + \frac{\partial \beta}{\partial r} Z_- , \alpha Z_+ + \beta Z_- \right\rangle. \quad \blacksquare \end{aligned}$$

The complex scalar functions ρ and σ describe the first order geometric behaviour of the congruence of lines. In particular, the real part of ρ is the *divergence*, the imaginary part is the *twist* and σ is the *shear* of the congruence (see [2] for details). By Proposition 1 these are invariant under translations of the origin. For line congruences in \mathbb{R}^3 the evolution of these quantities along the line can be more directly derived. In particular, they satisfy the *Sachs equations* [9]

$$\frac{\partial \rho}{\partial r} = \rho^2 + \sigma \bar{\sigma} \quad \frac{\partial \sigma}{\partial r} = (\rho + \bar{\rho})\sigma. \tag{4.3}$$

Note that if the shear or the twist vanish at some point on the line, they vanish at every point on the line.

Following Penrose and Rindler [9] these can be integrated in terms of the initial values of ρ and σ at $r=0$:

$$\rho = \frac{\rho_0 - (\rho_0 \bar{\rho}_0 - \sigma_0 \bar{\sigma}_0)r}{1 - (\rho_0 + \bar{\rho}_0)r + (\rho_0 \bar{\rho}_0 - \sigma_0 \bar{\sigma}_0)r^2} \quad \sigma = \frac{\sigma_0}{1 - (\rho_0 + \bar{\rho}_0)r + (\rho_0 \bar{\rho}_0 - \sigma_0 \bar{\sigma}_0)r^2}.$$

The shear has the following interpretation. Consider a circle in the plane orthogonal to the line γ_0 at some point. Lie-propagation along the line can alter this circle in a number of ways: if the shear is zero it will remain a circle, and if the shear is non-zero it will become an ellipse. In particular, $|\sigma|$ measures the eccentricity of the ellipse, while $\phi = Arg(\sigma)$ measures the inclination of the semi-major and semi-minor axes (details can be found in [2]).

The twist can be understood as follows:

Definition 3. A line congruence is *integrable* if locally there exists an embedded surface S in \mathbb{R}^3 such that S is orthogonal to the lines of the congruence.

Proposition 4. [2] *A line congruence is integrable if and only if ρ is real (the twist vanishes).*

Proposition 5. *The orthogonal surface S to an integrable line congruence is given by $r = r(v, \bar{v})$, where*

$$\bar{\partial}r = \frac{2F\bar{\partial}\bar{\xi} + 2\bar{F}\bar{\partial}\xi}{(1 + \xi\bar{\xi})^2}.$$

PROOF. It is not hard to show that the parametric surface in \mathbb{R}^3 , obtained by inserting $r = r(v, \bar{v})$ in (3.1) and (3.2), is orthogonal to the congruence if and only if the above condition holds. ■

Definition 4. The *curvature* of a congruence is defined to be $K = \rho\bar{\rho} - \sigma\bar{\sigma}$.

This definition reflects the fact that, in the twist-free case, the curvature of a congruence is the curvature of the one parameter family of surfaces in \mathbb{R}^3 orthogonal to the congruence. More generally,

Theorem 1. *A line congruence is locally the graph of a section of the bundle $\pi: \mathbb{T} \rightarrow \mathbb{P}^1$ if and only if the curvature is non-zero.*

PROOF. From equations (4.1) and (4.2) we find that

$$K = \frac{\partial_{\xi} \bar{\partial}_{\bar{\xi}} - \partial_{\bar{\xi}} \bar{\partial}_{\xi}}{\partial^{-} F \bar{\partial}^{-} \bar{F} - \partial^{+} F \bar{\partial}^{+} \bar{F}}.$$

Thus, the left hand side will vanish at a point $\gamma \in \Sigma$ if and only if the Jacobian relating ξ and ν is zero at γ , that is, the tangent space to Σ at γ has a vertical component. ■

5. Globally convex congruences

We say a congruence is *globally convex* if it is the graph of a global section of the canonical bundle $\pi: \mathbb{T} \rightarrow \mathbb{P}^1$. A globally convex congruence is a topological 2-sphere that generalises the concept of a closed convex surface.

Example: The derivative of the action of $PSL(2, \mathbb{C})$ on \mathbb{P}^1 generates a holomorphic vector field on \mathbb{P}^1 that is a global section of the canonical bundle. This 6-parameter family of line congruences splits into twisting and twist-free congruences. The former contains the standard overtwisted contact structure on \mathbb{R}^3 (see [1]), while the latter consists of the holomorphic \mathbb{P}^1 generated by lines through a point in \mathbb{R}^3 .

Globally convex congruences have nice properties. For example,

Proposition 6. *Suppose that $\Sigma \subset \mathbb{T}$ is a globally convex congruence, then every point in \mathbb{R}^3 is contained in some line of the congruence, i.e. the congruence is spacefilling.*

PROOF. Consider any point $p \in \mathbb{R}^3$. By a translation we can make p the origin. Now the resulting surface Σ is a vector field on S^2 and therefore must have a zero. In terms of the lines on \mathbb{R}^3 , a zero represents a line with zero perpendicular distance from the origin, that is a line passing through the origin. Thus every point p must have a line passing through it. ■

Theorem 2. *Let Σ be a globally convex congruence with curvature K . Then:*

$$\int_{\Sigma} K d\mu = 4\pi,$$

where $d\mu$ at $\gamma \in \Sigma$ is the pull-back via π of the volume form induced on the plane orthogonal to γ by the Euclidean metric on \mathbb{R}^3 .

PROOF. Suppose we choose coordinates $(\xi, \bar{\xi})$ and twistor function F to describe an open subset of Σ .

The basis of 1-forms $\{\theta^0, \theta^+, \theta^-\}$ dual to the vector basis in Proposition 2 has coordinates

$$\theta^+ = \frac{\sqrt{2}}{K(1 + \xi \bar{\xi})} (\rho d\xi - \bar{\sigma} d\bar{\xi}).$$

Now, $d\mu = i\pi^*(\theta^+ \wedge \theta^-)$ and so we get that

$$\int_{\Sigma} K d\mu = \int_{\Sigma} K i\pi^*(\theta^+ \wedge \theta^-) = 2 \int_{\mathbb{P}^1} \frac{d\xi d\bar{\xi}}{(1 + \xi\bar{\xi})^2} = 4\pi. \quad \blacksquare$$

Consider a line congruence in \mathbb{R}^3 given by an oriented surface $\Sigma \subset \mathbb{T}$.

Definition 5. A point $\gamma \in \Sigma$ is *complex* if the complex structure acting on \mathbb{T} preserves $T_{\gamma}\Sigma$, the tangent space to Σ at γ . A complex point $\gamma \in \Sigma$ is called *positive* or *negative* if the complex structure induced on $T_{\gamma}\Sigma$ by \mathbb{J} agrees or disagrees with the given orientation on Σ , respectively.

In the general case a line congruence $\Sigma \subset \mathbb{T}$ will have places where it is complex and others where it fails to be. We can perturb such a surface so that the complex points are isolated. Moreover, there is a well-defined index for each isolated complex point and we identify these as follows:

Proposition 7. *A point on Σ is complex if and only if the shear vanishes along the line. The index I of an isolated complex point γ_0 is minus the winding number of the semi-major axes of shear as we go around γ_0 .*

PROOF. Let $f: \Sigma \rightarrow \mathbb{T}$ be a C^2 smooth immersion with differential $df: T\Sigma \rightarrow T\mathbb{T}$. Let j be a conformal structure on Σ compatible with ν . We define the sections $\delta^+f \in \Omega^{1,0}(\Sigma) \otimes T^{1,0}\mathbb{T}$ and $\delta^-f \in \Omega^{0,1}(\Sigma) \otimes T^{1,0}\mathbb{T}$ by

$$\delta^{\pm}f = \frac{1}{2}(df \mp \mathbb{J} \circ df \circ j).$$

Then $\delta^+f \wedge \delta^-f \in \Omega^2(\Sigma) \otimes \det T^{1,0}\mathbb{T}$, works out to be

$$\delta^+f \wedge \delta^-f = (\partial\bar{\xi}\bar{\partial}\eta - \bar{\partial}\xi\partial\eta) \nu \wedge d\bar{\nu}$$

A point is complex if and only if this 2-form vanishes, see [6]. By simplifying the numerator of σ in (4.2), this is equivalent to the vanishing of the shear.

The index of a complex point is equal to the intersection index of $\delta^+f \wedge \delta^-f$ with the zero section of $\Omega^2(\Sigma) \otimes \det T^{1,0}\mathbb{T}$. This is just the winding number of the complex function $\partial\bar{\xi}\bar{\partial}\eta - \bar{\partial}\xi\partial\eta$, i.e. the winding number of $\bar{\sigma}$ about its isolated zero. \blacksquare

In the integrable case the argument of the shear is the angle between the real axis of ν with a principal curvature direction of the orthogonal surface in \mathbb{R}^3 . We have the following generalisation of the fact that the total number of isolated umbilics (counted with index) on an closed convex surface is 4.

Theorem 3. *The total number of shear-free lines (counted with index) on a globally convex congruence with only isolated shear-free lines is 4.*

PROOF. Let d_+ be the sum of the indices over all positive complex points of Σ . Define d_- similarly for negative complex points. Then by [7],

$$\begin{aligned}d_+ + d_- &= \chi(T\Sigma) + \chi(N\Sigma) \\d_+ - d_- &= c_1(TM)[\Sigma],\end{aligned}$$

where $N\Sigma$ is the normal bundle and $T\Sigma$ is the tangent bundle of Σ , χ is the Euler number of the appropriate bundle, and $c_1(TM)$ is the first Chern class of the tangent bundle to M .

In the case of a globally convex congruence with the graph orientation, we have $\chi(T\Sigma) = 2$, $\chi(N\Sigma) = 2$ and $c_1 = (T\mathbb{T})[\Sigma] = 4$. The theorem then follows. ■

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