FINITE GROUPS WITH CENTRAL AUTOMORPHISM GROUP OF MINIMAL ORDER

By M.J. CURRAN*
Department of Mathematics and Statistics, University of Otago, Dunedin, New Zealand

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ABSTRACT

For any group G, the centre of the inner automorphism group, $Z(Inn(G))$, is always contained in the group $Aut_c(G)$ of central automorphisms of G. In this paper we consider finite groups for which this lower bound is attained, that is $Aut_c(G) = Z(Inn(G))$, and give a characterisation of such groups.

1. Introduction

In this paper G always denotes a finite group and $Z(G)$ its centre. An automorphism σ of G is said to be central if σ commutes with every automorphism in $Inn(G)$, the group of inner automorphisms of G, or equivalently, if $g^{-1}\sigma(g)$ lies in $Z(G)$, for all g in G. The central automorphisms, denoted $Aut_c(G)$, fix the commutator subgroup G' of G pointwise and form a normal subgroup of the full automorphism group $Aut(G)$.

The central automorphism group is as large as possible when all automorphisms are central, that is $Aut_c(G) = Aut(G)$. In this case G must be nilpotent of class 2, so without loss of generality (see [2]) we may restrict attention to when G is a p-group. Non-abelian p-groups G such that $Aut_c(G) = Aut(G)$ have been well studied. See Curran and McCaughan [2] for references.

It is also possible to consider the opposite extreme, where the central automorphism group is as small as possible. Clearly $Z(Inn(G)) \leq Aut_c(G)$, for any group G. In this paper, we consider non-abelian groups G for which $Aut_c(G) = Z(Inn(G))$, that is $Aut_c(G)$ can be considered to be of minimal order. When G is arbitrary, $Aut_c(G)$ and $Z(Inn(G))$ may coincide because both these subgroups of $Aut(G)$ are trivial. However, the situation is more interesting when G is a p-group, because then both subgroups are non-trivial. For this reason we concentrate on p-groups, although more general groups are also considered.

In Section 2 we consider the well-known Adney-Yen map, used to count the number of central automorphisms of a group, and give a special case of this used to prove the following main result of Section 3:

Theorem 1.1. *Let G be a finite non-abelian p-group. If $Aut_c(G) = Z(Inn(G))$ then $Z(G) \leq G'$, and furthermore, $Aut_c(G) = Z(Inn(G))$ if and only if $Hom(G/G', Z(G)) \approx Z(G/Z(G))$.*

*E-mail: jcurran@maths.otago.ac.nt

To conclude the paper, some examples of p-groups that satisfy the condition of this theorem are given.

2. Preliminary definitions and results

Throughout this paper all groups are assumed to be finite and the following notation is used: $Hom(G, A)$ denotes the group of homomorphisms of G into an abelian group A; C_m the cyclic group of order m; C_m^r the direct product of r copies of C_m; $exp(G)$ the exponent of G. If G is a non-abelian p-group, terms in the upper central series of G are given by $1 < Z_1(G) = Z(G) < Z_2(G) < \ldots$, and terms in the lower central series by $G > \gamma_2(G) = G' > \gamma_3(G) > \ldots$. Also, a non-abelian group G that has no non-trivial abelian direct factor is said to be purely non-abelian.

We begin this section by recalling the Adney-Yen map: if $\sigma \in Aut_c(G)$, define the map $\theta : Aut_c(G) \to Hom(G, Z(G))$ by $\theta(\sigma) = f_\sigma$, where $f_\sigma(g) = g^{-1}\sigma(g)$, for all $g \in G$. Adney and Yen introduced the map θ and proved the following two results in [1].

Theorem 2.1. *Let $\theta : Aut_c(G) \to Hom(G, Z(G))$ be the Adney-Yen map. Then*
(*i*) *θ is always one-to-one.*
(*ii*) *θ is onto if and only if G is purely non-abelian.*

Since for any group $Hom(G, Z(G)) \approx Hom(G/G', Z(G))$, the size of $Aut_c(G)$ can be determined from the well known result below:

Corallary 2.2. *If G is purely non-abelian, then $|Aut_c(G)| = |Hom(G/G', Z(G))|$.*

For ease of notation, denote by C^* the subgroup $C_{Aut_c(G)}(Z(G))$ of central automorphisms fixing $Z(G)$ pointwise. The next result, used in Section 3, shows that not only can we determine the size of C^* but we can also determine its structure.

Theorem 2.3. *For any non-abelian group G the restriction of the Adney-Yen map $\theta : C^* \to Hom(G, Z(G))$ is a homomorphism and $C^* \approx Hom(G/G'Z(G), Z(G))$.*

Proof. Suppose $\sigma, \tau \in C^*$. Then for any $g \in G$, $f_{\sigma\tau}(g) = g^{-1}\sigma(\tau(g)) = g^{-1}\sigma(gg^{-1}\tau(g)) = g^{-1}\sigma(g)\sigma(g^{-1}\tau(g)) = g^{-1}\sigma(g)\ g^{-1}\tau(g)$, since $g^{-1}\tau(g) \in Z(G)$ and is fixed by σ. That is, $f_{\sigma\tau}(g) = f_\sigma(g) f_\tau(g)$, so $\theta(\sigma\tau) = \theta(\sigma)\theta(\tau)$ and θ is a homomorphism.

In general, θ may not be onto $Hom(G, Z(G))$, but if we define $\bar{\theta} : C^* \to Hom(G/G'Z(G)), Z(G))$ by $\bar{\theta}(\sigma) = \bar{f}_\sigma$, where $\bar{f}_\sigma(gG'Z(G)) = f_\sigma(g)$ for any $g \in G$, then it is straightforward to verify $\bar{\theta}$ is a well-defined isomorphism (even if G has an abelian factor). For example $\bar{\theta}$ is onto, for given any $f \in Hom(G/G'Z(G), Z(G))$, define σ_f: $G \to G$ by $\sigma_f(g) = gf(\bar{g})$, where $\bar{g} = gG'Z(G)$. Now $\sigma_f \in End(G)$ and if $g \in Ker(\sigma_f)$ then $g = f(\bar{g})^{-1} \in Z(G)$, so $\bar{g} = 1$. Therefore $1 = \sigma_f(g) = gf(\bar{g}) = g$, so $Ker(\sigma_f) = 1$ and $\sigma_f \in Aut(G)$. Further σ_f is clearly central, fixes $Z(G)$ and $\theta(\sigma_f) = f$. ■

The following properties of C^*, although not needed in Section 3, are of interest.

Theorem 2.4. *Let G be any group and* $C^* = C_{Aut_c(G)}(Z(G))$.

(*i*) *If G is abelian or* $Z(G) = 1$ *then* $C^* = 1$.

(*ii*) *If G is a p-group, then* $C^* = 1 \Leftrightarrow G$ *is abelian*.

(*iii*) C^* *is abelian and if G is a p-group so is* C^*.

(*iv*) $C^* \geq Z(Inn(G))$ *and* $C^* = Z(Inn(G)) \Leftrightarrow Hom(G/G'Z(G), Z(G)) \approx Z(Inn(G))$.

(*v*) *If G is nilpotent of class 2, then* $C^* \approx Hom(G/Z(G), Z(G))$ *and* $C^* \geq Inn(G)$.

(*vi*) *If* $Z(G) \leq G'$, *then* $Aut_c(G) = C^* \approx Hom(G/G', Z(G))$.

PROOF.

(i) If G is abelian then $Aut_c(G) = Aut(G)$, and the only automorphism fixing $Z(G) = G$ is the identity map. If $Z(G) = 1$, then $Aut_c(G) = 1$, so $C^* = 1$.

(ii) If G is a p-group, and $C^* = 1$, then by (2.3) $G = G'Z(G)$. Thus $G = \langle G', Z(G)\rangle \leq Z(G)$, since $G' \leq \Phi(G)$, and so G is abelian.

(iii) From (2.3).

(iv) Every inner automorphism fixes $Z(G)$ pointwise, so $C^* \geq Inn(G) \cap Aut_c(G) = Z(Inn(G))$.

(v) If G is nilpotent of class 2, then $G' \leq Z(G)$ and the isomorphism follows from (2.3). Further $Inn(G)$ is abelian, so $C^* \geq Z(Inn(G)) = Inn(G)$.

(vi) $Aut_c(G)$ always fixes G', so if $Z(G) \leq G'$ then $Aut_c(G)$ fixes $Z(G)$ and thus $Aut_c(G) = C^*$. The isomorphism follows from (2.3). ∎

Remarks 2.5. Unlike C^* in (2.4 iii), $Aut_c(G)$ may not be abelian and may not be a p-group when G is a p-group.

A group G such that $Z(G) \leq G'$ is called a stem group by Hall, who showed in [4] that such a group is an important building block of its isoclinic family of related groups. (2.4 vi) shows that for a stem group the structure of $Aut_c(G)$ is particularly simple. Jamili and Mousavi proved this result directly in [5, proposition 2.3] without considering C^*.

Example 2.6. Let $G = \langle a, b : a^4 = 1 = b^4, a^b = a^{-1}\rangle$. The size of $Aut_c(G)$ is given by (2.2) but not its structure. Thus $|Aut_c(G)| = |Hom(G/G', Z(G))| = |Hom(C_2 \times C_4, C_2 \times C_2)| = 2^4$. However, the structure of C^* is given by (2.3). In fact, G is nilpotent of class 2 here, so by (2.4 v) $C^* \approx Hom(G/Z(G), Z(G)) \approx Hom(C_2 \times C_2, C_2 \times C_2) \approx C_2^4$. Since $|C^*| = |Aut_c(G)|$, $Aut_c(G) = C^*$ and the structure of $Aut_c(G)$ is also determined. This example can be generalised:

Proposition 2.7. *Let G be a p-group with* $Z(G)$ *elementary abelian of rank m and* $Z(G) \leq \Phi(G)$. *Then* $Aut_c(G) = C^*$ *and is elementary abelian of order* p^{mn}, *where* $n = rank(G/G')$.

PROOF. Since $Z(G) \leq \Phi(G)$, $rank(G/G'Z(G)) = rank(G/G')$, so $Hom(G/G', Z(G)) \approx C_p^{mn} \approx Hom(G/G'Z(G), Z(G))$, and thus, since G is purely non-abelian, $Aut_c(G) = C^*$. ■

Finally, as in (2.6) and (2.7) above and later results, we assume standard simple theorems on $Hom(A, B)$, where A and B are abelian groups. (See, for example, Curran and McCaughan [2, lemma C]. In particular, the following result is used.

Lemma 2.8. *Let A and B be abelian groups with C a proper subgroup or quotient of A, and D a proper subgroup or quotient of B, such that* $|A|/|C| = n = |B|/|D|$, *for some* $n > 1$. *Then* $Hom(C, D)$ *is isomorphic to a proper subgroup of* $Hom(A, B)$.

PROOF. Assume n is a prime p and write A, B, C and D as a direct product of their Sylow subgroups. Then the lemma reduces to applying lemma D in [2] to the Sylow p-subgroups of A, B, C and D. ■

3. Central automorphism groups of minimal order

We first show that if G is a non-abelian group such that $Aut_c(G) = Z(Inn(G))$, then G is usually a purely non-abelian group.

Lemma 3.1. *Let G be a non-abelian group such that* $Aut_c(G) = Z(Inn(G))$. *Then either G is purely non-abelian or else G has a purely non-abelian subgroup N, with* $|Z(N)|$ *odd, such that* $G \approx C_2 \times N$.

PROOF. Suppose on the contrary $G \approx A \times N$, where N is purely non-abelian, A is non-trivial abelian and either $A \neq C_2$ or $A = C_2$ and $|Z(N)|$ is even. In these cases we claim G has a central automorphism that is not inner, so $Aut_c(G) \neq Z(Inn(G))$.

For if $A \neq C_2$ and $\theta \in Aut(A) = Aut_c(A)$ is non-trivial, then for any $(a, n) \in A \times N$, the map $(a, n) \to (\theta(a), n)$ gives an automorphism of G that is central but not inner. If $A = C_2$ and $|Z(N)|$ is even, take $z \in Z(N)$ with z of order 2. Then the map $(1, n) \to (1, n)$ and $(a, n) \to (a, zn)$ defines an automorphsim of G that is central but not inner, since $(a, 1) \to (a, z)$. ■

Example 3.2. There are groups of the form $G = C_2 \times N$ given in (3.1) for which $Aut_c(G) = Z(Inn(G))$. For example, let N be either non-abelian group of order p^3, where p is an odd prime. Then $Aut(G) = Aut(N)$, $Inn(G) = Inn(N) \approx C_p^2$ so $Inn(G) \leq Aut_c(G) = Aut_c(N)$. However, by (2.2) $|Aut_c(N)| = p^2$, so $Z(Inn(G)) = Inn(G) = Aut_c(G)$.

Remark 3.3. Clearly the converse of (3.1) is false. For consider the two groups of order $12: G = \langle a, b : a^3 = 1 = b^4, b^{-1}ab = a^{-1} \rangle$, the purely non-abelian dicyclic group, and $G = C_2 \times S_3$. In both cases $Inn(G) \approx S_3$, so $Z(Inn(G)) = 1$, but $Aut_c(G) \approx C_2$.

Corollary 3.4. *If G is a non-abelian p-group such that $Aut_c(G) = Z(Inn(G))$ then G is purely non-abelian.*

We now give conditions for $Aut_c(G)$ to be of minimal size:

Theorem 3.5. *Let G be any non-abelian group but not of form $G = C_2 \times N$, where N is purely non-abelian and $|Z(N)|$ is odd.*

(*i*) *If $Aut_c(G) = Z(Inn(G))$ then $Z(G) \leq G'$.*

(*ii*) *$Aut_c(G) = Z(Inn(G))$ if and only if $Hom(G/G', Z(G)) \approx Z_2(G)/Z_1(G)$.*

PROOF. First assume $Aut_c(G) = Z(Inn(G))$. Then by (3.1) G is purely non-abelian, and so by (2.2) the Adney-Yen map θ: $Aut_c(G) \to Hom(G, Z(G))$ is a bijection. But also $Aut_c(G) \leq Inn(G)$, which fixes $Z(G)$ pointwise, so $Aut_c(G) = C^*$, and thus by (2.3) θ is a homomorphism. Therefore $Aut_c(G) \approx Hom(G, Z(G)) \approx Hom(G/G', Z(G))$. On the other hand $Aut_c(G) = Z(Inn(G)) \approx Z_2(G)/Z_1(G)$, and the right hand side of (ii) follows.

Further, $Im\theta = \{f_\sigma : \sigma \in Z(Inn(G))\}$, where for any $g \in G$, $\sigma(g) = g^x$, some $x \in Z_2(G)$, so $f_\sigma(g) = g^{-1}g^x = [g, x] \in G' \cap Z(G)$. Thus by (2.3) $C^* \approx Hom(G/G'Z(G), G' \cap Z(G))$. That is, $Hom(G/G', Z(G)) \approx Aut_c(G) = C^* \approx Hom(G/G'Z(G), G' \cap Z(G))$, and therefore $Z(G) \leq G'$. For if not, $|G/G'|/|G/G'Z(G)| = |G'Z(G)|/|G'| = |Z(G)|/|G' \cap Z(G)| \neq 1$, so by (2.8), $|Hom(G/G', Z(G))| > |Hom(G/G'Z(G), G' \cap Z(G))|$ is a contradiction. This gives (i).

Now consider the converse of (ii). For any group G, $Z(Inn(G)) \leq Aut_c(G)$ and by (2.2) $|Aut_c(G)| \leq |Hom(G/G', Z(G))|$. Thus $|Z(Inn(G)| \leq |Aut_c(G)| \leq |Hom(G/G', Z(G))|$. But by assumption $|Z(Inn(G)| = |Hom(G/G', Z(G))|$, so we have equality throughout and thus $Z(Inn(G)) = Aut_c(G)$. ■

Examples 3.6. The restriction on the group in the statement of (3.5) is necessary. If $G = C_2 \times N$ is the group from (3.2), then $Aut_c(G) = Z(Inn(G))$, but neither conditions (i) nor (ii) of (3.5) hold. Even if G is purely non-abelian, (3.5 ii) may hold trivially. For example, if $Z(G) = 1$ then all the groups $Aut_c(G)$, $Z(Inn(G))$, $Hom(G/G', Z(G))$ are trivial. However, in the case of a p-group they are not.

Corollary 3.7. *Let G be a non-abelian p-group.*

(*i*) *If $Aut_c(G) = Z(InnG)$, then $Z(G) \leq G'$.*

(*ii*) *$Aut_c(G) = Z(Inn(G)) \Leftrightarrow Hom(G/G', Z(G)) \approx Z_2(G)/Z_1(G)$.*

Corollary 3.8. *If G is a non-abelian p-group with $Z_2(G)/Z_1(G)$ cyclic then $Aut_c(G) > Z(Inn(G))$. In particular, this holds if G is of maximal class.*

PROOF. $Hom(G/G', Z(G))$ has rank at least 2, so cannot be isomorphic to $Z_2(G)/Z_1(G)$. Hence the result follows from (3.7 ii). ■

Corollary 3.9. *If G is a non-abelian p-group with $Z(G) \approx C_p^s (s \geq 1)$ and rank r, then $Aut_c(G) = Z(Inn(G))$ if and only if $Z_2(G)/Z_1(G) \approx C_p^{rs}$.*

PROOF. $Hom(G/G', Z(G)) \approx Hom(C_{p^{n_1}} \times \ldots \times C_{p^{n_r}}, C_p^s)$ (each $n_i \geq 1$) $\approx C_p^{rs}$. ■

The following result of Curran and McCaughan [2] also follows:

Corollary 3.10. *If G is a non-abelian p-group, then $Aut_c(G) = Inn(G)$ if and only if $G' = Z(G)$ and G' is cyclic.*

PROOF. If $Aut_c(G) = Inn(G)$, then $Inn(G)$ is abelian, so $G' \leq Z(G)$. But since $Inn(G)$ is abelian, $Aut_c(G) = Z(Inn(G))$, so by (3.7) $Z(G) \leq G'$. That is $G' = Z(G)$. But now $Aut_c(G) \approx Hom(G/G',\ Z(G)) = Hom(G/Z(G),\ G') \approx G/Z(G)$ only if G' is cyclic, since $exp(G/Z(G)) = exp(G')$ for a p-group of nilpotency class 2. The converse is similar. ■

We finish with some examples of groups that satisfy the condition of (3.7 ii):

Examples 3.11. The following p-groups satisfy the conditions of (3.9) with $r = 2$ and $s = 1$:

(i) Let $G = \langle a, b, c{:}a^2 = b^8 = c^2 = 1 = [a, c],\ b^a = b^5,\ b^c = ba \rangle$. Then G is of rank 2, order 32 and $Z_1(G) = \langle b^4 \rangle,\ Z_2(G) = \langle b^2, a \rangle$.

(ii) Let $G = \langle a, b, c : a^4 = b^{2^n} = c^2 = 1, a^c = ab^{-1},\ b^a = b^{-1+2^{n-1}}, b^c = b^{-1} \rangle, n \geq 3$. Then $\gamma_i(G) = \langle b^{2^{i-2}} \rangle,\ 2 \leq i \leq n+1,\ Z_1(G) = \langle b^{2^{n-1}} \rangle,\ Z_i(G) = \langle b^{2^{n-i}}, a^2 \rangle,\ 2 \leq i \leq n$. Thus G is of rank 2, order 2^{n+3} and nilpotency class $n+1$. In particular in (i) and (ii) $Z_1(G) \approx C_2$ and $Z_2(G)/Z_1(G) \approx C_2^2$.

(iii) Let $G = \langle a, b : a^{p^n} = 1 = b^{p^{n+1}}, [b, a] = b^p \rangle$, $n \geq 2$, p odd. Then $\gamma_i(G) = \langle b^{p^{i-1}} \rangle,\ 2 \leq i \leq n+1, Z_1(G) = \langle b^{p^n} \rangle,\ Z_i(G) = \langle a^{p^{n-i+1}}, b^{p^{n-i+1}} \rangle,\ 2 \leq i \leq n$. Thus G is of rank 2, order p^{2n+1} and nilpotency class $n+1$. In particular, $Z_1(G) \approx C_p$ and $Z_2(G)/Z_1(G) \approx C_p^2$.

(iv) Let $G = \langle a, b, a_1, \ldots, a_{2n}{:}a^p = b^p = a_1^p = \ldots = a_{2n}^p = 1,\ [a, b] = a_1,\ [a_1, a] = a_2,$ $[a_i, b] = a_{i+2}, 1 \leq i \leq 2n-2,\ [a_{2i-1}, a] = a_{2i},\ 2 \leq i \leq n,$ rest commute$\rangle$, $p > n \geq 2$. Then $\gamma_2(G) = Z_{n+1}(G) = \langle a_1, \ldots, a_{2n} \rangle,\ \gamma_{2+i}(G) = Z_{n+1-i}(G) = \langle a_{2i}, \ldots, a_{2n} \rangle,\ i \leq i \leq n-1,\ \gamma_{n+1}(G) = Z_1(G) = \langle a_{2n} \rangle$. Thus G is metabelian of rank 2, order p^{2n+2} and nilpotency class $n+2$. In particular, $Z_1(G) \approx C_p$ and $Z_2(G)/Z_1(G) \approx C_p^2$.

Thus by (3.9), the groups G above all satisfy $Aut_c(G) = Z(Inn(G))$.

Examples 3.12. GAP [3] was used to search for all groups of orders 2^n and 3^m, $n \leq 7$, $m \leq 6$ for which $Aut_c(G) = Z(Inn(G))$ and $Inn(G)$ was non-abelian (so G was of nilpotency class at least 3). The following table gives the group number in the gap library and their structure.

Group order	Group numbers	Nil class	$Z_1(G)$	$Z_2(G)/Z_1(G)$	G/G'
2^5	6–8	3	C_2	C_2^2	$C_2 \times C_4$
2^6	41–43, 46	4	C_2	C_2^2	$C_2 \times C_4$
2^7	71–74, 87–91, 98	4	C_2	C_2^2	$C_2 \times C_4$
	150–152, 158	5	C_2	C_2^2	$C_2 \times C_4$
	527–528, 560–562, 634–637, 641–647	3	C_2	C_2^3	$C_2 \times C_2 \times C_4$
	740–741, 764, 780–781, 801–802, 813–814, 836	3	C_2	C_2^3	$C_2 \times C_2 \times C_2$
	1615–1616, 1620 –1621	3	C_2	C_2^4	$C_2 \times C_2 \times C_2 \times C_4$
3^5	22	3	C_3	C_3^2	$C_3 \times C_9$
3^6	33–39, 44–47, 56–57	4	C_3	C_3^2	$C_3 \times C_3$
	65–77	4	C_3	C_3^2	$C_3 \times C_9$

Note. In the table above (3.11 i) is group number 7 order 2^5, (3.11 ii) is group number 41 when $n=3$ and number 150 when $n=4$, (3.11 iii) is group number 22 when $p=3$ and $n=2$ and (3.11 iv) is group number 34 when $p=3$ and $n=2$.

Example 3.13. Example (iii) in (3.11) can be generalised. For example, let $G=\langle a,b{:}a^{p^n}=1=b^{p^{n+2}},\ [b,a]=b^{p^2}\rangle$, $n\geq 3$, p odd. If $n=2k-1$ or $2k$ $(k\geq 2)$, then $\gamma_i(G)=\langle b^{p^{2(i-1)}}\rangle$, $2\leq i\leq k+1$, $Z_1(G)=\langle b^{p^n}\rangle$, $Z_i(G)=\langle a^{p^{n-2i+2}},\ b^{p^{n-2i+2}}\rangle$, $2\leq i\leq n$. Thus G is of rank 2, order p^{2n+2} and nilpotency class $k+1$. In particular, $Hom(G/G',Z(G))\approx Hom(C_{p^2}\times C_{p^n},C_{p^2})\approx C_{p^2}\times C_{p^2}\approx Z_2(G)/Z_1(G)$, so by (3.7 ii) $Aut_c(G)=Z(Inn(G))$.

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