CYCLICITY AND UNICELLULARITY OF THE DIFFERENTIATION OPERATOR ON BANACH SPACES OF FORMAL POWER SERIES

By

B. Yousefi*
Department of Mathematics, Shiraz University, Shiraz 71454, Iran

and

A.I. Kashkuli
Department of Mathematics, Yasuj University, Yasuj, Iran

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Abstract

We investigate compactness, cyclicity and unicellularity of the differentiation operator on certain weighted sequence spaces.

1. Introduction

On most of the classical Banach function spaces the differentiation operator is closed and densely defined, but unbounded. On certain weighted sequence spaces, however, it can be presented as an everywhere-defined and bounded linear operator. In this note we examine its behavior on a class of such spaces, generalising definitions due to Shields [3]. Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0)=1$ and $1 \leq p < \infty$. We consider the space of sequences $f=\{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z)=\sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^p(\beta)$ denote the space of such formal power series. Let $\hat{e}_n(n)=\delta_k(n)$. So $e_k(z)=z^k$ and then $\{e_k\}_k$ is a basis such that $\|e_k\|=\beta(k)$. For $1<p<\infty$, $H^p(\beta) \cong L^p(\mu)$ where μ is the σ-finite measure defined on the positive integers by

$$\mu(K)=\sum_{n \in K} (\beta(n))^p, \qquad K \subseteq \mathbf{N} \cup \{0\}.$$

So $H^p(\beta)$ is a reflexive Banach space and the dual of $H^p(\beta)$ is $H^q\left(\beta^{\frac{p}{q}}\right)$ where $\frac{1}{p}+\frac{1}{q}=1$

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*Corresponding author; e-mail: byousefi@hafez.shirazu.ac.ir

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and $\beta^{\frac{p}{q}} = \left\{\beta(n)^{\frac{p}{q}}\right\}_n$ [2; 4]. Also, if $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H^q\left(\beta^{\frac{p}{q}}\right)$, then $\|g\|_q^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p$. The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and, respectively, $\beta(n) = 1$, $\beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$. If $\lim_n \frac{\beta(n+1)}{\beta(n)} = 1$, or $\liminf_n \beta(n)^{\frac{1}{n}} = 1$, then $H^p(\beta)$ consists of functions analytic on the open unit disc. Bounded analytic structure of $H^p(\beta)$ was studied in [7]. If the sequence $\left\{\frac{\beta(n)}{\beta(n+1)}\right\}_n$ decreases to zero, then each element of $H^p(\beta)$ is an entire function. It is convenient and helpful to introduce the notation $\langle f, g\rangle$ or $\langle g, f\rangle$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^p(\beta)^*$. Note that $\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$. Let X be a Banach space. We denote by $B(X)$, the set of bounded operators on the Banach space X. Let $A \in B(X)$ and $x \in X$. We say that x is a *cyclic vector* of A if

$$X = \operatorname{span}\{A^n x : n = 0, 1, 2, \ldots\}.$$

Here span$\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$. An operator $A \in B(X)$ is called cyclic if it has a cyclic vector. Also, an operator A in $B(X)$ is called a unicellular operator on X, if the set of its invariant subspaces, *LatA*, is linearly ordered by inclusion. Cyclicity, strict cyclicity and unicellularity of the multiplication operator were studied in [4; 5; 6; 9]. Composition operators on $H^p(\beta)$ were studied in [8]. In this paper we study the differentiation operators on $H^p(\beta)$.

2. Main results

In what follows, D denotes the differentiation operator on the Banach space $H^p(\beta)$. The functions $\beta_n = \frac{e_n}{\beta(n)}$ form a basis for $H^p(\beta)$ with $\|\beta_n\|_p = 1$ for all $n = 0, 1, 2, \ldots$. Note that $D\beta_0 = 0$ and for $n \geq 1$, $D\beta_n = \frac{n\beta(n-1)}{\beta(n)}\beta_{n-1}$. Thus the operator D is everywhere defined and bounded if and only if the sequence $\left\{\frac{n\beta(n-1)}{\beta(n)}\right\}_{n=1}^{\infty}$ is bounded and in this case for all $k \in \mathbf{N}$, $\|D^k\| = \sup_n w_n w_{n+1} \ldots w_{n+k-1}$ where $w_n = \frac{n\beta(n-1)}{\beta(n)}$ for all $n \geq 1$. Indeed $\|D^k\| = \sup_n n(n+1)\ldots(n+k-1)\frac{\beta(n-1)}{\beta(n+k-1)}$, $k \in N$. From now on we suppose that the sequence $\left\{\frac{\beta(n)}{\beta(n+1)}\right\}_n$ decreases to zero and that D is a bounded operator on $H^p(\beta)$. We give sufficient conditions for cyclicity and unicellularity of the differentiation operator on the Banach space $H^q\left(\beta^{\frac{p}{q}}\right)$. We also investigate the compactness of D on $H^p(\beta)$.

Proposition 1. *The operator D is compact on $H^p(\beta)$ if and only if the sequence $\left\{\frac{n\beta(n-1)}{\beta(n)}\right\}_n$ converges to zero.*

Proof. Suppose first that $\frac{n\beta(n-1)}{\beta(n)} \to 0$. Fix a positive integer m, and define

$$D_m f = \sum_{k=0}^{m-1} (k+1)\hat{f}(k+1)e_k$$

for all $f = \sum_{n=0}^{\infty} \hat{f}(n)e_n \in H^p(\beta)$. Then

$$\begin{aligned}\|(D-D_m)f\|_p &= \left\|\sum_{k=m}^{\infty}(k+1)\hat{f}(k+1)e_k\right\|_p\\ &\leq \sup_{k\geq m}\frac{(k+1)\beta(k)}{\beta(k+1)}\|f\|_p\end{aligned}$$

for all $f=\sum_{n=0}^{\infty}\hat{f}(n)e_n\in H^p(\beta)$. Since $\frac{n\beta(n-1)}{\beta(n)}\to 0$, we get $\|D-D_m\|\to 0$. Thus D is the limit in norm of a sequence of finite rank operators, and hence it is compact. Conversely, let D be compact on $H^p(\beta)$. Then for each $g=\sum_{k=0}^{\infty}\hat{g}(k)\,e_k\in H^q\left(\beta^{\frac{p}{q}}\right)$ we have

$$\left\langle\frac{e_n}{\beta(n)},g\right\rangle=\hat{g}(n)\beta(n)^{p-1}=\hat{g}(n)\beta(n)^{\frac{p}{q}}$$

for all $n\in\mathbf{N}$. Since $\|g\|_q^q=\sum|\hat{g}(n)|^q\beta(n)^p$, we get $|\hat{g}(n)|\beta(n)^{\frac{p}{q}}\to 0$ as $n\to\infty$. Hence $\frac{e_n}{\beta(n)}\to 0$ weakly, and so

$$\left\|D\left(\frac{e_n}{\beta(n)}\right)\right\|=\frac{n\beta(n-1)}{\beta(n)}\to 0,$$

since D is compact [1, proposition 3.3]. This completes the proof. ■

Proposition 2. *Suppose that* $\left\{\frac{\beta(n)}{\beta(n+1)}\right\}_n$ *is monotonically decreasing and* $r=\lim\frac{n\beta(n-1)}{\beta(n)}$. *Then the spectrum of D is equal to* $\{z:|z|\leq r\}$.

PROOF. First note that $\frac{n\beta(n-1)}{\beta(n)}\geq r$ for each n, and so

$$\beta(n)\leq\frac{n\beta(n-1)}{r}\leq\frac{n(n-1)}{r^2}\beta(n-2)\leq\cdots\leq\frac{n!}{r^n}.$$

Now since $\left\{\frac{n\beta(n-1)}{\beta(n)}\right\}_n$ is decreasing, we have $\|D^k\|=\frac{k!}{\beta(k)}$. Thus

$$\begin{aligned}\lim_k\|D^k\|^{\frac{1}{k}}&=\lim_k\left(\frac{k!}{\beta(k)}\right)^{\frac{1}{k}}\\ &=\lim_k\left(\frac{(k+1)!}{\beta(k+1)}\Big/\left(\frac{k!}{\beta(k)}\right)\right)\\ &=\lim_k\frac{(k+1)\beta(k)}{\beta(k+1)}=r,\end{aligned}$$

and so the spectral radius of D is equal to r. Hence $\sigma(D)\subseteq\{z:|z|\leq r\}$ where $\sigma(D)$ denotes the spectrum of D. Now let $|\lambda|<r$. Then since $\beta(n)\leq\frac{n!}{r^n}$, we get

$$\left(\frac{\beta(n)}{n!}\right)^p\leq\left(\frac{1}{r^n}\right)^p,$$

which implies that

$$\begin{aligned}\|e^{\lambda z}\|_p^p &= \sum_{n=0}^{\infty} \frac{|\lambda|^{np}}{(n!)^p} (\beta(n))^p \\ &\leq \sum_{n=0}^{\infty} \left(\frac{|\lambda|^p}{r^p}\right)^n,\end{aligned}$$

and that the last series converges, since $\frac{|\lambda|}{r} < 1$. So $e^{\lambda z} \in H^p(\beta)$. But $De^{\lambda z} = \lambda e^{\lambda z}$, thus $\lambda \in \sigma(D)$ and so $\{z : |z| < r\} \subset \sigma(D)$. This completes the proof. ■

A little calculation shows that the adjoint operator $D^* : H^p(\beta)^* \to H^p(\beta)^*$ is defined by $D^* e_n = (n+1)\left(\frac{\beta(n)}{\beta(n+1)}\right)^p e_{n+1}$. In the following we denote by H_m, the closed linear span of $\{e_n : n \geq m\}$ that is a subspace of $H^p(\beta)^*$. Recall that $H^p(\beta)^* = H^q\left(\beta^{\frac{p}{q}}\right)$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|e_n\|_{H^p(\beta)} = \beta(n)$ and that $\|e_n\|_{H^q\left(\beta^{\frac{p}{q}}\right)} = \beta(n)^{\frac{p}{q}}$ for all n.

Theorem 3. *Let the sequence* $\left\{\frac{n\beta(n-1)}{\beta(n)}\right\}_n$ *decrease and belong to* ℓ^p. *If* $f = \sum \hat{f}(k)\, e_k \in (H^p(\beta))^*$ *with* $\hat{f}(0) \neq 0$, *then* f *is a cyclic vector for* D^*.

Proof. Note that for all positive integers k and n we have

$$(D^*)^n e_k = (k+1)(k+2)\ldots(k+n)\left(\frac{\beta(k)}{\beta(n+k)}\right)^p e_{k+n}.$$

Now for $n \geq 1$ we get

$$(D^*)^n f = \sum_{k=0}^{\infty} \hat{f}(k)(k+1)\ldots(k+n)\left(\frac{\beta(k)}{\beta(n+k)}\right)^p e_{k+n}.$$

Without loss of generality we may assume that $\hat{f}(0) = 1$. Put $y_0 = 1$ and

$$y_n = \frac{\beta(n)^p}{n!}(D^*)^n f, \qquad n \geq 1.$$

Thus we have

$$y_n = e_n + \frac{\beta(n)^p}{n!}\sum_{k=1}^{\infty} \hat{f}(k)(k+1)\ldots(k+n)\left(\frac{\beta(k)}{\beta(n+k)}\right)^p e_{k+n},$$

and so

$$\|y_n - e_n\|_q = \left(\sum_{k\geq 1} \frac{\beta(n)^{pq}}{(n!)^q}(k+1)^q \ldots (k+n)^q |\hat{f}(k)|^q \left(\frac{\beta(k)}{\beta(n+k)}\right)^{pq} \beta(n+k)^p\right)^{\frac{1}{q}}.$$

Since $\left\{\frac{n\beta(n-1)}{\beta(n)}\right\}_n$ is decreasing, we get

$$\frac{1}{\beta(n)^{p-1}}\|e_n-y_n\|_q \le \beta(1)\left((n+1)\frac{\beta(n)}{\beta(n+1)}\right)\left(\sum_{k\ge 1}|\hat{f}(k)|^q\left(\beta(k)^{\frac{p}{q}}\right)^q\right)^{\frac{1}{q}}$$

$$\le \beta(1)\left((n+1)\frac{\beta(n)}{\beta(n+1)}\right)\|f\|_q.$$

Thus

$$\sum_{n\ge 1}\frac{1}{\beta(n)^{\frac{p^2}{q}}}\|e_n-y_n\|_q^p \le \beta(1)^p\|f\|_q^p\sum_{n\ge 1}\left((n+1)\frac{\beta(n)}{\beta(n+1)}\right)^p < \infty,$$

and so there exists a positive integer n_0 such that

$$\lambda=\beta(1)^p\|f\|_q^p\sum_{n\ge n_0}\left((n+1)\frac{\beta(n)}{\beta(n+1)}\right)^p<1.$$

Therefore for any finite linear combinations

$$\varphi=\sum d_k y_{n_0+k}/\beta(n_0+k)^{\frac{p}{q}}, \quad \psi=\sum d_k e_{n_0+k}/\beta(n_0+k)^{\frac{p}{q}},$$

by the Hölder inequality we have

$$\|\varphi-\psi\|_q \le \sum|d_k|\left((\|y_{n_0+k}-e_{n_0+k}\|_q/\beta(n_0+k)^{\frac{p}{q}}\right)$$

$$\le\left(\sum|d_k|^q\right)^{\frac{1}{q}}\left(\sum_{n\ge n_0}\|y_n-e_n\|_q^p/\beta(n)^{\frac{p^2}{q}}\right)^{\frac{1}{p}}$$

$$\le \lambda^{\frac{1}{p}}\|\psi\|_q.$$

We now see that that the operator S defined from $\mathrm{span}\{e_n\}_{n\ge n_0}$ to $H^q\left(\beta^{\frac{p}{q}}\right)$ by $Se_n=y_n \quad (n\ge n_0)$ extends to a bounded operator on $H_{n_0}=\mathrm{span}\{e_n: n\ge n_0\}$ with $\|S-I\|\le 1$, where $\mathrm{span}\{\cdot\}$ denotes the closed linear span of $\{\cdot\}$ in $H^q\left(\beta^{\frac{p}{q}}\right)$. Thus S is invertible on H_{n_0}, and so the sequence $\{y_n\}_{n\ge n_0}$ inherits the basis property of $\{e_n\}_{n\ge n_0}$ for H_{n_0}. Thus $\{y_n\}_{n\ge n_0}$ is a complete set, i.e., spanning H_{n_0}. It is clear now that $H_{n_0-1}=\mathrm{span}\{y_n: n\ge n_0-1\}$ where $H_{n_0-1}=\mathrm{span}\{e_n: n\ge n_0-1\}$. By continuing this process we conclude that $\mathrm{span}\{y_n: n\ge 0\}=H^q\left(\beta^{\frac{p}{q}}\right)$, and so

$$\mathrm{span}\left\{\frac{\beta(n)^p}{n!}(D^*)^n f: n\ge 0\right\}=H^q\left(\beta^{\frac{p}{q}}\right),$$

which implies that

$$\mathrm{span}\{(D^*)^n f: n\ge 0\}=(H^p(\beta))^*.$$

Thus indeed f is a cyclic vector for D^*. ■

Theorem 4. *Let the sequence* $\left\{\frac{\beta(n-1)}{\beta(n)}\right\}_n$ *be a decreasing sequence in* ℓ^p. *Then* D^* *is unicellular on* $H^q\left(\beta^{\frac{p}{q}}\right)$.

Proof. Suppose that $\mathcal{M} \neq \{0\}$ is a closed subspace of $H^q\left(\beta^{\frac{p}{q}}\right)$ that is invariant for D^*. Put

$$n_0 = \inf\{n : \hat{f}(n) \neq 0,\ f \in \mathcal{M}\}.$$

Then $\mathcal{M} \subseteq H_{n_0}$. Now we show that indeed $\mathcal{M} = H_{n_0}$. Note that there exists $f \in \mathcal{M}$ with $\hat{f}(n_0) \neq 0$. By applying Theorem (3) with $(H^p(\beta))^*$ replaced by H_{n_0}, we conclude that the vectors $\{(D^*)^n f : n \geq 0\}$, all of which belong to $\mathcal{M}$, span a dense subspace of H_{n_0}. Thus $\mathcal{M} = H_{n_0}$ and so D^* is unicellular. ■

Corollary 1. Let the sequence $\left\{n\frac{\beta(n-1)}{\beta(n)}\right\}_n$ be a decreasing sequence in ℓ^p. Then every function in $H^p(\beta)$ that is not a polynomial is a cyclic vector for D.

Proof. Suppose that $f \in H^p(\beta)$ is not cyclic for D. We show that f is a polynomial. Put

$$\mathcal{M} = \operatorname{span}\{D^n f : n = 0, 1, 2, \ldots\}.$$

Since f is not cyclic, $\mathcal{M}$ is a closed, proper, D-invariant subspace of $H^p(\beta)$. Put

$$\mathcal{N} = \{F \in (H^p(\beta))^* : F(x) = 0 \quad \forall x \in \mathcal{M}\}.$$

If $F \in \mathcal{N}$ and $f \in \mathcal{M}$, then

$$(D^* F, f) = (F, Df) = 0,$$

since $Df \in \mathcal{M}$. Thus $\mathcal{N}$ is a nontrivial D^*-invariant subspace of $H^p(\beta)^*$. By Theorem 4, there exists $n_0 \in \mathbf{N}$ such that $\mathcal{N} = H_{n_0} = \operatorname{span}\{e_n : n \geq n_0\}$. So $\hat{f}(n) = \langle f, e_n\rangle = 0$ for all $n \geq n_0$. This implies that $f(z) = \sum_{n=0}^{n_0-1} \hat{f}(n) z^n$, which is a polynomial. ■

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