STRONG APPROXIMATE CONTINUITY PROPERTIES OF CERTAIN CONJUGATE FUNCTIONS

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Abstract

The conjugate function $\tilde{F}(x) = -\lim_{\eta\to 0}\frac{1}{2\pi}\int_{\eta}^{\pi}[F(x+t) - F(x-t)]\cot\frac{t}{2}\,dt$ of an integrable and 2π-periodic function F exists finitely for almost all x. As our principal result we prove here that if $\tilde{F}(0)$ is finite and F is monotonic in some neighbourhood of 0, then $\tilde{F}$ satisfies strong continuity-type properties at 0. Using an auxiliary result concerning the existence of tangential limits for the Poisson integral, we derive some consequences of our main theorem for the boundary behaviour of analytic and univalent functions.

1. Introduction and statement of results

The *conjugate function* $\tilde{F}$ of a 2π-periodic function F, with $F \in L(-\pi, \pi)$, is defined by

$$\begin{aligned}\tilde{F}(x) &= -\lim_{\eta\to 0}\frac{1}{2\pi}\int_{\eta}^{\pi}[F(x+t) - F(x-t)]\cot\frac{t}{2}\,dt\\ &= \lim_{\eta\to 0}\frac{1}{2\pi}\int\limits_{\eta\le|t-x|\le\pi} F(t)\cot\frac{x-t}{2}\,dt.\end{aligned}$$

It is classical that if $F \in L(-\pi, \pi)$, then $\tilde{F}(x)$ exists and is finite for almost all x. Conjugate functions of continuous functions are not necessarily continuous or even bounded—the conjugate of an odd, continuous function for which $F(x) = (\log 2/x)^{-1}$ for $0 < x < 1$, for instance, is not finite at 0—but it is known [4, pp. 105–7] that if F is Dini-continuous on an interval (a, b), *i.e.*, $\int_0^d \frac{\omega(t)}{t}\,dt < \infty$ for some $d > 0$, where

$$\omega(t) = \sup\{|F(x) - F(y)| : x, y \in (a, b), |x - y| < t\}$$

is the *modulus of continuity* of F, then $\tilde{F}$ exists and is continuous at each point of (a, b).

In our main result here, we are concerned with the connection between certain local conditions on a function F at a fixed point, and strong continuity-type properties of the conjugate function at the same point. Without loss of generality we take the fixed point to be 0. To state our main theorem we need some definitions. We write, for a fixed constant α, and $\varepsilon > 0$,

$$E_\varepsilon(F, \alpha) = \{x : |F(x) - \alpha| \ge \varepsilon\}$$

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and

$$E_\varepsilon(F,\ \alpha;\delta) = E_\varepsilon(F,\ \alpha) \cap [-\delta, \delta], \quad \delta > 0.$$

A function F for which $m(E_\varepsilon(F, F(0); \delta)) = o(\delta)$ as $\delta \to 0$ for each fixed $\varepsilon > 0$, where $m(\cdot)$ denotes Lebesgue measure, is said to be *approximately continuous* at 0. It was shown by Denjoy [1] that every measurable function is approximately continuous almost everywhere.

Theorem 1. *Suppose that $F \in L(-\pi, \pi)$ and that F is monotonic in some neighbourhood of 0. Suppose also that $\tilde{F}(0)$ exists. Then, for each fixed $\varepsilon > 0$,*

$$m\,(E_\varepsilon(\tilde{F},\ \tilde{F}(0); \delta)) = o(\delta^\lambda), \quad \delta \to 0, \tag{1.1}$$

for every $\lambda > 1$.

The conjugate of a function satisfying the hypotheses of Theorem 1 thus satisfies a strong form of approximate continuity at 0. We observe also that the conclusion of Theorem 1 clearly extends to integrable functions of the form $F = G + \psi$, where G is monotonic and ψ is Dini-continuous in some neighbourhood of 0. We note as well that it is easy to show that the conclusion of the theorem can be stated in the following equivalent form: *there is a set $E \subset [-\pi, \pi]$ such that $m(E \cap [-\delta, \delta]) = o(\delta^\lambda), \delta \to 0$, for every $\lambda > 1$ and*

$$\lim_{x \to 0,\, x \in E^c} \tilde{F}(x) = \tilde{F}(0), \quad E^c = [-\pi, \pi] \backslash E.$$

We next prepare to describe some applications of Theorem 1. To this end we need an auxiliary result concerning the boundary behaviour of the Poisson integral. For a 2π-periodic function $F \in L(-\pi, \pi)$ and $re^{i\theta} \in U = \{z : |z| < 1\}$, the Poisson integral of F is defined by

$$u\,(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P\,(r, \theta - t)\, F(t)\, dt, \tag{1.2}$$

where $P(r, x) = (1 - r^2)/(1 - 2r\cos x + r^2)$ is the Poisson kernel for the unit disc U. We shall write $u = P[F]$ to denote the relationship (1.2).

We also need the notion of a tangential $\mathcal{T}_\lambda$*-limit*. We set

$$\Omega_c(\lambda) = \{re^{i\theta} \in U : |\theta| \le c\,(1 - r)^{1/\lambda}\}$$

for $\lambda > 1$ and $c > 0$. These regions $\Omega_c(\lambda)$ make tangential contact with the boundary ∂U of the unit disc U at 1. We say that a function $f : U \to \mathbf{C}$ has a tangential $\mathcal{T}_\lambda$-limit equal to α at 1 if $f(z) \to \alpha$ as $z \to 1$ inside $\Omega_c(\lambda)$ for every $c > 0$.

Theorem 2. *Suppose that $F \in L(-\pi, \pi), u = P\,[F]$, and that α and λ are fixed constants with $\lambda > 1$. Let χ_ε denote the characteristic function of the set $E_\varepsilon(F, \alpha)$ and suppose that, for each fixed $\varepsilon > 0$,*

$$\int_{-\delta}^{\delta} \chi_\varepsilon(t)|F(t)-\alpha|\,dt = o(\delta^\lambda), \qquad \delta \to 0. \tag{1.3}$$

Then $u = P[F]$ has $\mathcal{T}_\lambda$-limit α at $z = 1$.

Remark. Since

$$\int_{-\delta}^{\delta} |F(t)-\alpha|\,dt \le \int_{-\delta}^{\delta} \chi_\varepsilon(t)|F(t)-\alpha|\,dt + 2\varepsilon\delta,$$

it is clear that (1.3) implies that

$$\int_{-\delta}^{\delta} |F(t)-\alpha|\,dt = o(\delta),\ \delta \to 0. \tag{1.4}$$

It is well known that if (1.4) holds, then $u = P[F]$ has a non-tangential (or angular) limit at 1.

We now write $E_\varepsilon(\delta)$ for $E_\varepsilon(F, \alpha; \delta)$, and note that, if $F \in L^\infty(-\pi, \pi)$,

$$\int_{-\delta}^{\delta} \chi_\varepsilon(t)|F(t)-\alpha|\,dt \le (\|F\|_\infty + |\alpha|)\, m(E_\varepsilon(\delta)),$$

where $\|F\|_\infty$ is the essential supremum norm of F, while if $F \in L^p(-\pi, \pi)$, $p > 1$, then, by Hölder's inequality with $q = p/(p-1)$,

$$\begin{aligned}\int_{-\delta}^{\delta} \chi_\varepsilon(t)|F(t)-\alpha|\,dt &\le \left(\int_{-\delta}^{\delta} \chi_\varepsilon(t)^q ds\right)^{1/q} \left(\int_{-\delta}^{\delta} |F(t)-\alpha|^p\,dt\right)^{1/p} \\ &= o\,(m(E_\varepsilon(\delta))^{(p-1)/p}, \qquad \delta \to 0.\end{aligned}$$

We thus have the following corollary of Theorem 2.

Corollary 1. *Assume that $\lambda > 1$ and that, for each fixed $\varepsilon > 0$, either*

(i) $F \in L^\infty(-\pi, \pi)$ and

$$m(E_\varepsilon(\delta)) = o(\delta^\lambda), \qquad \delta \to 0,$$

or (ii) $F \in L^p(-\pi, \pi)$, where $p > 1$, and

$$m(E_\varepsilon(\delta)) = O(\delta^{\lambda p/(p-1)}), \qquad \delta \to 0.$$

Then $u = P[F]$ has a tangential $\mathcal{T}_\lambda$-limit at 1.

We remark that the first part of Corollary 1 can be strengthened for functions in H^∞, that is, the class of functions that are *analytic* and bounded in U. Note that if $g \in H^\infty$, then the boundary function $g^*(\theta) = \lim_{r\to 1} g(re^{i\theta})$ exists for almost all θ, by Fatou's theorem, and $g = P[g^*]$ [3, p. 34]. Doob [2; see also 10] has shown that if, for each fixed $\varepsilon > 0$,

$$m(E_\varepsilon(g^*, \alpha; \delta)) = O(\delta^\lambda), \qquad \delta \to 0,$$

then $g = P[g^*]$ has a tangential $\mathcal{T}_\lambda$-limit at 1.

We need some final notation before presenting some simple consequences of our results for analytic functions and univalent functions. We write $v = Q[F]$ to denote the relationship

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(r, \theta - t)F(t)\, dt,$$

where $Q(r, x) = 2r\sin x/(1 - 2r\cos x + r^2)$ is the conjugate Poisson kernel. Note that, if $F \in L(-\pi, \pi)$, then the function h defined by

$$h(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}F(t)\, dt, \qquad z \in U,$$

i.e., the function $h = P[F] + iQ[F]$, is analytic in U.

Theorem 3. *Suppose that* $F \in L(-\pi, \pi)$ *and that* F *is monotonic in some neighbourhood of* 0. *Set* $h = P[F] + iQ[F]$ *and suppose that* $v(r) = O(1)$ *as* $r \to 1$, *where* $v = Q[F] = \operatorname{Im} h$. *Then the analytic function* h *has a finite* $\mathcal{T}_\lambda$*-limit for every* $\lambda > 1$ *at* 1.

The proofs of Theorems 2 and 3 are given in Section 3, but we show here that Theorem 3 can be applied to the class S of functions $f(z) = z + a_2z^2 + \ldots$ that are analytic and univalent in U. To this end, we set $g(z) = i\log\frac{f(z)}{z}, z \in U$, and note the familiar fact that g belongs to the Hardy space H^p for every positive p (since $f \in H^p$ for $p < 1/2$ [3, p. 50]). Hence $g^* \in L(-\pi, \pi)$, where $g^*(\theta) = \lim_{r\to 1} g(re^{i\theta})$, and writing

$$\operatorname{Re} g^*(\theta) = -\lim_{r\to 1} \arg[f(re^{i\theta})/re^{i\theta}] = \theta - \arg f(e^{i\theta}) = F(\theta),$$

we then have $\operatorname{Re} g = P[F]$ and $\operatorname{Im} g = Q[F]$. The following result for functions in S is now an easy consequence of Theorem 3.

Corollary 2. *Suppose that* $f \in S$ *and that* $|f(r)| = O(1)$ *as* $r \to 1$. *Suppose also that arg* $f(e^{i\theta})$ *is monotonic in a neighbourhood of* 0. *Then* f *has a finite* $\mathcal{T}_\lambda$*- limit for every* $\lambda > 1$ *at* 1.

If f is a *starlike* univalent function, that is, $f \in S$ and the image domain $f(U)$ contains the line segment $[0, w]$ whenever it contains w, then [9, section 2.2] $\arg f(e^{i\theta})$ is an increasing function of θ, and Corollary 2 thus implies that such functions f have $\mathcal{T}_\lambda$-limits for every $\lambda > 1$ at $e^{i\theta}$, *i.e.*, $f(ze^{i\theta})$ has a $\mathcal{T}_\lambda$-limit at 1, for every θ for which $|f(re^{i\theta})| = O(1)$ as $r \to 1$. This result has been established by a different method in [11], where it has also been shown that the result is sharp with respect to the type of tangential limit obtained. As a final observation here, we note that it is known [12, p. 78; see also 7] that every function $f \in S$ has $\mathcal{T}_\lambda$-limits for every $\lambda > 1$ at all points $e^{i\theta} \in \partial U$, with the possible exception of a set of θ of logarithmic capacity zero.

2. Proof of Theorem 1

In proving Theorem 1, we may assume, without loss of generality, that $F(0) = 0$ and that F is increasing in $(-a, a)$ where $0 < a < 1$. The proof of Theorem 1 is based on a number of lemmas.

We need some notation for our first lemma. We say that a function f has a weak Lebesgue point (a *WL*-point) at 0 if f is integrable in a neighbourhood of 0 and

$$\frac{1}{\delta}\int_{-\delta}^{\delta} [f(t) - f(0)]\, dt \to 0 \quad \text{as} \quad \delta \to 0.$$

Lemma 1. *The conjugate function* $\tilde{F}$ *has a WL-point at* 0.

Proof of Lemma 1. Let χ_a denote the characteristic function of the interval $(-a, a)$ and its 2π-translates, and set

$$F_a = F\chi_a, \quad G_a = F - F_a.$$

Since $G_a = 0$ in $(-a, a)$, G_a is trivially Dini-continuous on $(-a, a)$ and consequently, as noted in Section 1, $\tilde{G}_a$ is continuous in that interval and hence has a *WL*-point at 0.

To complete the proof of Lemma 1, it is sufficient to show that $\tilde{F}_a$ has a *WL*-point at 0. We assume, as we may, that $F(a) - F(-a) < 2\pi$ and we let g be an increasing function on $\mathbf{R}$ for which

$$g(t) = F(t) \quad \text{for} \quad t \in (-a, a), \qquad g(t + 2\pi) - g(t) = 2\pi, \quad t \in \mathbf{R}.$$

We set $h(t) = g(t) - t$, $t \in \mathbf{R}$, and note that h is periodic. We also set $h_a = h\chi_a$ and write $\psi_a = \psi\chi_a$, where ψ is the 2π-periodic function with $\psi(t) = t, -\pi \le t < \pi$. If we further set $H_a = \psi_a - (1 - \chi_a)h$, then we have

$$F_a = h_a + \psi_a = h - (1 - \chi_a)h + \psi_a = h + H_a.$$

We note next that

$$\tilde{h}(0) = -\lim_{\eta \to 0} \frac{1}{2\pi}\int_{\eta}^{\pi} [h(t) - h(-t)] \cot\frac{t}{2}\, dt$$

is finite, because $\tilde{F}(0)$ is finite, and this implies that h, which can only have a simple discontinuity at 0, must in fact be continuous there. Hence, by a standard result [6, p. 28],

$$V(r) = -\frac{1}{2\pi}\int_{1-r}^{\pi} [h(t) - h(-t)] \cot\frac{t}{2}\, dt + o(1)$$

as $r \to 1$, where $V = Q[h]$, and it follows that $V(r) \to \tilde{h}(0)$ as $r \to 1$. Next, $\tilde{h} \in L^2(-\pi, \pi)$ since $h \in L^\infty(-\pi, \pi)$, and [6, p. 31] we have $V = Q[h] = P[\tilde{h}]$. We also have, for almost all x,

$$\begin{aligned}\tilde{h}(x) &= -\lim_{\eta\to 0}\frac{1}{2\pi}\int_{\eta}^{\pi}[g(x+t)-g(x-t)-2t]\cot\frac{t}{2}\,dt\\ &\le \frac{1}{\pi}\int_{0}^{\pi} t\cot\frac{t}{2}\,dt \le 2,\end{aligned}$$

where we have used the fact that g is increasing. It follows that $V = P[\tilde{h}]$ is bounded above in U and, since V has a finite radial limit at 1, we deduce from a classical tauberian theorem of Loomis [8] that $\tilde{h}$ has a WL-point at 0. Finally, we note that, by the argument of the first part of this proof, $\tilde{H}_a$ also has a WL-point at 0. The conclusion of Lemma 1 now follows from the linearity property of conjugation.

We next set, for $0 < \delta \le \delta_1 = \min\{a, 1/3\}$,

$$I_\delta = [-\delta, \delta], \quad J_\delta = [-\delta^{1/4}, \delta^{1/4}],$$
$$F_1 = F\chi_S, \quad F_2 = F - F_1,$$

where S is the union of J_δ and its 2π-translates, and χ_S denotes its characteristic function. Then $\tilde{F} = \tilde{F}_1 + \tilde{F}_2$, and we note that $\tilde{F}(0), \tilde{F}_1(0)$ and $\tilde{F}_2(0)$ are all finite. ■

Lemma 2. *With F_2 and I_δ defined as above, we have*

$$|\tilde{F}_2(t) - \tilde{F}_2(0)| \le B\delta^{1/2}, \qquad t \in I_\delta, \tag{2.1}$$

where B is a constant that depends only on F.

Proof of Lemma 2. Let $t \in I_\delta$. Then

$$\begin{aligned}&2\pi|\tilde{F}_2(t) - \tilde{F}_2(0)|\\ &\quad= \left|\lim_{\eta\to 0}\int_{\eta\le|x-t|\le\pi} F_2(x)\cot\frac{x-t}{2}\,dx - \lim_{\eta\to 0}\int_{\eta\le|x|\le\pi} F_2(x)\cot\frac{x}{2}\,dx\right|\\ &\quad= \left|\int_{\delta^{1/4}\le|x|\le\pi} F(x)\left[\cot\frac{x-t}{2} - \cot\frac{x}{2}\right]dx\right|\\ &\quad\le |t|\int_{\delta^{1/4}\le|x|\le\pi}\frac{|F(x)|}{\sin^2\frac{|x|}{4}}\,dx \le (2\pi)^2\frac{|t|}{\delta^{1/2}}\,\|F\|_1,\end{aligned}$$

where we have used the mean value theorem for derivatives and the fact that $|x-t| \ge |x|/2$ for $t \in I_\delta$ and $|x| \ge \delta^{1/4}$. It is clear that (2.1) follows and Lemma 2 is proved. ■

The proof of our next lemma involves functions of *bounded mean oscillation*. To define such functions, we first write, for a 2π-periodic, locally integrable function φ and a bounded interval I,

$$I(\varphi) = \frac{1}{m(I)}\int_I \varphi(x)\,dx,$$

for the average of φ over I. Then φ is said to have bounded mean oscillation, and we write $\varphi \in \mathrm{BMO}$, provided that

$$\sup_I \frac{1}{m(I)} \int_I |\varphi(x) - I(\varphi)|\, dx = \|\phi\|_* < \infty,$$

where the supremum is over all bounded intervals.

Lemma 3. *With* F_1 *and* I_δ *defined as above, we have, for each fixed* $\varepsilon > 0$ *and every* $\lambda > 1$,

$$m(\{t \in I_\delta : |\tilde{F}_1(t) - I_\delta(\tilde{F}_1)| > \varepsilon\}) = o(\delta^\lambda), \quad \delta \to 0. \tag{2.2}$$

PROOF: We note first that $\tilde{F}_1$ is the conjugate of the bounded function F_1, so $\tilde{F}_1 \in \mathrm{BMO}$ and [4, p. 227]

$$\|\tilde{F}_1\|_* \le K \|F_1\|_\infty,$$

where K is an absolute constant. Next,

$$\tilde{F}(0) = -\lim_{\eta \to 0} \frac{1}{2\pi} \int_\eta^\pi \Delta_F(t) \cot \frac{t}{2}\, dt$$

is finite, where $\Delta_F(t) = F(t) - F(-t)$, and $\cot(t/2) - 2/t = O(1)$ as $t \to 0$, so $\int_0^\pi \Delta_F(t)/t\, dt$ is finite. Hence, since F is increasing in $(-a, a)$,

$$\frac{1}{2} \Delta_F(\eta) \log \frac{1}{\eta} \le \int_\eta^{\eta^{1/2}} \frac{\Delta_F(t)}{t}\, dt$$

for $\eta \in (0, a^2)$, and we deduce that

$$\Delta_F(\eta) = o\left(\left(\log \frac{1}{\eta}\right)^{-1}\right), \quad \eta \to 0.$$

As a consequence,

$$\begin{aligned} \|\tilde{F}_1\|_* \le K \|F_1\|_\infty &= K \sup\{|F(t)|;\ t \in J_\delta\} \\ &\le K \Delta_F(\delta^{1/4}) \\ &= o\left(\frac{1}{\log \frac{1}{\delta}}\right), \quad \delta \to 0. \end{aligned} \tag{2.3}$$

Finally, by a fundamental inequality of John and Nirenberg [4, p. 230; 5] for functions in BMO, we have, for any $\varepsilon > 0$,

$$m(\{t \in I_\delta : |\tilde{F}_1(t) - I_\delta(\tilde{F}_1)| > \varepsilon\}) \le C m(I_\delta) \exp\left(-\frac{c\varepsilon}{\|\tilde{F}_1\|_*}\right), \tag{2.4}$$

where C and c are absolute constants. We now choose $\delta_2 < \delta_1$ such that

$$\|\tilde{F}_1\|_* \leq \frac{\varepsilon c}{\lambda \log \frac{1}{\delta}}, \quad 0 < \delta < \delta_2.$$

Such a choice is possible by (2.3). Then

$$m(I_\delta) \exp\left(-\frac{c\varepsilon}{\|\tilde{F}_1\|_*}\right) \leq 2\delta \exp\left(-\lambda \log \frac{1}{\delta}\right) = 2\delta^{\lambda+1}$$

for $0 < \delta < \delta_2$, and, combined with (2.4), this clearly implies (2.2). The proof of Lemma 3 is complete. ■

We are now in a position to complete the proof of Theorem 1. We note first that, by (2.1),

$$|I_\delta(\tilde{F}_2) - \tilde{F}_2(0)| = \frac{1}{2\delta}\left|\int_{-\delta}^{\delta} [\tilde{F}_2(x) - \tilde{F}_2(0)]\, dx\right| \leq B\delta^{1/2} \to 0$$

as $\delta \to 0$. Consequently, using Lemma 1,

$$I_\delta(\tilde{F}_1) - \tilde{F}_1(0) = I_\delta(\tilde{F}) - \tilde{F}(0) - I_\delta(\tilde{F}_2) + \tilde{F}_2(0) = o(1) \tag{2.5}$$

as $\delta \to 0$. Next, by (2.1) and (2.5),

$$\begin{aligned} |\tilde{F}(t) - \tilde{F}(0)| &\leq |\tilde{F}_1(t) - I_\delta(\tilde{F}_1)| + |I_\delta(\tilde{F}_1) - \tilde{F}_1(0)| + |\tilde{F}_2(t) - \tilde{F}_2(0)| \\ &< |\tilde{F}_1(t) - I_\delta(\tilde{F}_1)| + \varepsilon, \end{aligned}$$

for all $t \in I_\delta$ with δ sufficiently small. Hence, for such δ,

$$\begin{aligned} E_{2\varepsilon}(\tilde{F}, \tilde{F}(0); \delta) &= \{t \in I_\delta : |\tilde{F}(t) - \tilde{F}(0)| \geq 2\varepsilon\} \\ &\subset \{t \in I_\delta : |\tilde{F}_1(t) - I_\delta(\tilde{F}_1)| > \varepsilon\}, \end{aligned}$$

and (1.1) now follows from (2.2). This completes the proof of Theorem 1. ■

3. Proofs of Theorems 2 and 3

We begin with the proof of Theorem 2. We assume, without loss of generality, that $\alpha = 0$. We then need to prove that

$$u(re^{i\theta_r}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta_r - t) F(t)\, dt \to 0, \quad r \to 1, \tag{3.1}$$

where $|\theta_r| \leq c(1-r)^{1/\lambda}$, c a positive constant, and (1.3) holds. To this end, let $\varepsilon \in (0,1)$ be given and choose $\delta \in (0,1)$ such that

$$\int_{-t}^{t} |F(s)|\, ds \leq \varepsilon t, \quad 0 < t \leq \delta. \tag{3.2}$$

Such a choice of δ is possible because, as noted earlier, the hypotheses of the theorem imply that $t^{-1}\int_{-t}^{t} |F(s)| ds \to 0$ as $t \to 0$. Choose r_0 in $(0,1)$ such that $2|\theta_r| \leq \delta$ for $r_0 \leq r < 1$. We note first that

$$\left|\int_{\delta\le|t|\le\pi} P(r,\theta_r - t)F(t)\,dt\right| \le A\frac{1-r}{\delta^2}\int_{|t|\ge\delta}|F(t)|\,dt \to 0 \tag{3.3}$$

as $r \to 1$, where we have used a standard estimate for the Poisson kernel. (Here, and below, A denotes an absolute constant.)

Next, $2\pi - \frac{1}{2}|t| \ge |t-\theta_r| \ge \frac{1}{2}|t|$ for $2|\theta_r| \le |t| \le \pi$, so $P(r, t-\theta_r) \le P(r, \frac{1}{2}t)$, and

$$\left|\int_{2|\theta_r|\le|t|\le\delta} P(r,\theta_r - t)F(t)\,dt\right| \le \int_{|t|\le\delta} P\left(r,\frac{t}{2}\right)|F(t)|\,dt \le A\varepsilon. \tag{3.4}$$

The last inequality here is an immediate consequence of (3.2) and the inequality (see [4, p. 23])

$$\int_{|t|\le\delta} P\left(r,\frac{t}{2}\right)|F(t)|\,dt \le MF(0)\int_{|t|\le\delta} P\left(r,\frac{t}{2}\right)dt,$$

where $MF(0) = \sup_{0<t\le\delta}(2t)^{-1}\int_{-t}^{t}|F(s)|\,ds$.

With χ_ε defined as in Theorem 2, and noting that $\int_{-\pi}^{\pi}P(r,x)\,dx = 2\pi$ and $|\theta_r| \le c(1-r)^{1/\lambda}$, we next have,

$$\begin{aligned}\left|\int_{|t|\le2|\theta_r|} P(r,\theta_r - t)F(t)\,dt\right| &\le \int_{|t|\le2|\theta_r|} \chi_\varepsilon(t)P(r,\theta_r - t)|F(t)|\,dt + 2\pi\varepsilon\\ &= o\left(\frac{|\theta_r|^\lambda}{1-r}\right) + 2\pi\varepsilon\\ &= o(1) + 2\pi\varepsilon, \end{aligned}\tag{3.5}$$

as $r \to 1$, where we have used (1.3) with $\alpha = 0$ and the inequality $P(r,x) \le 2/(1-r)$. Combining (3.5) with (3.4) and (3.3), we deduce, (3.1) and Theorem 2 is proved. ■

We now prove Theorem 3. Let $F = h + G_a + H_a$ denote the decomposition of F detailed in the proof of Lemma 1 in Section 2. By arguments used in the proof of that lemma, it follows, because of the monotonicity of F in a neighbourhood of 0, that $v(r) = O(1)$ implies that $\tilde{F}(0)$ and $\tilde{h}(0)$ exist and that F is continuous at 0. Next, as noted earlier, $\tilde{h} \in L^2(-\pi,\pi)$ and $Q[h] = P[\tilde{h}]$ so, by Theorem 1 and the second part of Corollary 1 (the case $p = 2$), $Q[h]$ has finite $\mathcal{T}_\lambda$-limits for every $\lambda > 1$ at 1. If $v_a = Q[F-h] = Q[G_a + H_a]$, then

$$2\pi v_a(re^{i\theta}) = \int_{a\le|t|\le\pi} Q(r,\theta - t)[F(t) - h(t)]\,dt + \int_{-a}^{a} Q(r,\theta - t)t\,dt,$$

and, noting that $Q(1,t) = \cot\frac{t}{2}$, it is easily shown that

$$2\pi v_a(re^{i\theta}) \to -\int_{a\le|t|\le\pi}[F(t) - h(t)]\cot\frac{t}{2}\,dt - \int_{-a}^{a} t\cot\frac{t}{2}\,dt,$$

as $re^{iq} \to 1$ in any way from inside U. It follows that $v = Q[F]$ has finite $\mathcal{T}_\lambda$-limits for every $\lambda > 1$ at 1. Since F is continuous at 0, $u = P[F]$ has unrestricted limits at 1 by a classical result for the Poisson integral [6, p. 11], and the proof of Theorem 3 is complete. ■

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