SPECTRA IN SOME INVERSE SEMIGROUP ALGEBRAS

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ABSTRACT

Let $\mathcal{S}$ be the semigroup with generators $s_j, s_j^* \, (j = 1, 2, \ldots, N)$ and defining relations $s_j^* s_j = 1, \;\; s_j^* s_k = \theta \; (j \neq k)$, where 1 is the identity and θ the zero. We determine the spectrum of the average of these generators in $\ell^1(\mathcal{S})/\mathbb{C}\theta$, showing it to be an ellipse together with its interior. The major and minor diameters of the ellipse are $1 + 1/N$ and $1 - 1/N$, respectively. So the interior is non-empty except in the degenerate case $N = 1$.

1. Introduction

The spectrum of an element in a Banach algebra, along with its functional calculus, constitutes a very powerful tool for the study of the Banach algebra. Yet it is often difficult to calculate the spectrum of a given element in a specific Banach algebra. For the case of $\ell^1(G)$ where G is the free group on n symbols, $n \geq 2$, the average of the generators and their inverses give examples of self-adjoint elements whose spectrum is an ellipse together with its interior; the necessary computations were carried out by Duncan and Williamson [6] by two different methods. Recently, Niels Laustsen asked the second author a similar question when the free group is replaced by an inverse semigroup related to the Cuntz O_n algebras. Laustsen was able to answer his question about whether the algebra was hermitian by using proposition 3.1.8 of Dales [4] together with a simple computation (see [5]). The method used does not determine the complete spectrum of the average of the generators and their inverses. Our aim here is to determine that complete spectrum. Of the two methods used in [6] only the second method seems to be feasible here, namely the use of generalised Chebyshev polynomials. This method seems to be particularly well suited to the Banach algebra $\ell^1(S)$ when the semigroup S has a well-behaved notion of the length of elements.

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Let $J = \{1, 2, \ldots, N\}$ and let $\mathcal{S}$ be the involution semigroup with generators s_j, s_j^* $(j \in J)$ subject to the relations:

$$s_j^* s_j = 1, \quad s_j^* s_k = \theta \quad (j \neq k)$$

where 1 is the identity and θ is the zero in the semigroup. The elements of $\mathcal{S}$ are the reduced words in s_j, s_j^* $(j \in J)$ and reductions occur only with the above relations. Then $\mathcal{S}$ is an inverse semigroup and * provides the generalised inverse in the usual way. In [3], Crabb and Munn point out that $\mathcal{S}$ is, in fact, a polycyclic semigroup in the sense of Nivat and Perrot [7]. In particular, when $N = 1$ we get the bicyclic semigroup whose Banach semigroup algebra is indeed hermitian by theorem 3.2.12 of Wordingham [8]. We include a proof of this. Note, for later reference, that in this case the ℓ^1-algebra has the following faithful representation Φ on a Hilbert space: define $\Phi(s_1)$ and $\Phi(s_1^*)$ to be the right and left shift operators on $\ell^2(\mathbb{N})$, and extend multiplicatively to $\mathcal{S}$ and then linearly to the algebra (see [3] or [5]).

As in the original Laustsen question, we shall work with the Banach algebra $\mathcal{A} = \ell^1(\mathcal{S})/\mathbb{C}\theta$. As is customary, elements of $\mathcal{S}$ will be identified with the corresponding point masses in $\mathcal{A}$. It is easy to see that the norm in $\mathcal{A}$ is just the ℓ^1 norm on $\mathcal{S}\backslash\{\theta\}$.

For $n \in \mathbb{N}$, let μ_n be the average of the words of length n, and let $\mu_0 = 1$. For simplicity we shall write μ_1 as μ. We shall prove that $\mathrm{Sp}(\mathcal{A}, \mu)$, the spectrum of μ in $\mathcal{A}$, is an ellipse together with its interior. Specifically,

$$\mathrm{Sp}(\mathcal{A}, \mu) = \left\{ \gamma + i\delta : \gamma,\ \delta \in \mathbb{R},\ \frac{\gamma^2}{A^2} + \frac{\delta^2}{B^2} \leq 1 \right\},$$

where $A = (N+1)/2N$ and $B = (N-1)/2N$; this degenerates to the line segment $[-1,\ 1]$ when $N = 1$. The main steps in the process are: firstly, to show thatSp$(\mathcal{A},\ \mu) = \mathrm{Sp}(\mathcal{B}, \mu)$ where $\mathcal{B}$ is the closed subalgebra of $\mathcal{A}$ generated by μ and 1; secondly, to establish that $\mathrm{Sp}(\mathcal{B}, \mu)$ is the ellipse plus interior defined above.

2. Preliminaries

Let $\mathcal{S}^\circ$ denote the set of words in $\mathcal{S}$ that use only the generators s_j $(j \in J)$, together with 1. In view of the defining relations, each word in $\mathcal{S}\backslash\{\theta\}$ can be expressed uniquely in the form uv^* where $u, v \in \mathcal{S}^\circ$. For $n \in \mathbb{N}$, let $\mathcal{W}_n$ be the set of words of length n in $\mathcal{S}$ and let

$$\mathcal{U}_n = \{uv^* \in \mathcal{W}_n : u, v \in \mathcal{S}^\circ,\ u \neq 1\}, \quad \mathcal{V}_n = \{v^* \in \mathcal{W}_n : v \in \mathcal{S}^\circ\},$$

so that $\mathcal{W}_n = \mathcal{U}_n \cup \mathcal{V}_n$ and $\mathcal{U}_n \cap \mathcal{V}_n = \emptyset$. Additionally, define $\mathcal{W}_0 = \{1\}$. Note that, if s is one of s_j $(j \in J)$, then $s\mathcal{W}_n \subseteq \mathcal{U}_{n+1}$, $s^*\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ and $s^*\mathcal{U}_n = \mathcal{W}_{n-1} \cup \{\theta\}$. For any finite subset $\mathcal{T}$ of $\mathcal{S}$, write $\Sigma\mathcal{T}$ for the sum of the elements of $\mathcal{T}$. Then, for $n \in \mathbb{N}$, $\Sigma\mathcal{W}_n = \Sigma\mathcal{U}_n + \Sigma\mathcal{V}_n$ and

$$\begin{aligned}
(s_1 + \cdots + s_N)\Sigma\mathcal{W}_n &= \Sigma\mathcal{U}_{n+1}, \\
(s_1^* + \cdots + s_N^*)\Sigma\mathcal{V}_n &= \Sigma\mathcal{V}_{n+1}, \\
(s_1^* + \cdots + s_N^*)\Sigma\mathcal{U}_n &= N\Sigma\mathcal{W}_{n-1}.
\end{aligned}$$

Since $2N\mu = s_1 + \cdots + s_N + s_1^* + \cdots + s_N^*$ it follows that

$$2N\mu\Sigma\mathcal{W}_n = \Sigma\mathcal{W}_{n+1} + N\Sigma\mathcal{W}_{n-1} \quad (n \geq 1).$$

The number of words in $\mathcal{S}$ of the form uv^* where $u, v \in \mathcal{S}^\circ$ with u of length r and v of length $n-r$, is $N^r N^{n-r} = N^n$. Hence $|\mathcal{W}_n| = (n+1)N^n$ and it follows that

$$\mu\mu_n = \frac{n+2}{2(n+1)}\mu_{n+1} + \frac{n}{2N(n+1)}\mu_{n-1} \quad (n \geq 1). \tag{1}$$

A simple induction argument shows that each μ_n is a polynomial of degree n in μ, and so the linear span of the μ_n ($n \geq 0$) is just the set of polynomials in μ. Call this set $\mathcal{P}$ and note that its closure is $\mathcal{B}$.

We can extend * to an involution on $\mathcal{A}$. For $a = \sum_{w \in \mathcal{S}} \alpha_w w$ where the $\alpha_w \in \mathbb{C}$, define

$$a^* = \sum_{w \in \mathcal{S}} \overline{\alpha_w} w^*$$

where $\bar{\alpha}$ denotes the complex conjugate of α. Note that if μ commutes with a then

$$\mu a^* = \mu^* a^* = (a\mu)^* = (\mu a)^* = a^*\mu^* = a^*\mu$$

so that μ also commutes with a^*. It is also useful to have the following notion of support. With a as above, define

$$\mathrm{supp}(a) = \{w \in \mathcal{S}\backslash\{\theta\} : \alpha_w \neq 0\}.$$

3. Main result

Let

$$\sigma = s_1 + \cdots + s_N + s_1^* + \cdots + s_N^* \quad (= 2N\mu).$$

Then, for $n \in \mathbb{N}$ and $z \in \mathcal{A}$ with $\mathrm{supp}(z) \subseteq \mathcal{W}_n$, write $\sigma z = \vec{z} + \bar{z}$ where $\mathrm{supp}(\vec{z}) \subseteq \mathcal{W}_{n+1}$ and $\mathrm{supp}(\bar{z}) \subseteq \mathcal{W}_{n-1}$.

Lemma 1. *Let $n \in \mathbb{N}$ and let $z \in \mathcal{A}$ with $\mathrm{supp}(z) \subseteq \mathcal{W}_n$ and $z = z^*$. Then*

$$\|\bar{z} - \bar{z}^*\| \leq \|\vec{z} - \vec{z}^*\|.$$

If $\vec{z} = \vec{z}^*$ then $z \in \mathcal{P}$.

PROOF. Let $z = \sum_{w \in \mathcal{W}_n} \alpha_w w$ where the $\alpha_w \in \mathbb{C}$. Note that, since $z = z^*$, $\overline{\alpha_w} = \alpha_{w^*}$. We see that

$$\vec{z} = \sum_{w \in \mathcal{W}_n} \alpha_w (s_1 + \cdots + s_N) w + \sum_{w \in \mathcal{V}_n} \alpha_w (s_1^* + \cdots + s_N^*) w$$

and

$$\vec{z}^* = \sum_{w \in \mathcal{W}_n} \overline{\alpha_w} w^* (s_1^* + \cdots + s_N^*) + \sum_{w \in \mathcal{V}_n} \overline{\alpha_w} w^* (s_1 + \cdots + s_N).$$

From the reduced forms of the words involved we deduce that the coefficient of $s_j w$ in $\vec{z}$ is α_w and the coefficient of the same word in $\vec{z}^*$ is $\overline{\alpha_{(s_j\tilde{w})^*}}$ where $\tilde{w}$ is the word obtained from w by deleting the last generator in its reduced form (when w has length 1, $\tilde{w}=1$). Since $\overline{\alpha_{(s_j\tilde{w})^*}} = \alpha_{(s_j\tilde{w})}$ we get

$$\|\vec{z}-\vec{z}^*\| \geq \sum_{\substack{w\in\mathcal{W}_n\\ j\in J}} |\alpha_w - \alpha_{s_j\tilde{w}}|. \tag{2}$$

Next, we have

$$\overleftarrow{z} = \sum_{w\in\mathcal{U}_n} \alpha_w(s_1^* + \cdots + s_N^*)w \quad \text{and} \quad \overleftarrow{z}^* = \sum_{w\in\mathcal{U}_n} \overline{\alpha_w} w^*(s_1 + \cdots + s_N).$$

For any $w\in\mathcal{W}_{n-1}$, the coefficients of w in $\overleftarrow{z}$ and $\overleftarrow{z}^*$ are $\sum_{j\in J}\alpha_{s_j w}$ and $\sum_{j\in J}\overline{\alpha_{s_j w^*}}$, respectively. Hence

$$\begin{aligned}\|\overleftarrow{z}-\overleftarrow{z}^*\| &= \sum_{w\in\mathcal{W}_{n-1}} \left|\sum_{j\in J}(\alpha_{s_j w} - \overline{\alpha_{s_j w^*}})\right| \leq \sum_{\substack{w\in\mathcal{W}_{n-1}\\ j\in J}} |\alpha_{s_j w} - \overline{\alpha_{s_j w^*}}| \\ &= \sum_{\substack{w\in\mathcal{W}_{n-1}\\ j\in J}} |\alpha_{s_j w} - \alpha_{w s_j^*}| = \sum_{\substack{w\in\mathcal{W}_{n-1}\\ j\in J}} |\alpha_{w s_j^*} - \alpha_{s_j(\widetilde{w s_j^*})}| \leq \|\vec{z}-\vec{z}^*\|.\end{aligned}$$

Now suppose that $\vec{z}=\vec{z}^*$ and write s for the generator s_1. It follows from (2) that, for each $w\in\mathcal{W}_n$, $\alpha_w = \alpha_{s\tilde{w}}$. Applying this successively gives $\alpha_w = \alpha_{s^N}$. So $z = \alpha_{s^N}\sum_{w\in\mathcal{W}_n} w = \alpha\mu_n$ for some scalar α. Hence $z\in\mathcal{P}$. ■

To show that $\mathrm{Sp}(\mathcal{A},\mu) = \mathrm{Sp}(\mathcal{B},\mu)$ it is enough to prove the following.

Proposition 2. *Let $a\in\mathcal{A}$ with $a\sigma = \sigma a$. Then $a\in\mathcal{B}$.*

Proof. Suppose, to begin with, that $a = a^*$. Write

$$a = a_0 + a_1 + a_2 + \cdots \quad \text{and} \quad \sigma a = b_0 + b_1 + b_2 + \cdots$$

where $\mathrm{supp}(a_n)$ and $\mathrm{supp}(b_n)$ are subsets of $\mathcal{W}_n$. Since $(\sigma a)^* = \sigma a$, each $b_n = b_n^*$.

Assume towards a contradiction that a_k is the first term of a not in $\mathcal{P}$. Certainly $k\geq 1$. Then, by Lemma 1, $\overrightarrow{a_k} \neq \overrightarrow{a_k}^*$ so we can write $\|\overrightarrow{a_k} - \overrightarrow{a_k}^*\| = \varepsilon > 0$. Since $b_{k+1} = \overrightarrow{a_k} + \overleftarrow{a_{k+2}}$ it follows that $\|\overleftarrow{a_{k+2}} - \overleftarrow{a_{k+2}}^*\| = \varepsilon$. Lemma 1 then gives $\|\overrightarrow{a_{k+2}} - \overrightarrow{a_{k+2}}^*\| \geq \varepsilon$. Continuing we get

$$\|\overrightarrow{a_{k+2m}} - \overrightarrow{a_{k+2m}}^*\| \geq \varepsilon \quad (m = 0,1,2,\ldots).$$

But $\|\overrightarrow{a_n}\| \leq \|\overrightarrow{a_n}\| + \|\overleftarrow{a_n}\| = \|\overrightarrow{a_n} + \overleftarrow{a_n}\| = \|\sigma a_n\| \to 0$, which gives a contradiction. It follows that $a_n\in\mathcal{P}$ $(n = 0,1,2,\ldots)$ and hence $a = \lim_{n\to\infty}(a_0 + a_1 + \cdots + a_n)\in\mathcal{B}$.

Finally, remove the condition $a = a^*$. The above argument applies to both $b = a + a^*$ and $c = i(a - a^*)$. Hence $a = \frac{1}{2}(b - ic)\in\mathcal{B}$, as required. ■

Lemma 3. *For $n\geq 0$, let ρ_n be a polynomial of degree n such that* $\mathrm{supp}(\rho_n(\mu))\subseteq\mathcal{W}_n$. *Then*

$$\mathrm{Sp}(\mathcal{B},\mu)=\{\lambda\in\mathbb{C}:|\rho_n(\lambda)|\le\|\rho_n(\mu)\|\quad(n\ge 0)\}.$$

PROOF. The inclusion $\mathrm{Sp}(\mathcal{B},\mu)\subseteq\{\lambda\in\mathbb{C}:|\rho_n(\lambda)|\le\|\rho_n(\mu)\|\quad(n\ge 0)\}$ follows from elementary properties of the spectrum.

For the reverse inclusion, suppose $\lambda\notin\mathrm{Sp}(\mathcal{B},\mu)$. Then there is a polynomial p such that $\|p(\mu)\|<|p(\lambda)|$. Note that the linear span of the $\rho_n\,(n\ge 0)$ is $\mathcal{P}$. Let $p=\sum_{n=0}^{M}\beta_n\rho_n$. Then

$$\sum_{n=0}^{M}|\beta_n|\|\rho_n(\mu)\| = \|p(\mu)\| < |p(\lambda)|\le\sum_{n=0}^{M}|\beta_n||\rho_n(\lambda)|.$$

Hence there exists n such that $\|\rho_n(\mu)\|<|\rho_n(\lambda)|$ and the result follows. ■

Polynomials ρ_n $(n\ge 0)$ satisfying the conditions of Lemma 3 can be defined by $\rho_n(\mu)=\gamma_n\mu_n$ where the γ_n are non-zero scalars. We mention, although it is not required here, that such ρ_n are scalar multiples of generalised Chebyshev polynomials π_n in the sense of [6].

In the proof of the main result, below, U_n $(n\ge 0)$ denotes the Chebyshev polynomial defined by $U_n(\cos\phi)=\sin(n+1)\phi/\sin\phi$. Recall the recurrence relation

$$2xU_n(x)=U_{n+1}(x)+U_{n-1}(x)\quad(n\ge 1). \tag{3}$$

Theorem 4. *Let* $A=\dfrac{N+1}{2N}$ *and* $B=\dfrac{N-1}{2N}$. *Then* $\mathrm{Sp}(\mathcal{A},\mu)=E$ *where*

$$E=\left\{\gamma+i\delta:\gamma,\ \delta\in\mathbb{R},\ \frac{\gamma^2}{A^2}+\frac{\delta^2}{B^2}\le 1\right\}$$

when $N\ne 1$, *and* $E=[-1,\,1]$ *when* $N=1$.

PROOF. In view of Proposition 2, it is enough to prove that $\mathrm{Sp}(\mathcal{B},\mu)=E$. We give the proof for $N\ne 1$. Minor adjustments cover the remaining case.

For $n\ge 0$, define $\rho_n(\mu)=(n+1)N^{n/2}\mu_n$. Then, from (1), for $n\ge 1$, we get $2\sqrt{N}\mu\rho_n(\mu)=\rho_{n+1}(\mu)+\rho_{n-1}(\mu)$. Comparing this with (3), and ρ_j with U_j $(j=0,\,1)$, we deduce that, for $n\ge 0$, $\rho_u(\mu)=U_n(\sqrt{N}\mu)$. Hence $\lambda\in\mathrm{Sp}(\mathcal{B},\mu)$ if and only if, for all $n\ge 0$, $|\rho_n(\lambda)|\le\|\rho_n(\mu)\|$, *i.e.* $|U_n(\sqrt{N}\lambda)|\le(n+1)N^{n/2}$, *i.e.*

$$|\sin(n+1)\zeta|\le(n+1)N^{n/2}|\sin\zeta| \tag{4}$$

whenever $\cos\zeta=\sqrt{N}\lambda$. Let $\lambda=\gamma+i\delta$ and $\zeta=\xi+i\eta$, where $\gamma,\delta,\xi,\eta\in\mathbb{R}$, and let $\eta_0=\log\sqrt{N}$. Note that $\cosh^2\eta_0=\frac{1}{4}(N+2+1/N)=NA^2$ and similarly $\sinh^2\eta_0=NB^2$. Then

$$\begin{aligned}\lambda\in E\quad&\Leftrightarrow\quad\frac{\gamma^2}{A^2}+\frac{\delta^2}{B^2}\le 1\\ &\Leftrightarrow\quad\frac{\cos^2\xi\ \cosh^2\eta}{NA^2}+\frac{\sin^2\xi\ \sinh^2\eta}{NB^2}\le 1\end{aligned}$$

$$\Leftrightarrow \quad \frac{\cos^2\xi \ \cosh^2\eta}{\cosh^2\eta_0} + \frac{\sin^2\xi \ \sinh^2\eta}{\sinh^2\eta_0} \le 1$$

$$\Leftrightarrow \quad \cosh^2\eta \le \cosh^2\eta_0$$

$$\text{(since } \cosh^2\eta > \cosh^2\eta_0 \Leftrightarrow \sinh^2\eta > \sinh^2\eta_0\text{)}$$

$$\Leftrightarrow \quad |\eta| \le |\eta_0|.$$

Thus $\lambda \in E$ if and only if $|\eta| \le \log\sqrt{N}$.

To complete the proof we show that the latter condition is satisfied if and only if (4) holds for all $n \ge 0$. Inequality (4) is equivalent to

$$\sin^2(n+1)\xi + \sinh^2(n+1)\eta \le (n+1)^2 N^n(\sin^2\xi + \sinh^2\eta),$$

and this will hold if

$$\sin^2(n+1)\xi \le (n+1)^2 N^n \sin^2\xi \tag{5}$$

and

$$\sinh^2(n+1)\eta \le (n+1)^2 N^n \sinh^2\eta. \tag{6}$$

An elementary induction argument gives $|\sin(n+1)\xi| \le (n+1)|\sin\xi|$, and hence (5), for all $n \ge 0$ and all $\xi \in \mathbb{R}$. Another induction argument gives $\eta \mapsto \sinh(n+1)\eta/\sinh\eta$ increasing for $\eta \ge 0$. Hence it is enough to verify (6) for $\eta = \log\sqrt{N}$. Then (6) routinely reduces to $n+1 \le nN + N^{-n}$, which clearly holds for all $n \ge 0$. Finally, suppose that (4) holds for all $n \ge 0$. Then, for all $n \ge 1$

$$|\sin(n+1)\zeta|^{1/n} \le (n+1)^{1/n}\sqrt{N}|\sin\zeta|^{1/n}$$

and, for $\sin\zeta \ne 0$, letting $n \to \infty$ we get $e^{|\eta|} \le \sqrt{N}$, *i.e.* $|\eta| \le \log\sqrt{N}$. ■

4. Concluding remarks

1. When $N \ne 1$, $\mathrm{Sp}(\mathcal{A}, \mu)$ has non-empty interior and hence $\mathcal{A}$ is not hermitian. On the other hand, as was mentioned in the introduction, $\mathcal{A}$ *is* hermitian when $N = 1$, *i.e.* when $\mathcal{S}$ is the bicyclic semigroup. This follows from theorem 3.2.12 in [8]. However, as this reference is difficult to access, we offer here the following proof distilled from results in [8].

Proposition 5. *Let* $N = 1$. *Then* $\mathcal{A}$ *is hermitian.*

Proof. It is enough to show that $1 + a^*a$ is invertible for each $a \in \mathcal{A}$. See, for example, Bonsall and Duncan [2, remark on page 226].

Write s for s_1 so that $s^*s = 1$ and $\mathcal{S} = \{s^m s^{*n} : m, n = 0, 1, 2, \ldots\}$. Let $\mathcal{C}$ be the ℓ^1-algebra of the free group on one generator g and let $\psi: \mathcal{A} \to \mathcal{C}$ be the homomorphism defined by $\psi(s) = g$ and $\psi(s^*) = g^{-1}$. We show that the kernel of ψ is the closed left ideal $\mathcal{I}$ of $\mathcal{A}$ generated by $\{p_n = e_n - e_{n+1} : n = 0, 1, 2, \ldots\}$ where $e_n = s^n s^{*n}$. Since $\psi(p_n) = 0$ $(n = 0, 1, 2, \ldots)$ it follows that $\mathcal{I}$ is contained in the kernel. For the reverse inclusion, let $x \in \mathcal{A}$ with $\psi(x) = 0$. We can write $x = \sum \alpha_{kn} s^k e_n$ where $\alpha_{kn} \in \mathbb{C}$ and $\sum |\alpha_{kn}| < \infty$, the sum being taken over

$k \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots,$ where $s^{-m} = s^{*m}$ $(m \in \mathbb{N})$. Then $\psi(x) = \sum \alpha_{kn} g^k = 0$ so that$\sum_n \alpha_{kn} = 0$. Hence $x = \sum \alpha_{kn} s^k q_n$ where $q_n = e_n - 1 = -p_0 - \cdots - p_{n-1} \in \mathcal{I}$. Since $\mathcal{I}$ is a closed left ideal of $\mathcal{A}$ it follows that $x \in \mathcal{I}$.

Now let $a \in \mathcal{A}$ and suppose that $\mathcal{A}(1 + a^*a) \neq \mathcal{A}$. Let $\mathcal{L}$ be a maximal left ideal of $\mathcal{A}$ that contains $\mathcal{A}(1 + a^*a)$.

Case 1. Suppose $\mathcal{I} \subseteq \mathcal{L}$. Then $\mathcal{L}/\mathcal{I}$ is a proper left ideal of $\mathcal{A}/\mathcal{I}$. Since $\mathcal{A}/\mathcal{I}$ is isomorphic to $\mathcal{C}$ and $\psi(1 + a^*a)$ is invertible, $\mathcal{C}$ being hermitian, we have a contradiction.

Case 2. Suppose $\mathcal{I} \not\subseteq \mathcal{L}$. Then $p = p_n \notin \mathcal{L}$ for some n. Note that $(1 - e_1)s = 0 = s^*(1 - e_1)$ so that $p_0 \mathcal{A} p_0 = \mathbb{C} p_0$. Hence $p\mathcal{A}p = s^n p_0 s^{*n} \mathcal{A} s^n p_0 s^{*n} = \mathbb{C} s^n p_0 s^{*n} = \mathbb{C}p$. Since $\mathcal{L}$ is a maximal left ideal of $\mathcal{A}$, $\mathcal{L} + \mathcal{A}p = \mathcal{A}$. So there exist $q \in \mathcal{L}$ and $e \in \mathcal{A}p$ such that $q + e = 1$. Then $e \neq 0$ and $\mathcal{A}(1 - e) = \mathcal{A}q \subseteq \mathcal{L}$. If $x \in \mathcal{L}$ then $e^*x \in \mathcal{L}$, $e^*x(1 - e) \in \mathcal{L}$ and hence $e^*xe \in \mathcal{L}$. Also, $e^*xe \in p\mathcal{A}p = \mathbb{C}p$. Since $p \notin \mathcal{L}$, $e^*xe = 0$. Taking $x = 1 + a^*a$ we get $e^*e + (ae)^*ae = 0$. Since $\mathcal{A}$ has a faithful representation on a Hilbert space (Φ defined in the introduction) it follows that $e^*e = 0$ and hence that $e = 0$, which is a contradiction.

We must therefore have $\mathcal{A}(1 + a^*a) = \mathcal{A}$. So there exists $b \in \mathcal{A}$ such that $b(1 + a^*a) = 1$ and $(1 + a^*a)b^* = 1$. Hence $1 + a^*a$ is invertible. ∎

2. The recurrence formula (1) has coefficients that are asymptotically constant and hence a theorem of Poincaré (see, for example, Agarwal [1]) provides the order of magnitude of the solutions. This affords a proof only that $\mathrm{Sp}(\mathcal{A}, \mu)$ contains E.

3. Here t, $\lambda \in \mathbb{C}$ and μ is as before. Take $G(t, \lambda)$ as in [6], *i.e.*

$$G(t, \lambda) = \sum_{n=0}^{\infty} t^n \pi_n(\lambda).$$

The recurrence formula for π_n is

$$\lambda \pi_n(\lambda) = \pi_{n+1}(\lambda) + \frac{1}{4N} \pi_{n-1}(\lambda)$$

with $\pi_0(\lambda) = 1$ and $\pi_1(\lambda) = \lambda$. This leads easily to

$$\left[\left(1 + \frac{t^2}{4N}\right) - t\mu\right] G(t, \mu) = 1 \quad (|t| < 2).$$

Now let $\lambda(t) = 1/t + t/(4N)$. This maps the region $0 < |t| < 2$ conformally onto the complement of E and we get the resolvent formula

$$[\lambda(t) - \mu]^{-1} = \sum_{n=0}^{\infty} t^{n+1} \pi_n(\mu).$$

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