GENERALISATIONS OF MERGELYAN'S INEQUALITY

By Violeta Morari*
Department of Mathematics, Cork Institute of Technology

[Received 22 December 2003. Read 21 October 2004. Published 11 March 2005.]

Abstract

Mergelyan's inequality states that if K is a compact subset of $\mathbf{R}^2$ and $y \in \mathbf{R}^2$, then $\int_K \frac{dm(x)}{|x-y|} \leq 2\sqrt{\pi}\sqrt{Area(K)}$. In this paper we examine the case when K is a set of finite measure in $\mathbf{R}^n$ and when the integrand is raised to a power $\alpha < n$. We also study the case of equality.

1. Introduction and summary of results

In the course of working on problems in approximation theory, Mergelyan (see [3; 1]) stated and proved that if K is a compact subset of $\mathbf{R}^2$ and $y \in \mathbf{R}^2$, then

$$\int_K \frac{1}{|x-y|}\, dm(x) \leq 2\sqrt{\pi}\sqrt{Area(K)}, \tag{1.1}$$

where dm is the Lebesgue measure in $\mathbf{R}^2$, and $|x-y|$ denotes the euclidean distance between points $x,y \in \mathbf{R}^2$. Further details and proofs can also be found in [2]. Our purpose in the present paper is to generalise this inequality. We examine the case when K is a set of finite measure in $\mathbf{R}^n$ and the integrand in (1.1) is raised to a power $\alpha < n$. We also study the case of equality. We next formulate our principal results:

Theorem 1.1. *Let n be a positive integer. Let $\alpha < n$ and let dm_n denote Lebesgue measure on $\mathbf{R}^n$. Let A be a subset of $\mathbf{R}^n$ with $m_n(A) < \infty$. Let*

$$M_{n,\alpha} = \frac{n}{n-\alpha} \cdot \left(\frac{2}{n}\right)^{\frac{\alpha}{n}} \cdot \frac{\pi^{\frac{\alpha}{2}}}{\left[\Gamma\left(\frac{n}{2}\right)\right]^{\frac{\alpha}{n}}}.$$

Then we have the following estimates:
1. *If $0 < \alpha < n$, then*

$$\int_A \frac{dm_n(x)}{|x-y|^\alpha} \leq M_{n,\alpha} \cdot m_n(A)^{\frac{n-\alpha}{n}}, \quad \forall y \in \mathbf{R}^n. \tag{1.2}$$

2. *If $\alpha < 0$, then*

---

*E-mail: vmorari@cit.ie

$$\int_A \frac{dm_n(x)}{|x-y|^\alpha} \geq M_{n,\alpha}\cdot m_n(A)^{\frac{n-\alpha}{n}}, \quad \forall y \in \mathbf{R}^n. \tag{1.3}$$

If $\alpha = 0$, equality holds in 1.2 and 1.3 for any measurable A of finite measure. To avoid triviality, we assume for the rest of the paper that $\alpha \neq 0$.

Theorem 1.2. *Suppose that $\alpha < n$, $\alpha \neq 0$ and that A is a subset in $\mathbf{R}^n$ of finite measure. Equality holds in (1.2) and (1.3) for some $y \in \mathbf{R}^n$ if and only if $A = (B\backslash X) \cup Y$ where B is a ball centered at y with $m_n(B) = m_n(A)$ and both X and Y are sets in $\mathbf{R}^n$ of measure zero.*

2. Definitions and notations

Arbitrary points of the euclidean space $\mathbf{R}^n$ are represented by

$$x = (x_1, x_2, \ldots, x_n),$$

and the norm of x is

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2},$$

where $x_i \in \mathbf{R}$, $i = 1,2,\ldots,n$. In the sense of Lebesgue measure,

$$dm_n(x) = dx_1 \ldots dx_n.$$

We denote an open ball in $\mathbf{R}^n$ centred at a with radius r by $B(a, r)$. For $\alpha \in \mathbf{R}$ we define $k_{n,\alpha}$ on $\mathbf{R}^n$ in the following way:

$$k_{n,\alpha}(x) = \begin{cases} \frac{1}{|x|^\alpha}, & \text{if } \alpha \in \mathbf{R}, |x| \neq 0, \\ 0, & \text{if } \alpha < 0, |x| = 0, \\ \infty, & \text{if } \alpha > 0, |x| = 0. \end{cases}$$

3. Some preliminary lemmas

We want to show that $k_{n,\alpha}$ is locally integrable on $\mathbf{R}^n$. To do so we note that $k_{n,\alpha}$ is clearly measurable on $\mathbf{R}^n$, and, we first prove that $k_{n,\alpha}$ is integrable on an open ball in $\mathbf{R}^n$:

Lemma 3.1. *Let $r > 0$ and let $\alpha < n$. Then $k_{n,\alpha}$ is integrable on $B(0,r)$ and*

$$\int_{B(0,r)} \frac{1}{|x|^\alpha} dm_n(x) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \frac{r^{n-\alpha}}{n-\alpha}. \tag{3.1}$$

Proof. First suppose that $\alpha < 0$. Then $k_{n,\alpha}$ is a continuous function on $\bar{B}(0, r)$, and is thus Riemann integrable on $B(0, r)$. Then

$$\int_{B(0,r)} \frac{1}{|x|^{\alpha}} dm_n(x)$$

$$= \int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{r} \frac{\rho^{n-1}\sin\phi_{n-2}\sin^2\phi_{n-3}\cdots\sin^{n-2}\phi_1}{\rho^{\alpha}} d\rho\, d\phi_1\cdots d\phi_{n-1}$$

$$= I_0\cdot I_1\cdot\cdots\cdot I_{n-1}, \tag{3.2}$$

where

$$I_i = \int_0^{\pi} \sin^i\phi\, d\phi, \quad i = 1,2,\ldots,n-2, \tag{3.3}$$

and

$$I_{n-1} = \int_0^{r} \frac{\rho^{n-1}}{\rho^{\alpha}} d\rho = \frac{r^{n-\alpha}}{n-\alpha}. \tag{3.4}$$

The value of I_i is well known (see, e.g., p. 256 in [7]) and is equal to

$$I_i = \int_0^{\pi} \sin^i t\, dt = B\left(\frac{i+1}{2},\frac{i}{2}\right) = \frac{\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)}, \quad i = 1,2,\ldots,n-2.$$

So

$$I_0\cdot I_1\cdot\cdots\cdot I_{n-2} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}. \tag{3.5}$$

Thus, from formulas (3.4) and (3.5) we obtain (3.1).
Now, suppose that $0 \le \alpha < n$. Take $M \in \mathbf{N}$ such that $M\cdot r > 1$. Then for $k \ge M$ we define the sequence $\{g_k\}_{k=M}^{\infty}$ in the following way:

$$g_k(y) = \begin{cases} k^{\alpha}, & \text{if} \quad |y| \le \frac{1}{k}, \\ \frac{1}{|y|^{\alpha}}, & \text{if} \quad |y| > \frac{1}{k},\ y \in \mathbf{R}^n. \end{cases}$$

Since for any $k \in \mathbf{N}$, g_k is a continuous function on $B(0,r)$, it follows that g_k is Riemann integrable, and, consequently, it is Lebesgue integrable on $B(0,r)$. Also, $B\left(0,\frac{1}{k}\right) \subseteq B(0,r), \forall k \ge M$. So, since $\alpha < n$, then using (3.4) and (3.5) we obtain

$$\int_{B(0,r)} g_k(y)\, dm_n(y) = \int_{B\left(0,\frac{1}{k}\right)} k^{\alpha}\, dm_n(y) + \int_{B(0,r)\setminus B\left(0,\frac{1}{k}\right)} \frac{1}{|y|^{\alpha}} dm_n(y) \tag{3.6}$$

$$= k^{\alpha}\cdot m_n\left(B\left(0,\frac{1}{k}\right)\right)$$

$$+\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_{\frac{1}{k}}^{r} \frac{\rho^{n-1}\sin\phi_{n-2}\sin^2\phi_{n-3}\ldots\sin^{n-2}\phi_1}{\rho^{\alpha}} d\rho\, d\phi_1\cdots d\phi_{n-1} \tag{3.7}$$

$$= k^{\alpha}\cdot\frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}\cdot\frac{1}{k^n}+\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{n-\alpha}\left[r^{n-\alpha}-\left(\frac{1}{k}\right)^{n-\alpha}\right] \tag{3.8}$$

$$\to \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{n-\alpha}r^{n-\alpha} \quad \text{as} \quad k\to\infty.$$

So,

$$\lim_{k\to\infty}\int_{B(0,r)} g_k(y)\,dm_n(y)=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{n-\alpha}r^{n-\alpha}. \tag{3.9}$$

Since g_k is an increasing sequence of positive functions with limit $k_{n,\alpha}$, it follows from the Monotone Convergence Theorem that

$$\lim_{k\to\infty}\int_{B(0,r)} g_k(y)\,dm_n(y)=\int_{B(0,r)} k_{n,\alpha}(y)\,dm_n(y).$$

Hence, from (3.9) we obtain that $k_{n,\alpha}$ is integrable on $B(0,r)$ and that

$$\int_{B(0,r)} k_{n,\alpha}(y)\,dm_n(y)=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{r^{n-\alpha}}{n-\alpha}.$$

This finishes the proof. ■

From this it immediately follows that $k_{n,\alpha}$ is locally integrable and is also integrable on sets of finite measure:

Corollary 3.2. *Let $\alpha < n$. Then $k_{n,\alpha}$ is locally integrable on $\mathbf{R}^n$.*

Corollary 3.3. *Let B be a subset of $\mathbf{R}^n$ with $m_n(B) < \infty$. Let $0 \le \alpha < n$. Then*

$$\int_B k_{n,\alpha}(y)\,dm_n(y) < \infty.$$

PROOF. To prove this, take a ball $B(0,\delta)$ such that $m_n(B(0,\delta)) = m_n(B)$. Then

$$\int_B k_{n,\alpha}(y)\,dm_n(y)=\int_{B\backslash B(0,\delta)}\frac{1}{|y|^{\alpha}}dm_n(y)+\int_{B\cap B(0,\delta)}\frac{1}{|y|^{\alpha}}dm_n(y)=I_1+I_2. \tag{3.10}$$

Now, both I_1 and I_2 are finite, since I_2 is finite from Corollary 3.2, and

$$I_1 \le \frac{1}{\delta^{\alpha}}\int_{B\backslash B(0,\delta)} dm_n(y) < \infty.$$

With that the proof is complete. ■

4. Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1
Since Lebesgue measure is translation invariant, and

$$\int_A \frac{dm_n(x)}{|x-y|^\alpha} = \int_{A-y} \frac{dm_n(x)}{|x|^\alpha},$$

it is enough to prove inequalities (1.2) and (1.3) when y is the zero vector.

Let $B(0,\delta)$ be the ball with radius $\delta = \left(\frac{n\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}} \cdot m_n(A)\right)^{\frac{1}{n}}$, so that $m_n(A) = m_n(B(0,\delta)) = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)} \cdot \delta^n$. Now, since $m_n(B(0,\delta)) = m_n(A)$, then

$$m_n(B(0,\delta)\backslash A) = m_n(A\backslash B(0,\delta)). \tag{4.1}$$

We first consider the proof of formula (1.2)
Let $0 < \alpha < n$. Then

$$\delta^\alpha \le |x|^\alpha, x \in A\backslash B(0,\delta),$$
$$\delta^\alpha > |x|^\alpha, x \in B(0,\delta)\backslash A,$$

and so

$$\sup_{x\in A\backslash B(0,\delta)} \frac{1}{|x|^\alpha} \le \frac{1}{\delta^\alpha} \le \inf_{x\in B(0,\delta)\backslash A} \frac{1}{|x|^\alpha}.$$

Hence,

$$\int_{A\backslash B(0,\delta)} \frac{dm_n(x)}{|x|^\alpha} \le \int_{B(0,\delta)\backslash A} \frac{dm_n(x)}{|x|^\alpha}. \tag{4.2}$$

Now expressing the integral in (1.2) as a sum of integrals over $A \cap B(0,\delta)$ and $A(0,\delta)$ and using (4.2) we have

$$\int_A \frac{dm_n(x)}{|x|^\alpha} = \int_{A\cap B(0,\delta)} \frac{dm_n(x)}{|x|^\alpha} + \int_{A\backslash B(y,\delta)} \frac{dm_n(x)}{|x|^\alpha} \tag{4.3}$$

$$\le \int_{A\cap B(0,\delta)} \frac{dm_n(x)}{|x|^\alpha} + \int_{B(0,\delta)\backslash A} \frac{dm_n(x)}{|x|^\alpha} \tag{4.4}$$

$$= \int_{B(0,\delta)} \frac{dm_n(x)}{|x|^\alpha} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\delta^{n-\alpha}}{n-\alpha} \tag{4.5}$$

$$= M_{n,\alpha} \cdot m_n(A)^{\frac{n-\alpha}{n}}. \tag{4.6}$$

Thus

$$\int_A \frac{dm_n(x)}{|x|^\alpha} \le M_{n,\alpha} \cdot m_n(A)^{\frac{n-\alpha}{n}}. \tag{4.7}$$

Observe that for the case $n=2$ and $\alpha=1$, we obtain Mergelyan's inequality. This finishes the proof of (1.2).
We now prove (1.3). If $\alpha<0$, then

$$\sup_{x\in B(0,\delta)\setminus A}\frac{1}{|x|^{\alpha}}\leq\frac{1}{\delta^{\alpha}}\leq\inf_{x\in A\setminus B(0,\delta)}\frac{1}{|x|^{\alpha}},$$

and so we obtain

$$\int_{B(0,\delta)\setminus A}\frac{dm_n(x)}{|x|^{\alpha}}\leq\int_{A\setminus B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}}. \tag{4.8}$$

Thus

$$\int_{A}\frac{dm_n(x)}{|x|^{\alpha}}=\int_{A\cap B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}}+\int_{A\setminus B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}} \tag{4.9}$$

$$\geq\int_{A\cap B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}}+\int_{B(0,\delta)\setminus A}\frac{dm_n(x)}{|x|^{\alpha}} \tag{4.10}$$

$$=\int_{B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}}=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{\delta^{n-\alpha}}{n-\alpha} \tag{4.11}$$

$$=M_{n,\alpha}\cdot m_n(A)^{\frac{n-\alpha}{n}}.$$

With that Theorem 1.1 is now proved. ■

The proof of inequality (1.2) in the first part of Theorem 1.1 is modelled on the one given by T.W. Gamelin and D. Khavinson in [2] for the case $n=2$ and $\alpha=1$. Inequality (1.3) appears to be new even in the case for $n=2$. More details can be found in [5].

Proof of Theorem 1.2
Again, by translation invariance of Lebesgue measure, it is enough to prove the result when $y=0$, and we proceed under this assumption. First, examine the case when $0<\alpha<n$. So, let $A=(B\setminus X)\cup Y$, for some sets X and Y, where $m_n(X)=m_n(Y)=0$ and $B(0,\delta)$ is the ball centered at 0 that has the same volume as A. We examine the case when $(B\setminus X)\cap Y=\emptyset$.
Then

$$\int_{A}\frac{dm_n(x)}{|x|^{\alpha}}=\int_{(B(0,\delta)\setminus X)\cup Y}\frac{dm_n(x)}{|x|^{\alpha}} \tag{4.12}$$

$$=\int_{B(0,\delta)}\frac{dm_n(x)}{|x|^{\alpha}} \tag{4.13}$$

$$=M_{n,\alpha}\cdot m_n(A)^{\frac{n-\alpha}{n}}.$$

This concludes the proof in this case. Next we examine the case when $(B\backslash X) \cap Y \neq \emptyset$ Here $(B\backslash X) \cup Y$ can be represented as the union of two disjoint sets in the following way:

$$(B\backslash X) \cup Y = (B\backslash X) \cup (Y\backslash(B\backslash X)). \tag{4.14}$$

Now since $B\backslash X$ and $Y\backslash(B\backslash X)$ are disjoint, and since $m_n(Y\backslash(B\backslash X)) = 0$, the previous case applies and the proof follows.

Conversely, suppose that for some $A \subset \mathbf{R}^n, m_n(A) < \infty$, the following equality holds:

$$\int_A \frac{1}{|x|^\alpha} dm_n(x) = M_{n,\alpha} \cdot m_n(A)^{\frac{n-\alpha}{n}}. \tag{4.15}$$

We want to show that $A = (B\backslash X) \cup Y$ for the ball $B = B(0, \delta)$ where $m_n(B(0, \delta)) = m_n(A)$ and some sets of measure zero X and Y. Define X and Y in the following way:

$$X = B\backslash A, \ \ Y = A\backslash B.$$

Now it is easy to see that

$$A = (A\backslash B) \cup (A \cap B) = (A\backslash B) \cup (B\backslash(B\backslash A)) = Y \cap (B\backslash X).$$

All that remains to be shown is that $m_n(A\backslash B) = m_n(B\backslash A) = 0$. By (4.1) it is enough to show that $m_n(B\backslash A) = 0$.
Firstly, since $m_n(A) = m_n(B)$, from (4.15) we have:

$$\begin{aligned}\int_A \frac{1}{|x|^\alpha} dm_n(x) &= M_{(n,\alpha)} \cdot m_n(A)^{\frac{n-\alpha}{n}} \\ &= M_{(n,\alpha)} \cdot m_n(B)^{\frac{n-\alpha}{n}} \\ &= \int_{B(0,\delta)} \frac{1}{|x|^\alpha} dm_n(x).\end{aligned}$$

Thus,

$$\int_{A\backslash B(0,\delta)} \frac{dm_n(x)}{|x|^\alpha} = \int_{B(0,\delta)\backslash A} \frac{dm_n(x)}{|x|^\alpha}.$$

On the other hand,

$$\begin{aligned}\int_{A\backslash B(0,\delta)} \frac{1}{|x|^\alpha} dm_n(x) \leq \frac{1}{\delta^\alpha} \int_{A\backslash B(0,\delta)} dm_n(x) = \frac{1}{\delta^\alpha} \int_{B(0,\delta)\backslash A} dm_n(x) \\ \leq \int_{B(0,\delta)\backslash A} \frac{1}{|x|^\alpha} dm_n(x).\end{aligned} \tag{4.16}$$

So, for the equality to hold, we need

$$\int_{B(0,\delta)\backslash A} \left(\frac{1}{|x|^\alpha} - \frac{1}{\delta^\alpha} \right) dm_n(x) = 0. \tag{4.17}$$

Since for $x \in B \backslash A$ we have

$$\frac{1}{|x|^{\alpha}} - \frac{1}{\delta^{\alpha}} > 0, \tag{4.18}$$

from (4.17) and (4.18) we now have

$$m_n(B(0,\delta)\backslash A) = 0.$$

Thus $m_n(A\backslash B(0,\delta))$ is also zero.

We omit the proof for $\alpha < 0$ since it is similar to the one above. This finishes the proof of Theorem 1.2. ■

Other results characterising balls using notions from potential theory may be found, among other places, in [6, section 7].

ACKNOWLEDGEMENTS

The results of this paper form part of a thesis submitted to University College Cork for the degree of Doctor of Philosophy. I thank my supervisor, Professor Finbarr Holland, for his encouragement and advice during the preparation of that thesis.

REFERENCES

[1] A. Browder, *Introduction to function algebras*, Benjamin, New York, 1969.
[2] T.M. Gamelin and D. Khavinson, The isoperimetric inequality and rational approximation, *American Mathematical Monthly* **96** (1) (1989), 18–30.
[3] S.N. Mergelyan, Uniform approximations to functions of a complex variable, *American Mathematical Society Translations* **101** (1954), 3–99.
[4] N. du Plessis, *An introduction to potential theory*, Oliver & Boyd, Edinburgh, 1970.
[5] V. Morari, Generalisations of Mergelyan's inequality, with applications, unpublished PhD thesis, University College Cork, 2002.
[6] I. Netuka and J. Vesely, Mean value property and harmonic functions, *Classical and Modern Potential Theory and Applications* 430, Kluwer, Dordrecht, 1994, 294–391.
[7] E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press, 1969.