

# A NOTE ON RESTRICTIONS OF LINEAR RELATIONS

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## ABSTRACT

This note is devoted to the answer to the following question: If  $X$  and  $Y$  are normed spaces and  $T$  is a linear relation from  $X$  into  $Y$  such that every closed infinite-dimensional subspace  $M$  of  $D(T)$  contains a closed infinite-dimensional subspace  $N$  for which the restriction of  $T$  to  $N$  is  $F_+$  (resp. Tauberian), is  $T$  an  $F_+$ -relation (resp. Tauberian)?

## 1. Introduction

Throughout this note,  $X$  and  $Y$  are infinite-dimensional normed spaces,  $B_X$  the closed unit ball of  $X$ ,  $X'$  the first dual space and  $\tilde{X}$  the completion of  $X$ . We write  $J$  for the canonical injection of a given normed space into its second dual space. A linear relation  $T: X \rightarrow Y$  is a mapping from a subspace  $D(T) \subseteq X$ , called the domain of  $T$ , into  $P(Y) \setminus \{\emptyset\}$  (the collection of nonempty subsets of  $Y$ ) such that  $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$  for all non zero scalars  $\alpha, \beta \in K$  and  $x_1, x_2 \in D(T)$ . The class of such relations is denoted by  $LR(X, Y)$ . If  $T$  maps the points of its domain to singletons, then  $T$  is said to be an operator. Continuous defined everywhere operators are referred to as bounded operators.

Let  $M$  be a subspace of  $X$ . Then  $M^\perp := \{x' \in X': x'(x) = 0 \text{ for all } x \in M\}$ . If  $T \in LR(X, Y)$ , then the restriction  $T|_{M \cap D(T)}$  is defined by its graph  $G(T|_{M \cap D(T)}) := \{(m, y) : m \in M \cap D(T), y \in Tm\}$  and the conjugate  $T'$  of  $T$  is given by  $G(T') := G(-T^{-1})^\perp \subseteq Y' \times X'$  where  $\langle (y, x), (y', x') \rangle := \langle x, x' \rangle + \langle y, y' \rangle$  and  $T^{-1}$  is the inverse of  $T$  whose graph is  $G(T^{-1}) := \{(y, x) \in Y \times X : (x, y) \in G(T)\}$ . We write  $\underline{Q}$  for the quotient map from  $Y$  onto  $Y/T(0)$ .

$T \in LR(X, Y)$  is said to be injective if  $N(T) := T^{-1}(0) = \{0\}$ ; continuous if for each neighbourhood  $\Omega$  in  $R(T) := TD(T)$ ,  $T^{-1}(\Omega)$  is a neighbourhood in  $D(T)$  equivalent to  $\|T\| := \|\underline{Q}T\| < \infty$ ; open if its inverse is continuous; precompact if  $\underline{Q}TB_{D(T)}$  is totally bounded, and  $F_+$  if there exists a finite-codimensional subspace  $M$  of  $X$  for which  $\widetilde{T|_{M \cap D(T)}}$  is injective and open and  $T$  is called Tauberian if  $(T'')^{-1}JY \subseteq J(D(T))$ , [4].

Notice that when  $X$  and  $Y$  are Banach spaces and  $T$  is a bounded operator,  $T$  is  $F_+$  if and only if  $T$  is upper semi-Fredholm (that is, its null space  $N(T)$  is finite-dimensional and its range space  $R(T)$  is closed) [4, V. 1.7], and the definition of Tauberian linear relation is the standard one for bounded operators of Kalton and Wilansky [9].

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Bounded Tauberian operators in Banach spaces appeared in response to a problem in summability [6]. Since then, they have found application in many different situations: study of isomorphic properties of Banach spaces [11], factorisation of operators [5; 11], semigroups of generalised Fredholm operators [7; 8], etc. Tauberian operators between Fréchet spaces are discussed in Bonet and Ramanujan [1]. Unbounded Tauberian operators in normed spaces were introduced and studied in [2] and [3] and generalised to linear relations in [4].

From the beginning, Tauberian and  $F_+$  linear relations have been compared. Many results obtained show the analogy between both classes of linear relations (see [4]). The results presented in this note illustrate that not all the properties enjoyed by  $F_+$  linear relations are shared by Tauberian linear relations.

It is well-known that the restriction of a bounded upper semi-Fredholm (resp. Tauberian) operator in Banach spaces to any closed subspace is again an upper semi-Fredholm (resp. Tauberian) operator. Conversely, we have the following result.

**Proposition 1.** [10]. *Let  $X, Y$  be Banach spaces and let  $T: X \rightarrow Y$  be a bounded operator such that every closed infinite-dimensional subspace  $M$  of  $X$  contains a closed infinite-dimensional subspace  $N$  for which  $T|_N$  is upper semi-Fredholm. Then  $T$  is upper semi-Fredholm.*

The homologous results for  $F_+$  and Tauberian linear relations will now be investigated.

**Proposition 2.** *Let  $T \in LR(X, Y)$ . Then the following properties are equivalent:*

- (i) *There is no closed infinite-dimensional subspace  $M$  of  $D(T)$  such that  $T|_M$  is precompact.*
- (ii) *For every closed infinite-dimensional subspace  $M$  of  $D(T)$  there exists a closed infinite-dimensional subspace  $N$  of  $M$  for which  $T|_N$  is not precompact.*

PROOF. (i)  $\Rightarrow$  (ii) Note that the precompactness is a hereditary property [4, V. 2.2].

Assume (ii). Let  $M$  be a closed infinite-dimensional subspace of  $D(T)$  such that  $T|_M$  is precompact. Then, by hypothesis, there exists a closed infinite-dimensional subspace  $N$  of  $M$  for which  $T|_N$  is not precompact, contradicting the hereditary property of the precompactness. Hence (ii)  $\Rightarrow$  (i). ■

**Theorem 3.** *Let  $T \in LR(X, Y)$  be closed. Then the following properties are equivalent:*

- (i)  *$T$  is  $F_+$ .*
- (ii) *For every closed infinite-dimensional subspace  $M$  of  $D(T)$  there exists a closed infinite-dimensional subspace  $N$  of  $M$  such that  $T|_N$  is  $F_+$ .*

PROOF. Suppose that  $T \in F_+$  equivalently  $\inf\{\|T|_Z\|: Z \text{ infinite-dimensional subspace of } D(T)\} > 0$ , [4, V. 2.4]. Therefore  $T|_M \in F_+$  whenever  $M$  is an infinite-dimensional subspace of  $D(T)$ .

Conversely, assume (ii). Since a linear relation  $S \in LR\{X, Y\}$  is precompact if and only if  $\inf\{\|S|_Z\|: Z \text{ closed finite-codimensional subspace } X\} = 0$ , [4, V. 2.2], we have

that each closed infinite-dimensional subspace  $M$  of  $D(T)$  contains another such subspace  $N$  for which  $QT|_N$  is not precompact. Hence  $QT|_Z$  is not precompact whenever  $Z$  is a closed infinite-dimensional subspace of  $D(T)$  by Proposition 2. Now, as  $T$  is closed, from [4, V. 1.6] it follows that  $QT$ , and hence  $T$ , is  $F_+$ . ■

This theorem is a generalisation of Proposition 1 which was basically proved by Kato [10] with a different scheme of proof.

The example 4 below shows that the Theorem 3 for Tauberian linear relations is false even for bounded operators in Banach spaces.

Recall that if  $X$  and  $Y$  are Banach spaces, then  $X$  is called somewhat reflexive if every closed infinite-dimensional subspace of  $X$  contains an infinite-dimensional reflexive subspace and we say that  $X$  and  $Y$  are totally incomparable if no closed infinite-dimensional subspace of  $X$  is isomorphic to a subspace of  $Y$ .

*Example 4.* Let  $X$  and  $Y$  be totally incomparable Banach spaces and let  $X$  be somewhat reflexive non reflexive. Then there exists a bounded operator  $T: X \oplus Y \rightarrow Y$  such that each closed infinite-dimensional subspace  $M$  of  $X \oplus Y$  contains another such subspace  $N$  for which  $T|_N$  is Tauberian and  $T$  is not Tauberian.

Define  $T: X \oplus Y \rightarrow Y$  by  $T(x,y) := y$ . Then, as  $N(T) = X$  is not reflexive,  $T$  is not Tauberian by [7; 1]. Let  $M$  be a closed infinite-dimensional subspace of  $X \oplus Y$ . Let us consider two cases for  $M$ :

**Case I:**  $M$  has not infinite-dimensional reflexive subspaces. Then by [7; 3]  $T|_M$  is Tauberian if and only if  $T|_M$  is upper semi-Fredholm. But since  $X$  is somewhat reflexive,  $N(T|_M) = M \cap X$  is finite-dimensional; otherwise,  $N(T|_M)$  would contain an infinite-dimensional reflexive subspace, which contradicts our assumption over  $M$ . By a result of Rosenthal [12; 2], two Banach spaces  $U$  and  $V$  are totally incomparable if and only if for every Banach space  $W$  with closed subspaces  $A$  and  $B$  isomorphic to  $U$  and  $V$ , the sum  $A+B$  is closed. This property implies that  $N(T)+M$  is closed in  $X \oplus Y$  and thus  $R(T|_M) = TM$  is closed. Therefore  $T|_M$  is upper semi-Fredholm, as desired.

**Case II:**  $M$  contains an infinite-dimensional reflexive subspace, say  $N$ . Then  $T|_N$  is Tauberian by [7; 1].

It is well-known that the James' space is an example of somewhat reflexive non reflexive Banach space and since every closed infinite-dimensional subspace of  $l_1$  contains another such subspace that is isomorphic to  $l_1$  we conclude that the James' space and  $l_1$  are totally incomparable.

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