FINITE GROUPS WITH FEW AUTOMORPHISMS

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Abstract

We give a complete list of all *finite* groups with less than 48 automorphisms and investigate the rate of growth of $|G|$ with respect to $|\text{Aut } G|$ in this range. We also list *all* groups with less than 50 endomorphisms.

1. Introduction

In [3] De Vries and De Miranda gave a complete listing of all groups with at most eight automorphisms. In this paper we extend their results in the case of finite groups by giving a complete list of finite groups with less than 48 automorphisms. Since by [1], a group with only finitely many endomorphisms is itself finite, this allows us to give a complete list of *all* groups with at most 50 endomorphisms.

2. Notation

We use the following notation throughout. Aut G will denote the full group of automorphisms of G, and End G will denote the full semi-group of endomorphisms of G.

A_n = the alternating group on n symbols,
C_n or $n/1$ = the cyclic group of order n,
D_n = the dihedral group of order $2n$, $n>2$,
Q_n = the dicyclic group of order $4n$, $n \geq 2$, given by $<a, b{:}a^{2n}=1, b^2=a^n, b^{-1}ab=a^{-1}>$,
S_n = the symmetric group on n symbols
$\text{Hol}(G)$ = the holomorph of a given group G,
$SL(n,k)$ = the special linear group of $n \times n$ matrices over Z_k,

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$A\alpha B$ = a split extension of B by A,
$(n/1)^k$ = the direct product of k *copies of* $(n/1)$.

Throughout this paper the group of prime power order is identified by the notation used in [8]. With this notation the nth group of order p^k is denoted p^k/n. In particular the cyclic group is always denoted by $p^k/1$. The generators and defining relations for all p-groups encountered in this paper are widely available on GAP or CAYLEY.

For the group of order 24 we use the notation of [12]. We emphasise that all groups are finite throughout.

3. Preliminary results

The following results are central to this paper.

Lemma 1. [4] *If G has a non-abelian automorphism group of order pq, where* $p \neq q$ *are primes, then* $G \simeq G_2 \times C_2$ *or* S_3 *and* $Aut\ G \simeq S_3$.

Lemma 2. [4]

(1) *Aut G is cyclic if and only if G is cyclic of order* 1, 2, 4, p^r *or* $2p^r$ *where p is an odd prime and r is a positive integer;*

(2) *There is no group G such that* $Aut\ G \simeq C_m$ *where m is an odd number* >1.

Lemma 3. [6] *Let G be a finite group, then* $|Aut\ G| = p_1 p_2 \ldots p_n$, ($p_i$ *primes,* $p_1 < p_2 < \ldots < p_n$) *if and only if G is one of the following groups:*

(1) $C_2 \times C_2$ *and* $Aut\ G \simeq S_3$;

(2) *A cyclic group of order* p^α, $2p^\alpha$ *where* $\alpha = 0, 1, 2$ *and p is an odd prime with* $p-1$ *containing no square factor;*

(3) $<a,\ b \mid a^m = b^{p_n} = 1,\ a^{-1}ba = b^t>$ *where the order of t modulo* p_n *is m,* $m \mid 2p_2p_3 \ldots p_{n-1}$, $p_1 = 2$, $p_n = 2p_2p_3 \ldots p_{n-1} + 1$;

(4) $C_2 \times G$ *where G is as in* (3) *and has odd order.*

Lemma 4. [6] *Let G be a finite group with* $|Aut\ G| = pq^2$ *then one of the following must hold*

(1) $|Aut\ G| = 2^2 5$ *if and only if G is one of* $<a_1 b \mid a^5 = b^4 = 1,\ b^{-1}ab = a^k,\ k^2 \equiv -1$ $(mod\ 5)>$; D_{10}, C_{25}, C_{50}, C_{33}, C_{44}, C_{66};

(2) $|Aut\ G| = 2^2 3$ *if and only if G is one of* D_6, Q_3, $S_3 \times C_3$, C_{36}, $C_2 \times C_2 \times C_3$, C_{13}, C_{26}, C_{21}, C_{28}, C_{42};

(3) $|Aut\ G| = 2.3^2$ *if and only if G is one of* C_{27}, C_{54}, C_{19}, C_{38};

(4) $|Aut\ G| = 4p$, $p > 5$ *if and only if G is one of* C_{4p+1}, $C_{2(4p+1)}$, $C_{3(2p+1)}$, $C_{4(2p+1)}$, $C_{6(2p+1)}$, *where* $4p+1$, $2p+1$ *are primes;*

(5) $|Aut\ G| = 2q^2$, $q > 3$ *if and only if G is one of* C_{2q^2+1}, $C_{2(2q^2+1)}$ q^2+1 *prime.*

Lemma 5. [11] *Let G be a finite group, then* $|Aut\ G| = 2^3 p$ *(p an odd prime) if and only if G is*

1. Q_2, C_{8p+1}, $C_2 \times C_{8p+1}$, *where* $8p+1$ *is prime;*
2. $C_3 \times C_{25}$, $C_4 \times C_{25}$, $C_4 \times C_{27}$, $C_2 \times C_3 \times C_{25}$, $C_3 \times C_{4p+1}$, $C_2 \times C_3 \times C_{4p+1}$, $C_4 \times C_{4p+1}$, *where* $4p+1$ *is prime;*
3. $C_2 \times C_2 \times C_5$, $C_5 \times C_9$, $C_2 \times C_5 \times C_9$, $C_8 \times C_9$, $C_8 \times C_{2p+1}$, $C_5 \times C_{2p+1}$, $C_2 \times C_5 \times C_{2p+1}$, $C_3 \times C_4 \times C_{2p+1}$, *where* $2p+1$ *is prime;*
4. A_4, $SL(2, 3)$;
5. S_4, 24/14, Q_5, 24/15;
6. 24/5, 24/9, $5/1 \times S_3$, $6/1 \times S_3$, $C_3 \times Hol\ (5/1)$, $C_3 \times Q_3$, $C_3 \times D_5$, $C_2 \times Hol(5/1)$, $C_2 \times D_5$.

Lemma 6. [7] *If G is a finite group with* $|Aut\ G| = p^2 q^2$, *where p and q are primes, then* $p = 2$, *q is an odd prime and G is isomorphic to one of the following:*

1. *G in abelian, and one of the following occurs;*
    - (a) $|Aut\ G| = 36$ *and G is isomorphic to one of* $4/2 \times 7/1$, $4/2 \times 9/1$, 63/1, 126/1, 108/1;
    - (b) $|Aut\ G| = 100$ *and G is isomorphic to one of* 125/1 *or* 250/1;
    - (c) $|Aut\ G| = 4q^2, q \neq 3$ *nor* 5, *and* $4q^2+1$ *is prime then G is isomorphic to* $(4q^2+1)/1$ *or* $2(4q^2+1)/1$;
    - (d) $|Aut\ G| = 4q^2$, $q \neq 3$ *nor* 5, *and* $4q^2+1$ *is prime then G is isomorphic to* $3/1 \times (2q^2+1)/1$, $6/1 \times (2q^2+1)/1$ or $4/1 \times (2q^2+1)/1$;
2. $G \simeq S_3 \times 9/1$ *and* $|Aut\ G| = 36$;
3. $G \simeq S_3 \times 7/1$ *and* $|Aut\ G| = 36$.

Lemma 7. [5] *If G is a group whose automorphism group has order dividing 32, then G is one of the following:*

- (i) $G \simeq 1/1$, 2/1 and Aut $G \simeq (1/1)$;
- (ii) $G \simeq 3/1$, 4/1, 6/1 and Aut $G \simeq (2/1)$;
- (iii) $G \simeq 5/1$, 10/1 and Aut $G \simeq (4/1)$,
    $G \simeq 8/1$, 12/1 and Aut $G \simeq (2/1)^2$;
- (iv) $G \simeq 24/1$ and Aut $G \simeq (8/5) \simeq (2/1)^3$;
    $G \simeq 15/1$, 16/1, 20/1, 30/1 and Aut $G \simeq (8/2) \simeq (2/1) \times (4/1)$;
    $G \simeq 8/2$, 8/3 and Aut $G \simeq (8/3)$ (dihedral);
- (v) $G \simeq 17/1$, 34/1 and Aut $G \simeq (16/1)$;
    $G \simeq 32/1$ and Aut $G \simeq (16/5) \simeq (2/1) \times (8/1)$;
    $G \simeq 40/1$, 48/1, 60/1 and Aut $G \simeq (16/10) \simeq (2/1) \times (2/1) \times (4/1)$;
    $G \simeq 16/5$, 16/6, 16/8, $(2/1) \times (3/1) \times (4/1)$, $(8/3) \times (3/1)$ and Aut $G \simeq (16/11) \simeq (8/3) \times (2/1)$;
- (vi) $G \simeq 51/1$, 64/1, 68/1, 102/1 and Aut $G \simeq 32/16 \simeq (16/1) \times (2/1)$;
    $G \simeq 96/1$ and Aut $G \simeq (32/36) \simeq (2/1)^2 \times (8/1)$;
    $G \simeq 80/1$ and Aut $G \simeq (32/21) \simeq (2/1) \times (4/1)^2$;

$G \simeq 120/1$ and Aut $G \simeq (32/45) \simeq (2/1)^3 \times (4/1)$;
$G \simeq (2/1) \times (4/1) \times (5/1)$, $(8/3) \times (5/1)$ and Aut $G \simeq (32/25)$;
$G \simeq 16/3$, $16/4$ and Aut $G \simeq (32/27)$;
$G \simeq 16/7$, $16/9$ and Aut $G \simeq (32/43)$;
$G \simeq (16/6) \times (3/1)$, $(16/8) \times (3/1)$, $32/11$, $(2/1) \times (3/1) \times (8/1)$ and Aut $G \simeq (32/46)$;
$G \simeq 32/16$, $32/17$ and Aut $G \simeq (32/48)$.

Lemma 8. [2] *There is no finite group G with* $|G| > 2$ *such that* $|Aut\ G| = p^n$ *for p odd and* $n \leq 5$.

We also make use of the following lemmas whose proofs are elementary.

Lemma 9. C_n*, the cyclic group of order n, has precisely n endomorphisms and precisely* $\phi(n)$ *automorphisms, where* ϕ *is the Euler* ϕ*-function.*

Lemma 10. *The number of endomorphisms of a finite group G, with given image, H say, is* $n|Aut\ H|$*, where n is the number of normal subgroups, N of G, such that* $G/N \simeq H$.

Lemma 11. *The number of endomorphisms of a finite group G, with given kernel, N say, is* $k|Aut\ G/N|$*, where k is the number of subgroups of G which are isomorphic to G/N.*

Lemma 12. *If* $G \simeq A \times B$ *and both A and B are fully invariant then* $|End\ G| = |End\ A||End\ B|$.

Lemma 13. *An abelian p-group with invariants of order* $p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_n}$ *has*

$$\prod_{i=1}^{n} \phi(i, n, p) \text{ endomorphisms}$$

where $\phi(i, n, p) = (\prod_{j=1}^{i-1} p^{\alpha_j})(p^{\alpha_j})^{n-i+1}$ *for* $i > 1$ *and* $\phi(1, n, p) = (p^{\alpha_1})^n$ *is the number of images an element of order* p^{α_4} *has in G.*

4. The finite groups with less than 48 automorphisms

Applying the results listed in the previous section and calculating automorphism groups where necessary, we may now classify all finite groups with less than 48 automorphisms as listed in the following table. For each n in the first column of the table, we list in the second column all the groups K of order n which occur as the automorphism group of a finite group. In the third column, we then list for each such K all finite groups G such that Aut $G \simeq K$. If a group is absent from the second column it does not occur as an automorphism group. Incidently, we can now show that the smallest odd $n > 1$ appearing in column 1 as at least $405 = 3^4$. 5 and we have strong evidence to suggest that $n \geq 2187 = 3^7$. There does exist a group of order 3^6 with an automorphism group of order 3^7 [10]

4.1. Groups with fewer than 48 automorphisms

Table 1

Order	Aut *G*	G
1	1/1	1/1, 2/1
2	2/1	3/1, 4/1, 6/1
4	4/1	5/1, 10/1
	4/2	8/1, 12/1
6	6/1	7/1, 9/1, 14/1, 18/1
	S_3	4/2, S_3
8	8/2	15/1, 16/1, 20/1, 30/1
	8/3	8/2, 8/3
	8/5	24/1
10	10/1	11/1, 22/1
12	12/1	13/1, 26/1
	4/2 × 3/1	21/1, 28/1, 36/1, 42/1
	2/1 × S_3	Q_3, S_3 × 3/1, S_3 × 2/1
16	16/1	17/1, 34/1
	16/5	32/1
	16/10	40/1, 48/1, 60/1
	16/11	16/5, 16/6, 16/8, 8/2 × 3/1, 8/3 × 3/1
18	18/1	19/1, 38/1
	2/1 × 3/2	27/1, 54/1
20	20/1	25/1, 50/1
	4/2 × 5/1	33/1, 44/1, 66/1
	Hol(5/1)	Hol(5/1), D_5
22	22/1	23/1, 46/1
24	24/2	35/1, 39/1, 45/1, 52/1, 70/1, 78/1, 90/1
	24/3	56/1, 72/1, 84/1
	24/4	24/14, 24/9, 6/1 × S_3 3/1 × Q_3
	24/5	24/15
	24/9	5/1 × S_3, 5/1 × 4/2
	S_4	8/4, A_4, 24/5, S_4, $SL(2, 3)$
28	28/1	29/1, 58/1
30	30/1	31/1, 62/1
32	32/16	51/1, 64/1, 68/1, 102/1
	32/21	80/1
	32/25	8/2 × 5/1, 8/3 × 5/1
	32/27	16/3, 16/4
	32/36	96/1
	32/43	16/7, 16/9
	32/45	120/1
	32/46	16/6 × 3/1, 16/8 × 3/1 32/11, 16/5 × 3/1
	32/48	32/16, 32/17
36	36/1	37/1, 74/1
	4/1 × 9/1	57/1, 76/1, 108/1, 114/1
	4/2 × 9/2	63/1, 126/1
	6/1 × S_3	9/1 × S_3, 7/1 × S_3 9/1 × 4/2, 7/1 × 4/2

Table 1 (*Continued*)

Order	Aut G	G
40	40/1	41/1, 82/1
	$8/2 \times 5/1$	55/1, 75/1, 100/1 110/1, 150/1
	$8/5 \times 5/1$	88/1, 132/1
	$2/1 \times \mathrm{Hol}(5/1)$	$3/1 \times \mathrm{Hol}(5/1)$, $3/1 \times D_5$, $2/1 \times D_5$ Q_5, $2/1 \times \mathrm{Hol}(5/1)$
42	42/1	43/1, 49/1, 86/1, 98/1
	Hol(7/1)	D_7, $3/1\alpha 7/1$, Hol(7/1)
44	$4/2 \times 11/1$	69/1, 92/1, 138
46	46/1	47/1, 94/1

5. Rate of growth of $|G|$ with respect to $|Aut\ G|$

In [9] Ledermann and Neumann proved the following result.

Theorem 14. *There is a function $f(n)$ with the property that if a finite group G has order $G| \geq f(n)$, then G has at least n automorphisms. The least such $f(n)$ satisfies*

$$f(1) = 1, \quad f(2) = 3, \quad f(3) = f(4) = 7 \text{ and } f(n) \leq (n-1)^{n+(n-2)\log_2\ (n-1)}$$

The results stated in Table 1 allow us to extend the range of values for which the least $f(n)$ is known to $n = 48$.

Table 2

n	1	2	3	4	5	6	7	8	9	10	11	12
$f(n)$	1	3	7	7	13	13	19	19	31	31	31	31
n	13	14	15	16	17	18	19	20	21	22	23	24
$f(n)$	43	43	43	43	61	61	61	61	67	67	67	67
n	25	26	27	28	29	30	31	32	33	34	35	36
$f(n)$	91	91	91	91	91	91	91	91	121	121	121	121
n	37	38	39	40	41	42	43	44	45	46	47	48
$f(n)$	127	127	127	127	151	151	151	151	151	151	151	151

We note that for $n \leq 48$, $f(n)$ is determined by the largest t such that $\phi(t) \leq n$, arising from the cyclic groups. It would be interesting to discover if this continues to hold when n increases, or if not, at what stage it breaks down.

We also note the following:
(i) $f(p^i) = f(p^i + 1)$, for p an odd prime and $1 \leq i \leq 5$, (ii) $f(2n-1) = f(2n)$, $1 < n < 204$. The numerical evidence suggests that $f(n)$ is always an odd number. This is in fact the case.

Theorem 15. *The least $f(n)$ is odd for all n.*

Proof. Suppose that $f(n)$ is even for some n, then $|\mathrm{Aut}\ G|$ is odd and $|G|$ is odd. But $|\mathrm{Aut}(G \times G_2)| = |\mathrm{Aut}\ G|$ and $|G \times G_2|$ is even, a contradiction.

6. Groups with few endomorphisms

Alperin [1] has shown that a group is finite if and only if its set of endomorphisms is finite. As the number of endomorphisms of a group is larger than the number of automorphisms, finding the groups with at most 48 endomorphisms reduces to an examination of the finite groups with less than 48 automorphisms. These are known and are listed in the previous section. The number of endomorphisms of the groups with less than 48 automorphisms may be easily obtained by applying Lemmas 8 to 12. It is also easy to show that if $|\mathrm{Aut}\ G| \geq 48$ then $|\mathrm{End}\ G| > 50$. We recall that $|\mathrm{End}\ C_n| = n$.

We summarise these results in the following table:

6.1. The non-cyclic groups with at most 50 endomorphisms

$\|\mathrm{End}\ G\|$	G
10	S_3
16	4/2
20	Q_3
26	D_5
28	8/4
32	8/2
33	A_4, $SL(2, 3)$
36	8/3, 16/9
	Hol(5/1), $S_3 \times (3/1)$
40	24/14
48	16/6
50	$5/1 \times S_3$, D_7

We now define a function $e(n)$ by $|G| \geq e(n) \Rightarrow |\mathrm{End}\ G| \geq n$.

Lemma 16. *The least* $e(n) = n$ *for* $1 \leq n \leq 50$.

We note that it is the cyclic groups of order n which determine $e(n)$, for $n \leq 50$, so we make the following conjecture.

Conjecture. *The least* $e(n) = n$ *for all* n.

We note the amusing fact that End $A_4 \simeq$End $SL(2,3)$ as semi-groups.

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