HOMOTHETIC MOTIONS IN SEMI-EUCLIDEAN SPACE $\mathbb{E}_2^4$

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ABSTRACT

In this study, a Hamilton motion is defined in four-dimensional Euclidean space $\mathbb{E}_2^4$, and it is shown that it is a homothetic motion. Furthermore, it is found that the Hamilton motion defined by a regular curve of order r has only one acceleration centre of order $(r-1)$ at every instant t.

1. Introduction

Inoguchi [2] reformulated the Gauss–Codazzi equations in a form similar to the theory of integrable system in Minkowski 3-space $\mathbb{E}_1^3$. The main tool of this reformulation was split quaternion numbers (also called Gödel quaternions in the literature). We have expressed Hamilton motion by means of Hamilton operators in semi-Euclidean space $\mathbb{E}_2^4$. We show that the Hamilton motions in semi-Euclidean space $\mathbb{E}_2^4$, are homothetic motions.

We found that the homothetic motion has only one pole point at every instant t. Also, we prove that a defined motion has only one acceleration centre of higher order at every instant t.

2. Split quaternions

A split quaternion p is an expression of the form

$$p = a_0 + a_1 i + a_2 j + a_3 k,$$

where a_0, a_1, a_2 and a_3 are real numbers, and i, j, k are split quaternionic units that satisfy the non-commutative multiplication rules

$$i^2 = -1, \quad j^2 = k^2 = 1$$
$$ij = -ji = k, \quad jk = -kj = -i,$$

and

$$ki = -ik = j.$$

Let us denote the algebra of split quaternions by H' and its natural basis by $\{1, i, j, k\}$. An element of H' is called a split quaternion [2].

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Definition 2.1. $\mathbb{E}^n$ with the metric tensor

$$\langle u, v\rangle = -\sum_{i=1}^{v} v_i w_i + \sum_{j=v+1}^{n} v_j w_j \qquad v, w \in \mathbb{E}^n, \quad 0 \le v \le n$$

is called semi-Euclidean space and is denoted by $\mathbb{E}_v^n$ where v is called the index of the metric. The resulting semi-Euclidean space $\mathbb{E}_v^n$ is reduced to $\mathbb{E}^n$ if $v = 0$. For $n \geq 2$, $\mathbb{E}_1^n$ is called Minkowski n-space; if $n = 4$ it is the simplest example of a relativistic space time [3].

Definition 2.2. Let $\mathbb{E}_v^n$ be a semi-Euclidean space furnished with a metric tensor $\langle , \rangle$. A vector v to $\mathbb{E}_v^n$ is called

$$\begin{gathered} \text{spacelike if } \langle v, v\rangle > 0 \text{ or } v = 0, \\ \text{null (a light vector) if } \langle v, v\rangle = 0 \text{ and } v \neq 0, \\ \text{timelike if } \langle v, v\rangle < 0 \end{gathered}$$

(c.f. [3]).

In the case when $0 \le v \le n$, the signature matrix ε is the diagonal matrix $[\delta_{ij}\varepsilon_j]$ whose diagonal entries are

$$\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_v = -1$$

and

$$\varepsilon_{v+1} = \varepsilon_{v+2} = \ldots = \varepsilon_n = 1.$$

Hence (c.f. [3])

$$\varepsilon = \begin{bmatrix} -I_v & 0 \\ 0 & I_{n-v} \end{bmatrix}.$$

Definition 2.3. The set of all linear isometries $\mathbb{E}_v^n \to \mathbb{E}_v^n$ is the same as the set $O(v, n)$ of all matrices $A \in GL(n, \mathbb{R})$ preserving the scalar product

$$\langle v, w\rangle = \varepsilon v \cdot w, \qquad v, w \in \mathbb{E}_v^n$$

of $\mathbb{E}_v^n$, where '$\cdot$' is Euclidean inner product.

The group $O(v, n)$ is denoted by $O_v(n)$. Hence (c.f. [3])

$$O_v(n) = \{A \in GL(n, \mathbb{R}) / \langle Av, Aw\rangle = \langle v, w\rangle, \quad \forall v, w \in \mathbb{E}_v^n\}$$
$$SO_v(n) = \{A \in O_v(n) / \det A = 1\}.$$

Definition 2.4. The following conditions of an $n \times n$ matrix are equivalent

(i) $A \in O_v(n)$
(ii) $A^T = \varepsilon A^{-1} \varepsilon$
(iii) The columns [rows] of A form an orthonormal basis for $\mathbb{E}_v^n$ (first v vectors timelike)
(iv) A carries one (hence every) orthonormal basis for $\mathbb{E}_v^n$ to an orthonormal basis (c.f. [3]).

The matrix A is called a real semi-orthogonal matrix.

Hereafter we identify H' with the semi-Euclidean space $\mathbb{E}_2^4$, where

$$\mathbb{E}_2^4 = \{a = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 : \langle a, a\rangle = -a_0^2 - a_1^2 + a_2^2 + a_3^2\}.$$

3. Homothetic motions in semi-Euclidean space $\mathbb{E}_2^4$

The 1-parameter homothetic motions of a body in semi-Euclidean space $\mathbb{E}_2^4$ is generated by transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} fA & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix} \tag{3.1}$$

where $A \in SO_2(4)$. The matrix $B = fA$ is called a homothetic matrix. X, X_0 and C are $n \times 1$ real matrices. The homothetic scale f and the elements of A and C are continuously differentiable functions of a real parameter t; X and X_0 correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space R_0 and the fixed space R, respectively. At the initial time $t = t_0$ we consider the coordinate systems of R_0 and R as coincident. To avoid the case of affine transformation we assume that

$$f = f(t) \neq constant, \quad f = f(t) \neq 0$$

and to avoid the cases of a pure translation or a pure rotation we also assume that

$$\dot{f}A + f\dot{A}, \dot{C} \neq 0$$

where '.' indicates $\frac{d}{dt}$.

The equation (2.1) by differentiation with respect to t yields

$$\dot{X} = \dot{B}X_0 + \dot{C} + B\dot{X}_0, \quad B = fA \tag{3.2}$$

where $\dot{X}$ is the absolute velocity, $\dot{B}X_0 + \dot{C} = 0$ is the sliding velocity and $B\dot{X}_0$ is the relative velocity of the point P whose position vector is X. At the common fixed points of R_0 and R we have

$$\dot{X} = \dot{X}_0 = \dot{B}X_0 + \dot{C} = 0$$

and these points are called instantaneous pole points at the time t. In order to find the position of these points at the time t_0 we solve the equation

$$\dot{B}X_0 + \dot{C} = 0 \tag{3.3}$$

for the vector X_0.

4. Hamilton motions in semi-Euclidean space $\mathbb{E}_2^4$

Let $p = a_0 + a_1 i + a_2 j + a_3 k$ be a split quaternion and let $h_p : H' \to H' h_p(q) = pq$.

The matrix of h relative to the natural basis $\{1, i, j, k\}$ for H' is

$$H(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}. \tag{4.1}$$

Let us consider the following curve:

$$\alpha : I \subset \mathbb{R} \to \mathbb{E}_2^4, \quad \alpha(t) = [\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t)], \quad \forall t \in I.$$

We suppose that the unit velocity curve $\alpha(t)$ is a differentiable regular curve of order r. Let position vector of the curve be timelike. Let the curve be a unit velocity timelike curve ($\langle \alpha'(t), \alpha'(t) \rangle = -1$).

The operator D, called the Hamilton operator, corresponding to $\alpha(t)$ is defined by the following matrix:

$$D = H[\alpha(t)] = \begin{bmatrix} \alpha_0(t) & -\alpha_1(t) & \alpha_2(t) & \alpha_3(t) \\ \alpha_1(t) & \alpha_0(t) & \alpha_3(t) & -\alpha_2(t) \\ \alpha_2(t) & \alpha_3(t) & \alpha_0(t) & -\alpha_1(t) \\ \alpha_3(t) & -\alpha_2(t) & \alpha_1(t) & \alpha_0(t) \end{bmatrix} \tag{4.2}$$

(See [1; 4] for Hamilton operators).

The 1-parameter Hamilton motions of a body in semi-Euclidean space $\mathbb{E}_2^4$ are generated by transformation

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} D & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ 1 \end{bmatrix} \tag{4.3}$$

or equivalently

$$X = DX_0 + C, \tag{4.4}$$

where $D \in H[\alpha(t)]$ and X, X_0 and C are $n \times 1$ real matrices, A and C are continuously differentiable functions of a real parameter t; X and X_0 correspond to the position vectors of the same point P.

Theorem 4.1. *The Hamilton motion determined by equation* (4.4) *in semi-Euclidean space* $\mathbb{E}_2^4$ *is a homothetic motion.*

Proof. Because $\alpha(t)$ does not pass through the origin, the matrix D can be represented as

$$D = f \begin{bmatrix} \frac{\alpha_0(t)}{f} & -\frac{\alpha_1(t)}{f} & \frac{\alpha_2(t)}{f} & \frac{\alpha_3(t)}{f} \\ \frac{\alpha_1(t)}{f} & \frac{\alpha_0(t)}{f} & \frac{\alpha_3(t)}{f} & -\frac{\alpha_2(t)}{f} \\ \frac{\alpha_2(t)}{f} & \frac{\alpha_3(t)}{f} & \frac{\alpha_0(t)}{f} & -\frac{\alpha_1(t)}{f} \\ \frac{\alpha_3(t)}{f} & -\frac{\alpha_2(t)}{f} & \frac{\alpha_1(t)}{f} & \frac{\alpha_0(t)}{f} \end{bmatrix} = fA \tag{4.5}$$

where

$$\begin{aligned} f: I \subset \mathbb{R} &\to \mathbb{R} \\ t &\to f(t) = \|\alpha(t)\| = \sqrt{(\alpha_0^2(t) + \alpha_1^2(t) - \alpha_2^2(t) - \alpha_3^2(t))}. \end{aligned}$$

Since the position of the curve is timelike, $(\alpha_0^2(t) + \alpha_1^2(t) - \alpha_2^2(t) - \alpha_3^2(t)) > 0$ holds. In the equation (4.5), we find $A\varepsilon A^T\varepsilon = A^T\varepsilon A\varepsilon = I_4$ and $\det A = 1$, where $\varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$. Thus D is a homothetic matrix. Since $D = fA$ is a homothetic matrix, equation (4.4) determines a homothetic motion. ■

Theorem 4.2. *The derivation operator* $\dot{D} = \frac{dD}{dt}$ *of the Hamilton operator* $D = fA$ *is a real semi-orthogonal matrix.*

PROOF. By (4.2), $\dot{D}\varepsilon\dot{D}^T\varepsilon = \dot{D}^T\varepsilon\dot{D}\varepsilon = I_4$ and $\det \dot{D} = 1$. Thus, $\dot{D}^T = \varepsilon\dot{D}^{-1}\varepsilon$, $\dot{D}$ is a real semi-orthogonal matrix. ■

Theorem 4.3. *In semi-Euclidean space* $\mathbb{E}_2^4$, *a Hamilton motion determined by the derivation operator is a regular motion, and it is independent of* f.

PROOF. This motion is regular as $\det \dot{D} = 1$; also, the value of $\det \dot{D}$ is independent of f. ■

5. Pole point and pole curves of the motion in semi-Euclidean space $\mathbb{E}_2^4$

To find the pole points of the Hamilton motion determined by equation (4.4) in semi-Euclidean space $\mathbb{E}_2^4$ we have to solve the equation

$$\dot{D}X_0 + \dot{C} = 0. \tag{5.1}$$

Because, by Theorem 4.3, $\det \dot{D} = 1$, equation (5.1) has only one solution, i.e. $X_0 = \dot{D}^{-1}(-\dot{C})$ at every instant t. In this case the following theorem can be proven.

Theorem 5.1. *The pole point corresponding to each instant* t *in* R_0 *is the rotation by* $\dot{D}$ *of the speed vector* $\dot{C}$ *of the translation vector at that moment.*

PROOF. As the matrix $\dot{D}$ is semi-orthogonal, the matrix $\dot{D}^{-1}$ is orthogonal too. Thus, it makes a rotation. ■

6. Acceleration centres of order $(r-1)$ of the motions in semi-Euclidean space $\mathbb{E}_2^4$

Definition 6.1. In equation (4.4), the set of zeros of the equation of the sliding acceleration of order r is called the acceleration centre of order $(r-1)$.

In order to find the acceleration centre of order $(r-1)$ for the equation (4.4), according to definition r.n, we have to find the solutions of the equation

$$D^{(r)}X_0 + C^{(r)} = 0, \tag{6.1}$$

where $D^{(r)} = \frac{d^r D}{dt^r}$ and $C^{(r)} = \frac{d^r C}{dt^r}$. As the curve $\alpha(t)$ is a regular curve of order r, then

$$-\big(\alpha_0^{(r)}(s)\big)^2 - \big(\alpha_1^{(r)}(t)\big)^2 + \big((\alpha_2^{(r)}(t)\big)^2 + \big(\alpha_3^{(r)}(t)\big)^2 \neq 0, \quad a_i^{(r)}(t) = \frac{d^r a_i(t)}{dt^r}.$$

As

$$\det D^{(r)} = \Big[-\big(\alpha_0^{(r)}(t)\big)^2 - \big(\alpha_1^{(r)}(t)\big)^2 + \big(\alpha_2^{(r)}(t)\big)^2 + \big(\alpha_3^{(r)}(t)\big)^2\Big]^2,$$

it follows that $\det D^{(r)} \neq 0$. Therefore the matrix $D^{(r)}$ has an inverse, and, by equation (6.1), the acceleration centre of order $(r-1)$ at every instant t, is $X_0 = [D^{(r)}]^{-1}(-C^{(r)})$.

Example 6.1. $\alpha : I \subset \mathbb{R} \to \mathbb{E}_2^4$ is defined by $\alpha(t) = (\cosh t, \sqrt{3}t, \sinh t, t)$ for every $t \in I$. Let $C = (0, t, 0, 0)$.

Because $\alpha(t) = (\cosh t, \sqrt{3}t, \sinh t, t)$ does not pass through the origin, the matrix D can be represented as

$$D = H(\alpha(t)) = \sqrt{2t^2+1}\begin{bmatrix} \frac{\cosh t}{\sqrt{2t^2+1}} & -\sqrt{3}t & \frac{\sinh t}{\sqrt{2t^2+1}} & t \\ \sqrt{3}t & \frac{\cosh t}{\sqrt{2t^2+1}} & t & -\frac{\sinh t}{\sqrt{2t^2+1}} \\ \frac{\sinh t}{\sqrt{2t^2+1}} & t & \frac{\cosh t}{\sqrt{2t^2+1}} & -\sqrt{3}t \\ t & -\frac{\sinh t}{\sqrt{2t^2+1}} & \sqrt{3}t & \frac{\cosh t}{\sqrt{2t^2+1}} \end{bmatrix}$$

$$= (\sqrt{2t^2+1})A,$$

where

$$\begin{aligned} f : I \subset \mathbb{R} &\to \mathbb{R} \\ t &\to f(t) = \|\alpha(t)\| = \sqrt{|-(2t^2+1)|}. \end{aligned}$$

We find $A\varepsilon A^T\varepsilon = A^T\varepsilon A\varepsilon = I_4$, $\det A = 1$ and $\dot{D} = \frac{dD}{dt} \in SO_2(4)$.

In this case, the (4.4) motion is given by

$$X = \sqrt{2t^2+1}\begin{bmatrix} \frac{\cosh t}{\sqrt{2t^2+1}} & -\sqrt{3}t & \frac{\sinh t}{\sqrt{2t^2+1}} & t \\ \sqrt{3}t & \frac{\cosh t}{\sqrt{2t^2+1}} & t & -\frac{\sinh t}{\sqrt{2t^2+1}} \\ \frac{\sinh t}{\sqrt{2t^2+1}} & t & \frac{\cosh t}{\sqrt{2t^2+1}} & -\sqrt{3}t \\ t & -\frac{\sinh t}{\sqrt{2t^2+1}} & \sqrt{3}t & \frac{\cosh t}{\sqrt{2t^2+1}} \end{bmatrix} X_0 + \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}. \quad (6.2)$$

Hence geometrical path of pole points in the Hamilton motion is determined by equation (6.2) as

$$X_0 = \begin{bmatrix} \sqrt{3} \\ \sinh t \\ -1 \\ \cosh t \end{bmatrix}.$$

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