LOCAL OPERATORS ON BANACH MODULES

By J. BRAČIČ*
Faculty of Natural Sciences and Engineering, University of Ljubljana, Slovenia

[Received 7 April 2003. Read 10 February 2004. Published 17 December 2004.]

ABSTRACT

In this article we extend the notion of local operators to left Banach modules over a commutative Banach algebra. We get interesting results if the underlying algebra is spectrally separable. For (SD)-algebras, a special subclass of spectrally separable algebras, Shulman has shown that all local operators are multipliers. We introduce local (SD)-algebras, which are not necessarily commutative, and use Shulman's result to show that any local multiplier between left Banach modules over a local (SD)-algebra is a multiplier.

1. Introduction

Given two linear spaces $A(\Omega) \subseteq B(\Omega)$ of complex-valued functions on a non-empty set Ω, a linear mapping $T\colon A(\Omega) \to B(\Omega)$ is said to be *local* provided that

$$(Tf)g = 0, \quad \text{for all } f \in A(\Omega) \text{ and } g \in B(\Omega) \text{ with } fg = 0$$

(cf. [8]). Let Ω be a normal Hausdorff space and let $A(\Omega) \subseteq B(\Omega) \subseteq C(\Omega)$, where $C(\Omega)$ is the Banach algebra of all complex-valued continuous functions on Ω, be linear subspaces such that for all disjoint closed subsets F and G in Ω there exists a function $f \in A(\Omega)$ that satisfies $f \equiv 1$ on F and $f \equiv 0$ on G. Then, by lemma 1 in [8], a linear mapping $T\colon A(\Omega) \to B(\Omega)$ is local if and only if

$$\mathit{supp}(Tf) \subseteq \mathit{supp}\, f \qquad \text{for all } f \in A(\Omega),$$

where $\mathit{supp}\, f$ denotes the support of f. This equivalence allows us to extend the notion of local mapping to the context of Banach modules over a commutative Banach algebra.

The basic structure of this paper is as follows. In Section 2, we define local operators between Banach modules over a unital commutative Banach algebra. We show that local multipliers are always local operators. In Section 3, we confine ourselves to the situation where all modules are built over a spectrally separable algebra. We show, for instance, that in this case the set of all local operators on a given left Banach module $\mathcal{X}$ is a strongly closed subalgebra in $B(\mathcal{X})$, the algebra of all bounded linear operators on $\mathcal{X}$. In that section we also characterise local operators between left Banach modules by the use of the spectral and co-spectral submodules. In Section 4, we recall from [13] the basics about (SD)-algebras and then define local (SD)-algebras. We show that unital C^*-algebras and group algebras over discrete groups are examples of algebras of this type. We include Shulman's original proof of

---

*E-mail: janko.bracic@fmf.uni-lj.si

Mathematical Proceedings of the Royal Irish Academy, **104**A (2), 239–248 (2004) © Royal Irish Academy

the assertion, which says that all local operators between two Banach modules over an (SD) algebra are multipliers. In Section 5 we use this result in showing that local multipliers between left Banach modules over a local (SD)-algebra are multipliers. We refer the reader to [1; 3; 5; 10; 11; 12] for details about the notions that are used in this paper.

2. Local operators

Let $\mathcal{A}$ be a commutative Banach algebra and $\Sigma(\mathcal{A})$ its character space, i.e. the set of all nonzero multiplicative linear functionals on $\mathcal{A}$. (The reader is referred to [12] for basics about Banach algebras and to [3] for Banach modules.) For a nonempty subset $\mathcal{M}$ in a left Banach $\mathcal{A}$-module $\mathcal{X}$, the *annihilator* $ann_{\mathcal{A}}(\mathcal{M})$ of $\mathcal{M}$ is

$$ann_{\mathcal{A}}(\mathcal{M}) := \{a \in \mathcal{A};\ \ a \cdot x = 0 \ \text{ for all } \ x \in \mathcal{M}\}.$$

It is a closed ideal in $\mathcal{A}$. The hull

$$h(ann_{\mathcal{A}}(\mathcal{M})) := \{\varphi \in \Sigma(\mathcal{A});\ \ ann_{\mathcal{A}}(\mathcal{M}) \subseteq ker\ \varphi\}$$

is called the *spectrum* of $\mathcal{M}$ and is denoted by $sp_{\mathcal{A}}(\mathcal{M})$. In the case when $\mathcal{M}$ is a singleton $\{x\}$, we shall use the notation $sp_{\mathcal{A}}(x)$ instead of $sp_{\mathcal{A}}(\{x\})$ (we shall follow this convention also in some other cases). Note that spectra are hull-kernel closed subsets of the space of characters. It follows that they are compact if $\mathcal{A}$ is unital.

The algebra $\mathcal{A}$ is a Banach left $\mathcal{A}$-module, for the usual multiplication in $\mathcal{A}$, therefore the spectrum may be defined also for any subset of $\mathcal{A}$. It is easy to see that $sp_{\mathcal{A}}(a)$ $(a \in \mathcal{A})$ is the hull-kernel closure of the *co-zero set* $\omega(a) := \{\varphi \in \Sigma(\mathcal{A});\ \ \varphi(a) \neq 0\}$ if $\mathcal{A}$ is semisimple. In particular, $sp_{\mathcal{A}}(a) = supp(\hat{a})$, for $a \in \mathcal{A}$ when $\mathcal{A}$ is regular and semisimple. Here we have denoted by $\hat{a} : \Sigma(\mathcal{A}) \to \mathbb{C}$ the *Gelfand transform* of $a \in \mathcal{A}$.

The notion of the spectrum that has been already introduced is just another aspect of the *local Arveson spectrum* (cf. [10, section 4.12]). In the following proposition we list without proofs some properties of spectra.

Proposition 2.1. *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{X}$ be a Banach left $\mathcal{A}$-module.*

(i) *If $\mathcal{A}$ is unital then the spectrum $sp_{\mathcal{A}}(\mathcal{M})$ of a nonempty subset $\mathcal{M} \subseteq \mathcal{X}$ is empty if and only if $\mathcal{M} = \{0\}$.*

(ii) *For every $a \in \mathcal{A}$ and every $x \in \mathcal{X}$, we have the inclusions*

$$hk(\omega(a) \cap sp_{\mathcal{A}}(x)) \subseteq sp_{\mathcal{A}}(a \cdot x) \subseteq sp_{\mathcal{A}}(a) \cap sp_{\mathcal{A}}(x),$$

where $hk(E)$ denotes the hull-kernel closure of a subset $E \subseteq \Sigma(\mathcal{A})$.

(iii) $sp_{\mathcal{A}}(x_1 + \ldots + x_n) \subseteq \cup_{k=1}^{n} sp_{\mathcal{A}}(x_k)$, *for arbitrary $x_1, \ldots, x_n$ in $\mathcal{X}$.*

Now we shall extend the notion of local operator.

Definition 2.2. Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{X}$, $\mathcal{Y}$ be left Banach $\mathcal{A}$-modules. A bounded operator T from $\mathcal{X}$ into $\mathcal{Y}$ is called a *local operator* if

$$sp_{\mathcal{A}}(Tx) \subseteq sp_{\mathcal{A}}(x), \qquad \text{for all } x \in \mathcal{X}.$$

The set of all local operators from a left Banach $\mathcal{A}$-module $\mathcal{X}$ into a left Banach $\mathcal{A}$-module $\mathcal{Y}$ will be denoted by $LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. It follows immediately from Definition 2.2 and Proposition 2.1 that $LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is a linear subset in $B(\mathcal{X}, \mathcal{Y})$, the Banach space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. Moreover, if $\mathcal{X} = \mathcal{Y}$, then $LO_{\mathcal{A}}(\mathcal{X})$ is a subalgebra in $B(\mathcal{X})$, containing the identity operator.

Example 2.3. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ and $\mathcal{Y}$ left Banach $\mathcal{A}$-modules. Denote by $B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ the space of all bounded *multipliers* from $\mathcal{X}$ into $\mathcal{Y}$, that is

$$B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) = \{M \in B(\mathcal{X}, \mathcal{Y}); \quad M(a \cdot x) = a \cdot Mx \quad (a \in \mathcal{A}, \ x \in \mathcal{X})\}.$$

An operator $T \in B(\mathcal{X}, \mathcal{Y})$ is a *local multiplier* if, for each x in $\mathcal{X}$ there exists a multiplier $M_x \in B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ with $T_x = M_x x$. This concept is similar to the concept of a local derivation, which was introduced by Kadison [7] and, independently, by Larson and Sourour [9]. Of course, every multiplier is a local multiplier. It is also easy to see that every local multiplier $T \in B(\mathcal{X}, \mathcal{Y})$ is a local operator. Indeed, let T be a local multiplier, x an arbitrary vector in $\mathcal{X}$, and $M_x \in B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ such that $T_x = M_x x$. If a is in $ann_{\mathcal{A}}(x)$, then

$$a \cdot Tx = a \cdot M_x x = M_x(a \cdot x) = 0$$

and therefore $a \in ann_{\mathcal{A}}(Tx)$. Now, $ann_{\mathcal{A}}(x) \subseteq ann_{\mathcal{A}}(Tx)$ gives $sp_{\mathcal{A}}(Tx) \subseteq sp_{\mathcal{A}}(x)$.

Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{X}$ be a Banach left $\mathcal{A}$-module. The *spectral submodule* of $F \subseteq \sum(\mathcal{A})$ is $\mathcal{X}(F) := \{x \in \mathcal{X}; \ sp_{\mathcal{A}}(x) \subseteq F\}$, and the *co-spectral submodule* of F is $\mathcal{X}_F$, the closure of the set $\mathcal{X}_{(F)} := \{x \in \mathcal{X}; \ sp_{\mathcal{A}}(x) \cap F = \emptyset\}$. By Proposition 2.1, it is not hard to see that $\mathcal{X}(F)$ and $\mathcal{X}_F$ are indeed submodules in $\mathcal{X}$.

Theorem 2.4. *Let $\mathcal{A}$ be a unital commutative Banach algebra, $\mathcal{X}$ and $\mathcal{Y}$ be left Banach $\mathcal{A}$-modules, and $T \in B(\mathcal{X}, \mathcal{Y})$.*

- (i) *The operator T is local if and only if $T\mathcal{X}(F) \subseteq \mathcal{Y}(F)$, for all closed subsets $F \subseteq \sum(\mathcal{A})$.*
- (ii) *If T is a local operator, then $T\mathcal{X}_F \subseteq \mathcal{Y}_F$, for all closed subsets $F \subseteq \sum(\mathcal{A})$.*

PROOF. (i) It is obvious that $T\mathcal{X}(F) \subseteq \mathcal{Y}(F)$, for all closed subsets F in $\sum(\mathcal{A})$, if T is a local operator. On the other hand, if $T \in B(\mathcal{X}, \mathcal{Y})$ preserves the spectral submodules, then, for an arbitrary $x \in \mathcal{X}$, we have $x \in \mathcal{X}(sp_{\mathcal{A}}(x))$ and therefore $T_x \in \mathcal{Y}(sp_{\mathcal{A}}(x))$, which means $sp_{\mathcal{A}}(Tx) \subseteq sp_{\mathcal{A}}(x)$.

(ii) Let $F \subseteq \sum(\mathcal{A})$ be closed. It is easily seen that $T\mathcal{X}_{(F)} \subseteq \mathcal{Y}_{(F)}$. Now use the continuity of T. ■

3. Spectrally separable algebras

A unital commutative Banach algebra $\mathcal{A}$ is *spectrally separable* if, for every two distinct characters φ_1 and φ_2 in $\sum(\mathcal{A})$ there exist elements a_1 and a_2 in $\mathcal{A}$ such that

$a_1a_2=0$ and $\varphi_1(a_1)\neq 0\neq\varphi_2(a_2)$. Spectrally separable algebras are a special subclass of *strongly harmonic* algebras and were studied by Baskakov in [2] (for strongly harmonic algebras see [12, §7.4]).

It is easy to see that every spectrally separable algebra is regular. Namely, if φ_1 and φ_2 are two distinct characters in $\sum(\mathcal{A})$ and $a_1, a_2\in\mathcal{A}$ are such that $a_1a_2=0$ and $\varphi_1(a_1)\neq 0\neq\varphi_2(a_2)$, then $\omega(a_1)$ and $\omega(a_2)$ are disjoint hull-kernel open neighbourhoods of φ_1 and φ_2, respectively. It follows that the hull-kernel topology on $\sum(\mathcal{A})$ is Hausdorff, i.e. $\mathcal{A}$ is regular. On the other hand, every unital semisimple regular commutative Banach algebra is spectrally separable. The following proposition is a very useful characterisation of spectrally separable algebras (for the proof see [4, theorem 2.10]).

Proposition 3.1. (Partition of unity, cf. [2]) *Let $\mathcal{A}$ be a unital commutative Banach algebra. The algebra $\mathcal{A}$ is spectrally separable if and only if, for any open covering $\mathcal{U}=\{U_1,\ldots,U_n\}$ of $\sum(\mathcal{A})$, there are $a_1,\ldots,a_n$ in $\mathcal{A}$ such that $a_1+\ldots+a_n=1$ and $sp_{\mathcal{A}}(a_k)\subset U_k$, for all $k=1,\ldots,n$.*

If $\mathcal{X}$ is a Banach left module over a spectrally separable algebra $\mathcal{A}$ and $F\subseteq\sum(\mathcal{A})$ is a closed subset, then $\mathcal{X}(F)$ is a closed submodule in $\mathcal{X}$ ([4, proposition 2.7]). We shall use this fact in the proof of the following proposition.

Proposition 3.2. *Let $\mathcal{A}$ be a spectrally separable algebra and $\mathcal{X}$ be a left Banach $\mathcal{A}$-module. Then $LO_{\mathcal{A}}(\mathcal{X})$ is a strongly closed subalgebra in $B(\mathcal{X})$.*

Proof. Let $\{T_\lambda\}_{\lambda\in\Lambda}\subseteq LO_{\mathcal{A}}(\mathcal{X})$ be a net and assume that $T\in B(\mathcal{X})$ is such that $\|T_\lambda x-Tx\|\to 0$, for all $x\in\mathcal{X}$. Fix $x\in\mathcal{X}$. Then $T_\lambda x\in\mathcal{X}(sp_{\mathcal{A}}(x))$, by Definition 2.2. Since $\mathcal{X}(sp_{\mathcal{A}}(x))$ is closed it follows $Tx\in\mathcal{X}(sp_{\mathcal{A}}(x))$, that is $sp_{\mathcal{A}}(Tx)\subseteq sp_{\mathcal{A}}(x)$. ■

Let $\mathcal{X}$ be a Banach space and $\mathcal{X}^*$ its topological dual. If $\mathcal{U}$ is a nonempty subset in $\mathcal{X}$, let $\mathcal{U}^\perp$ be defined by $\mathcal{U}^\perp:=\{\xi\in\mathcal{X}^*;\langle\xi,x\rangle=0,\ \text{for all } x\in\mathcal{U}\}$. Similarly, for a nonempty set $\mathcal{W}$ in $\mathcal{X}^*$, let $\mathcal{W}_\perp$ be given by $\mathcal{W}_\perp:=\{x\in\mathcal{X};\langle\xi,x\rangle=0,\ \text{for all } \xi\in\mathcal{W}\}$. It is easily seen that $\mathcal{U}^\perp$ is a weak $*$ closed subspace in $\mathcal{X}^*$ and $\mathcal{W}_\perp$ is closed subspace in $\mathcal{X}$.

Assume that $\mathcal{X}$ is a left Banach module over a commutative Banach algebra $\mathcal{A}$. Then $\mathcal{X}^*$ is the *dual module* of $\mathcal{X}$ if the module multiplication on $\mathcal{X}^*$ is given by $\langle a\cdot\xi,x\rangle=\langle\xi,a\cdot x\rangle(a\in\mathcal{A},\ x\in\mathcal{X},\ \xi\in\mathcal{X}^*)$. It is easily seen that, for submodules $\mathcal{M}\subseteq\mathcal{X}$ and $\mathcal{N}\subseteq\mathcal{X}^*$, the subspaces $\mathcal{M}^\perp\subseteq\mathcal{X}^*$ and $\mathcal{N}_\perp\subseteq\mathcal{X}$ are submodules as well.

For spectrally separable algebras we have the following extension of Theorem 2.4.

Theorem 3.3. *Let $\mathcal{A}$ be a spectrally separable algebra and $\mathcal{X},\mathcal{Y}$ be left Banach $\mathcal{A}$-modules. For an operator $T\in B(\mathcal{X},\mathcal{Y})$, the following is equivalent.*

(i) *T is a local operator.*

(ii) *$T\mathcal{X}(F)\subseteq\mathcal{Y}(F)$, for all closed subsets F in $\sum(\mathcal{A})$.*

(iii) *$T\mathcal{X}_F\subseteq\mathcal{Y}_F$, for all closed subsets F in $\sum(\mathcal{A})$.*

(iv) *T^* is a local operator.*

PROOF. The equivalence of (i) and (ii) and the implication (i) $\Rightarrow$ (iii) are the assertions of Theorem 2.4.

(iii) $\Rightarrow$ (i). Let x be in $\mathcal{X}$. If a character φ is not in the spectrum $sp_{\mathcal{A}}(x)$, then there exists an open neighbourhood U of φ such that $\bar{U} \cap sp_{\mathcal{A}}(x) = \emptyset$ (since $\sum(\mathcal{A})$ is a compact Hausdorff space). Thus $x \in \mathcal{X}_{\bar{U}}$ and, by assumption, $Tx \in \mathcal{Y}_{\bar{U}}$. By Proposition 3.1, there exists an element $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$ and $sp_{\mathcal{A}}(a) \subseteq U$. It follows that a annihilates $\mathcal{Y}_{(\bar{U})}$, and consequently also $\mathcal{Y}_{\bar{U}}$. Thus, a is in $ann_{\mathcal{A}}(Tx)$ and $\varphi(a) \neq 0$, which implies $\varphi \notin sp_{\mathcal{A}}(Tx)$.

(i) $\Rightarrow$ (iv). Let $F \subseteq \sum(\mathcal{A})$ be closed. For an arbitrary $x \in \mathcal{X}_F$ and $\eta \subseteq \mathcal{Y}_F^{\perp}$, we have $\langle T^*\eta, x\rangle = \langle \eta, Tx\rangle = 0$, where the last equality is valid because of the equivalence (i) $\Leftrightarrow$ (iii). We have proven that $T^*\mathcal{Y}_F^{\perp} \subseteq \mathcal{X}_F^{\perp}$. Since $\mathcal{Y}_F^{\perp} = \mathcal{Y}^*(F)$ and $\mathcal{X}_F^{\perp} = \mathcal{X}^*(F)$, by [2, lemma 1], we conclude that T^* preserves the spectral submodules or, equivalently, T^* is a local operator.

(iv) $\Rightarrow$ (i). The proof of this implication follows from

$$\langle \eta, Tx\rangle = \langle T^*\eta, x\rangle = 0 \quad (x \in \mathcal{X}^*(F)_{\perp}, \ \ \eta \in \mathcal{Y}^*(F))$$

and is very similar to the proof of (i) $\Rightarrow$ (iv). ■

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ be a closed subalgebra in $\mathcal{A}$. In the sequel it is always assumed that every left Banach $\mathcal{A}$-module $\mathcal{X}$ is a left Banach $\mathcal{B}$-module in a natural way, i.e. the action of $\mathcal{B}$ on $\mathcal{X}$ is just the restriction to $\mathcal{B}$ of the action of $\mathcal{A}$.

Proposition 3.4. *Let $\mathcal{A}$ be a commutative Banach algebra with the identity 1 and let $\mathcal{B}$ be a closed subalgebra in $\mathcal{A}$ such that it is spectrally separable and $1 \in \mathcal{B}$. Then*

$$LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \subseteq LO_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}),$$

for every pair of left Banach $\mathcal{A}$-modules $\mathcal{X}$ and $\mathcal{Y}$.

PROOF. Denote by $j : \mathcal{B} \to \mathcal{A}$ the inclusion map and let j^* be the restriction of the adjoint map of j to $\sum(\mathcal{A})$. Let $\partial\mathcal{A}$ be the Shilov boundary of $\mathcal{A}$. Since $\mathcal{B}$ is regular the Shilov boundary of $\mathcal{B}$ coincides with $\sum(\mathcal{B})$ (see [12, theorem 3.2.10]). By theorem 3.2.4 in [12], $\sum(\mathcal{B}) \subseteq j^*(\partial\mathcal{A})$. It is well-known that j^* maps $\sum(\mathcal{A})$ into $\sum(\mathcal{B})$. We may conclude that $\sum(\mathcal{B}) = j^*(\partial\mathcal{A})$.

Let $\mathcal{Z}$ be a left Banach $\mathcal{A}$-module, and consequently a left Banach $\mathcal{B}$-module. Let us show that

$$sp_{\mathcal{B}}(z) = j^*(sp_{\mathcal{A}}(z)), \quad \text{for all } z \in \mathcal{Z}. \tag{3.1}$$

Choose and fix $z \in \mathcal{Z}$. If $\psi \in sp_{\mathcal{A}}(z)$, denote $\varphi := j^*(\psi)$. It is clear that $ann_{\mathcal{B}}(z) \subseteq ann_{\mathcal{A}}(z)$ and therefore $\langle \varphi, a\rangle = \langle \psi, j(a)\rangle = 0$, for every $a \in ann_{\mathcal{B}}(z)$. This proves the inclusion

$$j^*(sp_{\mathcal{A}}(z)) \subseteq sp_{\mathcal{B}}(z). \tag{3.2}$$

Note that, since j^* is continuous and $sp_{\mathcal{A}}(z)$ is a compact subset of $\sum(\mathcal{A})$ the set $j^*(sp_{\mathcal{A}}(z))$ is a compact subset of $sp_{\mathcal{B}}(z)$. Towards a contradiction assume that there exists $\varphi \in sp_{\mathcal{B}}(z) \backslash j^*(sp_{\mathcal{A}}(z))$. Then, by Proposition 3.1, there exists $b \in \mathcal{B}$ such that $\varphi(b) = 1$ and $sp_{\mathcal{B}}(b) \cap j^*(sp_{\mathcal{A}}(z)) = \emptyset$. Consider $\mathcal{A}$ as a left Banach $\mathcal{B}$-module (via the multiplication in $\mathcal{A}$) and use (3.2) with $z = b$. We get $j^*(sp_{\mathcal{A}}(b)) \subseteq sp_{\mathcal{B}}(b)$, which gives $j^*(sp_{\mathcal{A}}(b)) \cap j^*(sp_{\mathcal{A}}(z)) = \emptyset$, and consequently $sp_{\mathcal{A}}(b) \cap sp_{\mathcal{A}}(z) = \emptyset$. It follows that $b \cdot z = 0$, that is $b \in ann_{\mathcal{B}}(z)$. However this is impossible because of $\varphi(b) = 1$. Thus, (3.1) is valid.

For $T \in LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$, we have

$$sp_{\mathcal{B}}(Tx) = j^*(sp_{\mathcal{A}}(Tx)) \subseteq j^*(sp_{\mathcal{A}}(x)) = sp_{\mathcal{B}}(x)(x \in \mathcal{X}),$$

which proves $T \in LO_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$. ■

4. (SD) and locally (SD) algebras

Let $\mathcal{A}$ be a spectrally separable algebra and $F \subseteq \sum(\mathcal{A})$ be a non-empty closed set. By corollary 2.8 in [4], the hull of the co-spectral ideal $\mathcal{A}_F$ is F. We shall see that $\mathcal{A}_F$ is actually the smallest closed ideal in $\mathcal{A}$ whose hull is F. On the other hand, it is obvious that the kernel $k(F) := \cap_{\varphi \in F} ker\varphi$ is the largest ideal in $\mathcal{A}$ with hull F.

Proposition 4.1. *Let $\mathcal{A}$ be a spectrally separable algebra and F a closed subset in $\sum(\mathcal{A})$. The co-spectral ideal $\mathcal{A}_F$ is the smallest closed ideal in $\mathcal{A}$ whose hull is F.*

Proof. Assume that $\mathcal{I}$ is a closed ideal in $\mathcal{A}$ with $h(\mathcal{I}) = F$. First we shall prove the inclusion $\mathcal{I}^{\perp} \subseteq \mathcal{A}^*(F)$. To get a contradiction suppose that there is ξ in $\mathcal{I}^{\perp}$ such that the spectrum of ξ is not contained in F. Hence there exists a character φ in $sp_{\mathcal{A}}(\xi)$ such that $\varphi \in \sum(\mathcal{A}) \backslash F$. Since the character space of $\mathcal{I}$ is $\sum(\mathcal{A}) \backslash F$ ([12, theorem 7.1.7]) there exists $a \in \mathcal{I}$ such that $\varphi(a) \neq 0$, that is $\varphi \in \omega(a)$. For an arbitrary $b \in \mathcal{A}$ the product ab is in $\mathcal{I}$, thus $\langle a \cdot \xi, b \rangle = \langle \xi, ab \rangle = 0$, which gives $a \cdot \xi = 0$. However, by Proposition 2.1, this is impossible because of $\varphi \in \omega(a) \cap sp_{\mathcal{A}}(\xi)$.

By lemma 1 in [2], the submodules $\mathcal{A}_F^{\perp}$ and $\mathcal{A}^*(F)$ of $\mathcal{A}^*$ coincide. Thus, $\mathcal{I}^{\perp} \subseteq \mathcal{A}_F^{\perp}$ and consequently, since $\mathcal{I}$ and $\mathcal{A}_F$ are closed, $\mathcal{A}_F \subseteq \mathcal{I}$. ■

We shall say that a closed set $F \subseteq \sum(\mathcal{A})$ is a *set of synthesis* for $\mathcal{A}$ if $\mathcal{A}_F = k(F)$. Let $\mathcal{A}$ be a unital commutative Banach algebra. The *diagonal* of $\mathcal{A}$ is the set

$$D_{\mathcal{A}} := \left\{(\varphi, \varphi); \varphi \in \sum(\mathcal{A})\right\} \subseteq \sum(\mathcal{A}) \times \sum(\mathcal{A}) = \sum(\mathcal{A} \hat{\otimes} \mathcal{A}).$$

In [13] Shulman introduced the following class of algebras. A unital commutative regular semisimple Banach algebra $\mathcal{A}$ is in the class *(SD)* (or is an *(SD)-algebra*) if $D_{\mathcal{A}}$ is a set of synthesis for $\mathcal{A} \hat{\otimes} \mathcal{A}$. For instance, the following algebras are in the class (SD): the group algebra $\ell^1(G)$ of a discrete abelian group G, the algebra $\mathcal{C}(K)$ of all complex-valued continuous functions on a compact Hausdorff space K, the projective tensor product of two such algebras, i.e. $\mathcal{C}(K_1) \hat{\otimes} \mathcal{C}(K_2)$, and some others (see [13]).

We are going to extend the class (SD). First we shall relax the condition of semisimplicity: we shall say that a spectrally separable algebra $\mathcal{A}$ is in (SD) if $D_{\mathcal{A}}$ is a set of synthesis for $\mathcal{A} \hat{\otimes} \mathcal{A}$. Note that, by proposition 2.4 (ii) in [4], $\mathcal{A} \hat{\otimes} \mathcal{A}$ is spectrally separable whenever $\mathcal{A}$ has this property.

Definition 4.2. A Banach algebra $\mathcal{C}$ (not necessarily commutative) with identity 1 is a *local (SD)-algebra* (or, it is *locally in (SD)*) if there exists a set $\mathcal{C}_0$ of generators of the algebra $\mathcal{C}$ such that, for each $c \in \mathcal{C}_0$, there exists a closed subalgebra $\mathcal{A}_c \subseteq \mathcal{C}$ that is an (SD)-algebra and contains 1 and c.

Of course, every (SD)-algebra is locally in (SD). The following proposition shows that there exist non-(SD)-algebras that are locally in (SD).

Proposition 4.3. *(i) Every unital C^*-algebra $\mathcal{C}$ is locally in the class (SD).*

(ii) The group algebra $\ell^1(G)$ of a discrete (not necessarily abelian) group G is locally in (SD).

PROOF. (i) The set $\mathcal{C}_0$ of all hermitian elements in $\mathcal{C}$ generates $\mathcal{C}$. Let h be an arbitrary element in $\mathcal{C}_0$. It is well-known that the closed subalgebra $\mathcal{A}_h$ in $\mathcal{C}$ that is generated by h and the identity is an abelian C^*-algebra and that it is actually isometrically $*$-isomorphic to the algebra $\mathcal{C}(\sigma(h))$ of all continuous complex functions on the spectrum $\sigma(h)$ (see [11, theorem 2.1.13]). Since $\mathcal{C}(\sigma(h))$ is in the class (SD) the algebra $\mathcal{A}_h$ is in (SD) as well.

(ii) For each $h \in G$ let $\delta_h : G \to \mathbb{C}$ be the function defined by $\delta_h(h) = 1$ and $\delta_h(g) = 0$, for $g \in G$ and $g \neq h$. Of course, $\{\delta_h; h \in G\}$ is a set of generators of the algebra $\ell^1(G)$. Denote by $<h>$ the subgroup in G that is generated by $h \in G$. For $k \in <h>$, define $\tilde{\delta}_k$: $<h> \to \mathbb{C}$ with $\tilde{\delta}_k(k) = 1$ and $\tilde{\delta}_k(g) = 0$, for $g \in <h>$ and $g \neq k$. The mapping $\tilde{\delta}_k \mapsto \delta_k$ is a continuous homomorphism from $\ell^1(<h>)$ into $\ell^1(G)$. Since $\ell^1(<h>)$ is in the class (SD) we may conclude that, for each $\delta_h (h \in G)$, there is a closed subalgebra in $\ell^1(G)$ that is in the class (SD) and contains δ_h. ■

Note 4.4. Every unital commutative Banach algebra that is locally in (SD) is in fact an (SD)-algebra; however, we will not give the proof because we do not need this result.

The following theorem is proven in [13] for (semisimple) (SD)-algebras. However the proof works also if the semisimplicity is not assumed. For the sake of completeness we include Shulman's original proof.

Theorem 4.5. (Shulman, [13, theorem 5]). *Let $\mathcal{A}$ be an (SD) algebra. Every local operator between two left Banach $\mathcal{A}$-modules is a multiplier.*

PROOF. Let $\mathcal{X}$ and $\mathcal{Y}$ be left Banach $\mathcal{A}$-modules. Consider the following subsets in $\mathcal{A}\hat{\otimes}\mathcal{A}$:

$$\mathcal{M} := \{a \otimes 1 - 1 \otimes a;\ a \in \mathcal{A}\} \text{ and } \mathcal{N} := \{a \otimes b;\ sp_{\mathcal{A}}(a) \cap sp_{\mathcal{A}}(b) = \emptyset\}.$$

Let us show that $h(\mathcal{M}) = D_{\mathcal{A}} = h(\mathcal{N})$. The inclusion $D_{\mathcal{A}} \subseteq h(\mathcal{M})$ is evident. Assume therefore that $(\varphi, \psi) \in \sum(\mathcal{A}\hat{\otimes}\mathcal{A})$ is not in $D_{\mathcal{A}}$ i.e. $\varphi \neq \psi$. Then there is $a \in \mathcal{A}$ such that $\varphi(a) \neq \psi(a)$. It follows

$$(\varphi, \psi)(a \otimes 1 - 1 \otimes a) = \varphi(a) - \psi(a) \neq 0,$$

which means $(\varphi, \psi) \notin h(\mathcal{M})$. If $a \otimes b$ is in $\mathcal{N}$, then $(\varphi, \varphi)(a \otimes b) = \varphi(a)\varphi(b) = 0$ because φ cannot be simultaneously in $\omega(a)$ and $\omega(b)$. On the other hand, if φ and

ψ are two distinct characters on $\mathcal{A}$, then there exist disjoint open neighbourhoods U and V of φ and ψ, respectively. By Proposition 3.1, there exist a and b in $\mathcal{A}$ such that $\varphi(a)=1$, $\psi(b)=1$, $sp_{\mathcal{A}}(a) \subseteq U$, and $sp_{\mathcal{A}}(b) \subseteq V$. It follows $a \otimes b$ is in $\mathcal{N}$ and $(\varphi, \psi)(a \otimes b) \neq 0$.

Let $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}_{\mathcal{N}}$ be the closed ideals in $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by $\mathcal{M}$, respectively by $\mathcal{N}$. Then, of course, $h(\mathcal{I}_{\mathcal{M}}) = h(\mathcal{M}) = D_{\mathcal{A}}$ and, similarly, $h(\mathcal{I}_{\mathcal{N}}) = h(\mathcal{N}) = D_{\mathcal{A}}$. However, since $D_{\mathcal{A}}$ is a set of synthesis for $\mathcal{A}\hat{\otimes}\mathcal{A}$, we have $\mathcal{I}_{\mathcal{M}} = \mathcal{I}_{\mathcal{N}} = k(D_{\mathcal{A}})$.

The space $B(\mathcal{X}, \mathcal{Y})$ has a natural structure of a Banach $\mathcal{A}$-bimodule: for $a, b \in \mathcal{A}$ and $S \in B(\mathcal{X}, \mathcal{Y})$ the products $a \cdot S$ and $S \cdot b$ are given by

$$(a \cdot S)x = a \cdot Sx \quad \text{and} \quad (S \cdot b)x = S(b \cdot x) \quad (x \in \mathcal{X}).$$

It follows that $B(\mathcal{X}, \mathcal{Y})$ has also a natural structure of a left Banach $\mathcal{A}\hat{\otimes}\mathcal{A}$-module: the product of $\sum_{n=1}^{\infty} a_n \otimes b_n \in \mathcal{A}\hat{\otimes}\mathcal{A}$ and $S \in B(\mathcal{X}, \mathcal{Y})$ is

$$\left(\sum_{n=1}^{\infty} a_n \otimes b_n\right) \cdot S = \sum_{n=1}^{\infty} a_n \cdot S \cdot b_n.$$

For an arbitrary nonempty subset $\mathcal{U} \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$, define

$$ker\, \mathcal{U} := \left\{ S \in (\mathcal{X}, \mathcal{Y});\ \left(\sum_{n=1}^{\infty} a_n \otimes b_n\right) \cdot S = 0, \text{ for all } \sum_{n=1}^{\infty} a_n \otimes b_n \in \mathcal{U} \right\}.$$

If $\mathcal{I}_{\mathcal{U}}$ is the closed ideal in $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by $\mathcal{U}$, then $ker\, \mathcal{I}_{\mathcal{U}} = ker\, \mathcal{U}$.

Now we shall show that $ker\, \mathcal{M} = B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. Indeed, if S is in $ker\, \mathcal{M}$, then $(a \otimes 1 - 1 \otimes a) \cdot S = 0$, for all $a \in \mathcal{A}$, which means $(a \cdot S)x = S(a \cdot x)$, for all $a \in \mathcal{A}$ and all $x \in \mathcal{X}$. Thus $S \in B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. The opposite inclusion is evident.

If S is in $LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ and $a, b \in \mathcal{A}$ are arbitrary, then

$$sp_{\mathcal{A}}((a \otimes b \cdot S)x) = sp_{\mathcal{A}}(a \cdot S(b \cdot x)) \subseteq sp_{\mathcal{A}}(a) \cap sp_{\mathcal{A}}(b) \quad (x \in \mathcal{X}).$$

Hence, if S is in $LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ and $a \otimes b$ is in $\mathcal{N}$, then $sp_{\mathcal{A}}((a \otimes b \cdot S)x) = \emptyset$, for all $x \in \mathcal{X}$, which implies $a \otimes b \cdot S = 0$ and consequently $S \in ker\, \mathcal{N}$. Since the ideals $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}_{\mathcal{N}}$ coincide, the sets $ker\, \mathcal{M}$ and $ker\, \mathcal{N}$ coincide as well. Thus,

$$LO_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}) \subseteq ker\, \mathcal{N} = ker\, \mathcal{M} = B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}). \quad \blacksquare$$

At the end of [13] Shulman shows that a unital commutative Banach algebra $\mathcal{A}$, that is also semisimple and regular, must be an (SD)-algebra if all local operators between any pair of left Banach $\mathcal{A}$-modules are multipliers. It would be interesting to know if this is true also when semisimplicity is replaced by spectral separability. Note, however, that in general, it is hard to check if all local operators are multipliers. Thus it seems that it makes no sense looking for (SD)-algebras in this way. The next theorem, although not so sharp as Shulman's result, could be a more appropriate tool for testing a spectrally separable algebra to be in the class (SD).

Theorem 4.6. *Let $\mathcal{A}$ be a spectrally separable algebra. If every local operator $T \in B(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary left Banach $\mathcal{A}$-modules, is a multiplier, then each singleton $\{\varphi\}$, $\varphi \in \sum(\mathcal{A})$, is a set of synthesis for $\mathcal{A}$.*

PROOF. Let φ be an arbitrary character on $\mathcal{A}$ and consider the left Banach $\mathcal{A}$-modules $\mathcal{X} := \mathcal{A}/\mathcal{A}_{\{\varphi\}}$ and $\mathcal{Y} := \mathcal{A}/k(\varphi)$. Obviously, the module $\mathcal{Y}$ can be identified with $\mathbb{C}_\varphi$, which implies $sp_{\mathcal{A}}(y) = \{\varphi\}$, for every nonzero $y \in \mathcal{Y}$. The same is valid for the module $\mathcal{X}$: a nonzero $x \in \mathcal{X}$ has the spectrum $sp_{\mathcal{A}}(x) = \{\varphi\}$. It follows that $\{0\}$ and $\mathcal{X}$ are the only spectral submodules of $\mathcal{X}$. The same holds, of course, for $\mathcal{Y}$. Thus, each $T \in B(\mathcal{X}, \mathcal{Y})$ preserves spectral submodules and is therefore a local operator. By assumption, local operators are multipliers and therefore $B(\mathcal{X}, \mathcal{Y}) = B_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$. Since $\mathcal{Y} = \mathbb{C}_\varphi$, we may conclude, that each $\xi \in \mathcal{X}^*$ has the property $\langle \xi, a \cdot x \rangle = \varphi(a)\langle \xi, x \rangle$ $(a \in \mathcal{A},\ x \in \mathcal{X})$.

It is well known that $\mathcal{X}^* = (\mathcal{A}/\mathcal{A}_{\{\varphi\}})^* = \mathcal{A}_{\{\varphi\}}^{\perp}$. Let ξ be a nonzero functional in $\mathcal{A}_{\{\varphi\}}^{\perp}$ (i.e. $\xi \in \mathcal{X}^*$) and let $e \in \mathcal{A}$ be such that $\langle \xi, e \rangle = 1$. Then, for an arbitrary $a \in \mathcal{A}$, we have

$$\begin{aligned}\varphi(a) &= \varphi(a)\langle \xi, e \rangle = \langle \xi, a \cdot (e + \mathcal{A}_{\{\varphi\}}) \rangle = \langle \xi, ae + \mathcal{A}_{\{\varphi\}} \rangle = \langle \xi, e \cdot (a + \mathcal{A}_{\{\varphi\}}) \rangle \\ &= \varphi(e)\langle \xi, a \rangle = \langle \varphi(e)\xi, a \rangle.\end{aligned}$$

It follows $\varphi = \varphi(e)\xi$ and we conclude that $\mathcal{A}_{\{\varphi\}}$ has codimension 1 in $\mathcal{A}$. Since $\mathcal{A}_{\{\varphi\}} \subseteq k(\varphi)$, these two ideals coincide. ■

Corollary 4.7. *For every algebra $\mathcal{A}$ in the class (SD), the singletons $\{\varphi\}$, $\varphi \in \sum(\mathcal{A})$, are sets of synthesis for $\mathcal{A}$.*

PROOF. By Theorem 4.5, every local operator is a multiplier, hence, by the previous proposition, singletons are sets of synthesis. ■

Example 4.8. Let $C^1[0,1]$ be the Banach algebra of all complex-valued continuous functions on the interval [0,1] that have continuous derivatives. It is well-known that this algebra is semisimple and regular and that the points in $\sum(C^1[0,1]) = [0,1]$ are not sets of spectral synthesis for $C^1[0,1]$. Thus, $C^1[0,1]$ is an example of a spectrally separable algebra that is not in the class (SD). The following example, which is borrowed from [6], shows that Theorem 4.5 is not valid for $C^1[0,1]^*$.

Let δ_0 be a linear functional on $C^1[0,1]$ defined by $f \mapsto f(0)$. Then the operator $Tg := g'(0)\delta_0 (g \in C^1[0,1])$ is a local multiplier from $C^1[0,1]$ into $C^1[0,1]^*$. To show this it is enough to see that, for each $g \in C^1[0,1]$, there is an element $F_g \in C^1[0,1]^*$ with

$$\langle Tg, f \rangle = g'(0)f(0) = \langle F_g, fg \rangle \quad (f \in C^1[0,1]).$$

Namely, this means that T is a multiplication by the functional F_g locally at g. If $g(0) \neq 0$, we set $F_g = (g'(0)/g(0))\delta_0$, and if $g(0) = 0$, we set $F_g(f) = f'(0)$. It follows $\langle F_g, fg \rangle = (fg)'(0) = g'(0)f(0)$. Obviously, T is not a multiplier because of $T(fg) = (fg)'(0)\delta_0$ and $fTg = f(0)g'(0)\delta_0$.

5. Local multipliers

Since, in general, an algebra $\mathcal{A}$ that is locally in (SD) is not necessarily commutative, it is possible that $\sum(\mathcal{A})$ is empty. Therefore the definition of local operators makes no sense. On the other hand, the notion of local multiplier is still present. However, we have the following result.

Theorem 5.1. *If $\mathcal{C}$ is a unital Banach algebra that is locally in the class (SD), then every local multiplier $T \in B(\mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary left Banach $\mathcal{C}$-modules, is a multiplier.*

PROOF. Let $\mathcal{X}$ and $\mathcal{Y}$ be left Banach $\mathcal{C}$-modules and $T \in B(\mathcal{X}, \mathcal{Y})$ be a local multiplier. Choose $c \in \mathcal{C}_0$. Then there is a closed subalgebra $\mathcal{A}_c$ in $\mathcal{C}$, which is in (SD), and $c \in \mathcal{A}_c$. Of course, $\mathcal{X}$ and $\mathcal{Y}$ are left Banach $\mathcal{A}_c$-modules. If M is in $B_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$, then it is in $B_{\mathcal{A}_c}(\mathcal{X}, \mathcal{Y})$. This implies that T is a local multiplier and hence a local operator if $\mathcal{X}$ and $\mathcal{Y}$ are considered as left Banach $\mathcal{A}_c$-modules. By Theorem 4.5, operator T belongs to $B_{\mathcal{A}_c}(\mathcal{X}, \mathcal{Y})$. In particular we have $T(c \cdot x) = c \cdot Tx$, for all $x \in \mathcal{X}$. Since c was an arbitrary element in $\mathcal{C}_0$ it follows $T(c_1 \dots c_n \cdot x) = c_1 \dots c_n \cdot Tx$, for all $x \in \mathcal{X}$ and an arbitrary finite family $\{c_1, \dots, c_n\} \subseteq \mathcal{C}_0$. Thus, $T \in B_{\mathcal{C}}(\mathcal{X}, \mathcal{Y})$ because it is linear and continuous. ■

Corollary 5.2. *If $\mathcal{X}$ and $\mathcal{Y}$ are left Banach modules over a unital C*-algebra or over the group algebra of a discrete group, then every local multiplier $T \in B(\mathcal{X}, \mathcal{Y})$ is a multiplier.*

PROOF. The assertion follows by the previous theorem and Proposition 4.3. ■

REFERENCES

[1] G.R. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Localness of $A(\Psi)$ algebras, *Mathematical Proceedings of the Royal Irish Academy* **101**A (1) (2001), 61–70.

[2] A.G. Baskakov, Spectral synthesis in Banach modules over commutative Banach algebras, *Mathematical Notes* **34** (1983), 776–82. (Translation from Matematicheskie Zametki.)

[3] F.F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.

[4] J. Bračič, Unital strongly harmonic commutative Banach algebras, *Studia Mathematica* **149** (2002), 253–66.

[5] S. Dineen, *Complex analysis on infinite-dimensional spaces*, Monographs in Mathematics, Springer, London, 1999.

[6] B.E. Johnson, Local derivations on C^*-algebras are derivations, *Transactions of the American Mathematical Society* **353** (2000), 313–25.

[7] R.V. Kadison, Local derivations, *Journal of Algebra* **130** (1990), 494–509.

[8] R. Kantrowitz and M.M. Neumann, Disjointness preserving and local operators on algebras of differentiable functions, *Glasgow Mathematical Journal* **43** (2001), 295–309.

[9] D.R. Larson and A.R. Sourour, Local derivations and local automorphisms of $B(X)$ *Proceedings of Symposia in Pure Mathematics* **51** (1990), Part 2, 187–94.

[10] K.B. Laursen and M.M. Neumann, *An Introduction to Local Spectral Theory,* Clarendon Press, Oxford, 2000.

[11] G.J. Murphy, *C^*-algebras and operator algebras*, Academic Press, Boston, 1990.

[12] T.W. Palmer, *Banach Algebras and the General Theory of $*$-Algebras, Volume 1: Algebras and Banach Algebras*, Cambridge University Press, Cambridge, 1994.

[13] V.S. Shulman, Spectral Synthesis and the Fuglede-Putnam-Rosenblum Theorem [in Russian], *Teoriya Funktsii, Funktsional'nyi Analiz i ikh Prilozheniya* **54** (1990), 25–36.