AN UNCERTAINTY PRINCIPLE RELATED TO THE EUCLIDEAN MOTION GROUP

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ABSTRACT

We show that a well-known uncertainty principle for functions on the circle can be derived from an uncertainty principle for the Euclidean motion group.

1. Uncertainty principles related to Lie group representations

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let (π, H) be a unitary representation of G. Then each element $X \in \mathfrak{g}$ generates a closed, skew-adjoint operator $\pi(X)$ on H by

$$\pi(X)x = \lim_{t \to 0} \frac{\pi(\exp tX)x - x}{t},$$

with domain $D(\pi(X))$ consisting of all $x \in H$ for which the limit exists.

The uncertainty principle related to π says that for operators generated by X, Y and $[X, Y]$ the following holds

$$\|\pi(X)x\| \, \|\pi(Y)x\| \geq \frac{1}{2} |\langle \pi([X, Y])x, x \rangle|$$

for all $x \in D(\pi(X)) \cap D(\pi(Y)) \cap D(\pi([X, Y]))$.

We would like to advocate this as a natural way to achieve uncertainty principles. It was first proposed in Kraus [4] for Lie groups with three dimensions or fewer, and the dimension constraint was recently removed by Christensen in [2]. For example, the classical Heisenberg uncertainty principle for functions on $\mathbb{R}^n$ is easily derived in this way from the Schrödinger representation of the Heisenberg group (see [3, p. 212]).

It is the purpose of this note to point out that the uncertainty principle for the circle, which was motivated by Breitenberger [1] and further discussed in [5; 6; 7; 8; 9; 10] is obtained similarly from the principal series representation of the Euclidean motion group of $\mathbb{R}^2$.

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2. The Euclidean motion group and a unitary representation

Let G be the Euclidean motion group

$$G = \left\{ (r,z) = \begin{pmatrix} e^{ir} & z \\ 0 & 1 \end{pmatrix} | r \in \mathbb{R}, z \in \mathbb{C} \right\}$$

Its Lie algebra is

$$\mathfrak{g} = \left\{ \begin{pmatrix} ir & z \\ 0 & 0 \end{pmatrix} | r \in \mathbb{R}, z \in \mathbb{C} \right\}$$

Let H be the Hilbert space $H = L_2(\mathbb{T})$ of square integrable functions on the circle $\mathbb{T} = \{s \in \mathbb{C} | \, |s| = 1\}$, with inner product $\langle f, g \rangle = \int_{\mathbb{T}} f(t)\overline{g(t)}dt$. As in [11, chapter V] the following defines a unitary representation of G on H:

$$\pi_a(r,z)f(s) = e^{i\mathrm{Re}(z\overline{sa})} f(e^{-ir}s), \quad (s \in \mathbb{T})$$

where $a \in \mathbb{C}$. For simplicity we assume in the following that $a = 1$, which is sufficient for our purpose. The representation π_1 will be denoted π.

3. Operators generated from the representation

We now generate three operators from elements of the Lie algebra $\mathfrak{g}$. Let X, Y_1, $Y_2 \in \mathfrak{g}$ be

$$X = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

then

$$\exp(tX) = \begin{pmatrix} e^{it} & 0 \\ 0 & 1 \end{pmatrix}, \quad \exp(tY_1) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \exp(tY_2) = \begin{pmatrix} 1 & it \\ 0 & 1 \end{pmatrix}$$

and we then get

$$\pi(X)f(s) = \lim_{t \to 0} \frac{\pi(t,0)f(s) - f(s)}{t} = \lim_{t \to 0} \frac{f(e^{-it}s) - f(s)}{t} = -f'(s),$$

where $f'(s) = \frac{d}{dt} f(e^{it}s)$. This operator has domain

$$\{f \in L_2(\mathbb{T}) | t \mapsto f(e^{it}) \text{ absolutely continuous with } f' \in L_2(\mathbb{T})\} \tag{3.1}$$

Also

$$\pi(Y_1)f(s) = \lim_{t \to 0} \frac{\pi(0,t)f(s) - f(s)}{t} = \lim_{t \to 0} \frac{e^{i(t,s)}f(s) - f(s)}{t} = i\cos(\theta)f(s)$$

and

$$\pi(Y_2)f(s) = \lim_{t \to 0} \frac{\pi(0,it)f(s) - f(s)}{t} = \lim_{t \to 0} \frac{e^{i(it,s)}f(s) - f(s)}{t} = i\sin(\theta)f(s)$$

when $s = e^{i\theta}$. Both of these operators are defined on H.

4. The uncertainty principle

Since

$$[X, Y_1] = Y_2, \quad [X, Y_2] = -Y_1$$

the uncertainty principle gives

$$|\langle \pi(Y_1)f, f\rangle| = |\langle \pi([X, Y_2])f, f\rangle| \leq 2\|\pi(X)f\|\,\|\pi(Y_2)f\| \tag{4.1}$$

and

$$|\langle \pi(Y_2)f, f\rangle| = |\langle \pi([X, Y_1])f, f\rangle| \leq 2\|\pi(X)f\|\,\|\pi(Y_1)f\|. \tag{4.2}$$

Let $T = i\pi(X)$ denote the operator $Tf = -if'$ with domain (3.1), and let $S_1 = i\pi(Y_1)$, $S_2 = i\pi(Y_2)$, then T, S_1 and S_2 are selfadjoint and the unitary operator $S = S_1 + iS_2$ is given by $Sf(e^{i\theta}) = -e^{i\theta}f(e^{i\theta})$ with $D(S) = H$.

Now

$$\|Sf\|^2 = \|S_1 f\|^2 + \|S_2 f\|^2 = \|f\|^2$$

and

$$|\langle Sf, f\rangle|^2 = |\langle S_1 f, f\rangle|^2 + |\langle S_2 f, f\rangle|^2$$

so the uncertainty principles (4.1) and (4.2) give

$$|\langle Sf, f\rangle|^2 \leq 4\|f\|^2\|Tf\|^2.$$

As explained in [8, theorem 2.3.3] this is exactly the uncertainty principle of Breitenberger. Notice that in this case we have a 3-dimensional Lie group so that the example is covered by [4].

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