ON THE SYMMETRISATION OF NONHOLONOMIC JETS

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[Received 14 September 2004. Read 12 November 2004. Published 11 March 2005.]

Abstract

The canonical involution in the second order iterated tangent bundle is generalised for an arbitrary order and transfered to jet spaces. The classification of all symmetrised nonholonomic jets of the third order is given.

1. Introduction

Nonholonomic and semiholonomic jets (defined originally by Ehresmann in [2]) were studied in detail for the second-order case up to now. In this paper, we arrange newly nonholonomic jets by a symmetrisation. So, the paper can be read as a contribution to the jet calculus. In Section 2, we recall the basic facts from the theory of nonholonomic jets. The generalised involution is defined and investigated and we take advantage of the quasijet language (quasijets were introduced and studied at first by Pradines in [10]) for the description of its operating in jet bundles. In Section 3, we precise our concept of the symmetrisation, drawing inspiration from the second-order case, and introduce so-called symmetrised nonholonomic jets. We focus particularly on investigating the third-order case, with the intention being to construct a stepping stone for a study of the general order. Nevertheless, third-order jet spaces have an independent importance for the applications in several branches of mathematics and physics. For example, in [12] the author discusses the completion of a system of differential equations to an involutive one, and the obtained involutive system describes a 13-dimensional manifold in a third-order jet bundle. Further, in [3], where the equivalence classes of ternary cubics are studied under general complex linear changes of variables, the equivalence problem on the third-order jet space of dimension 12 is studied. However, nonholonomic jets present a very general concept from the applications point of view. On the contrary, semiholonomic jets play an important role in mathematical physics, in the variational calculus and in the theory of systems of partial differential equations (see [11]). The relations of semiholonomic jets with Stieffel and Grassmann bundles are demonstrated in [9]; the application of semiholonomic jets in the theory of Verma modules is presented in [13]. (We refer also to the interesting extension of the theory of nonholonomic jets to so-called (r, s, q)-jets of fibred manifold morphisms in [14].) Notwithstanding that this paper is titled on the symmetrisation of nonholonomic jets, it deals mostly with certain semiholonomic jets in fact! Several symmetries in varieties of

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Mathematics Subject Classification: 58A20, 58A32

nature are observed and that is why symmetrisations play an important role in actual physical theories. We consider the concept of symmetrisation as suitable also for the arrangement of jet spaces; it gives no inapplicable permutations of indexes, but it offers several fully geometrically interpreted operations.

The construction of the generalised involution, the classification of all symmetrised nonholonomic jets of the third order and the geometrical interpretation of them can be viewed as the main new results of the paper. All manifolds and maps are assumed to be of class C^∞.

2. Projections and involutions

2.1. Nonholonomic jets

We recommend the monograph of Kolář, Michor and Slovák, [5], for a good self-contained introduction to the jet theory. The bundle of all (holonomic) r-jets of M into N is denoted by $J^r(M,N)$. The bundle $\tilde{J}^r(M,N)$ of nonholonomic r-jets of M into N is defined by the induction. For $r=1$, the bundle of nonholonomic 1-jets $\tilde{J}^1(M,N) := J^1(M,N)$. By induction, let $\alpha : \tilde{J}^{r-1}(M,N) \to M$ denote the source projection and let $\beta : \tilde{J}^{r-1}(M,N) \to N$ denote the target projection of $(r-1)$-th nonholonomic jets. Then Z is said to be a *nonholonomic r-jet* with the source $x \in M$ and the target $\bar{x} \in N$, if there is a local section $\sigma : M \to \tilde{J}^{r-1}(M,N)$ such that $Z = j^1_x\sigma$ and $\beta(\sigma(x)) = \bar{x}$.

Every $Z \in \tilde{J}^r(M,N)$ induces a map $\mu Z : (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}}$ in the following way. For $r=1$ and $Z = j^1_x f$ is μZ defined as $T_x f$. By induction, let $Z = j^1_x\sigma$ for a local α-section $\sigma : M \to \tilde{J}^{r-1}(M,N)$. Then $\sigma(u) \in \tilde{J}^{r-1}_u(M,N)$, $\mu(\sigma(u)) : (\underbrace{T\dots T}_{(r-1)\text{-times}} M)_u \to (\underbrace{T\dots T}_{(r-1)\text{-times}} N)_{\beta(\sigma(u))}$ and we put $\mu Z = T_x\mu(\sigma(u))$. The constructed map $\mu Z : (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}}$ is a vector bundle morphism with respect to all vector bundle structures $\underbrace{T\dots T}_{r\text{-times}} \to \underbrace{T\dots T}_{(r-1)\text{-times}}$. However, it is not an arbitrary vector bundle morphism with this property (we introduce a concept of the quasijet for it; the relation will be precised in Proposition 4).

Let x^i be local coordinates on M, $i = 1,\dots,m = \dim M$, and let $\bar{x}^p$ be local coordinates on N, $p = 1,\dots,n = \dim N$. On $J^r(M,N)$ we have local coordinates x^i, $\bar{x}^p$ and the induced coordinates $a^p_{i_1\dots i_q}$, $q = 1,\dots,r$, $i_1,\dots,i_q = 1,\dots,m$. The induced coordinates are symmetric in all subscripts. Apart from that, on $\tilde{J}^r(M,N)$ we have the local coordinates x^i, $\bar{x}^p$ and the induced coordinates $b^p_{i_1\dots i_r}$, $i_1,\dots,i_r = 0,1,\dots,m$ (excluding the case $i_1 = \dots = i_r = 0$), which are not symmetric in the subscripts.

2.2. Projections in the iterated tangent bundle

We introduce the following denotation of projections in the iterated tangent bundle $\underbrace{T\dots T}_{r\text{-times}} M$. For every s, $0 < s \le r$, we denote by

$$\pi^s : \underbrace{T\dots T}_{s\text{-times}} M \to M$$

the canonical projection to the base. Further, we denote by

$$\pi^s_b := \pi^{\,s}_{\underbrace{T\dots T}_{b\text{-times}} M} : \underbrace{T\dots T}_{s\text{-times}} (\underbrace{T\dots T}_{b\text{-times}} M) \to \underbrace{T\dots T}_{b\text{-times}} M$$

the projection with $\underbrace{T\dots T}_{b\text{-times}} M$ as the base space; by

$${}_a\pi^s := \underbrace{T\dots T}_{a\text{-times}} \pi^s : \underbrace{T\dots T}_{a\text{-times}} (\underbrace{T\dots T}_{s\text{-times}} M) \to \underbrace{T\dots T}_{a\text{-times}} M$$

the induced projection originating by the posterior application of the functor $\underbrace{T\dots T}_{a\text{-times}}$; and by

$${}_a\pi^s_b := \underbrace{T\dots T}_{a\text{-times}} \pi^{\,s}_{\underbrace{T\dots T}_{b\text{-times}} M}$$

the general case containing both previous cases. If a or b equal zero, we do not write them.

2.3. Involutions in the iterated tangent bundle

We denote by ${}_a\chi^s_b$ the map assigning to an element of $\underbrace{T\dots T}_{r\text{-times}} M$ the fibre with respect to the projection ${}_a\pi^s_b : \underbrace{T\dots T}_{r\text{-times}} M \to \underbrace{T\dots T}_{(r-s)\text{-times}} M$. Now, we introduce the maps

$$\kappa_r : \underbrace{T\dots T}_{r\text{-times}} M \to \underbrace{T\dots T}_{r\text{-times}} M$$

by requirements

$$\pi^1_{r-1} \circ \kappa_r = {}_{r-1}\pi^1$$
$$\chi^1_{r-1} \circ \kappa_r = {}_{r-1}\chi^1.$$

We observe that $\kappa_1 = \mathrm{id}$ and κ_2 is the well-known canonical involution $\kappa : TTM \to TTM$. We call κ_r the *r-involution*.

Proposition 1. $\kappa_r^r = \mathrm{id}$.

PROOF. The local coordinates on M are x^i, the local coordinates on TM are x^i, y^i, the local coordinates on TTM are x^i, y^i, X^i, Y^i, etc., and $2^r m$ local coordinates on $\underbrace{T\dots T}_{r\text{-times}} M$ can be expressed symbolically by a sequence of m-tuples $(x^i_1, x^i_2, \dots, x^i_{2^r})$ or shortly by

$$\alpha = (1, 2, \dots, 2^r).$$

We observe that the application of κ_r gives

$$\kappa_r(\alpha) = (1, 3, \dots, 2^r - 1, 2, 4, \dots, 2^r).$$

Then the further application gives

$$\kappa_r^2(\alpha) = (\kappa_r \circ \kappa_r)(\alpha) = (1, 5, \dots, 2^r - 3, 2, 6, \dots, 2^r - 2, 3, 7, \dots, 2^r - 1, 4, 8, \dots, 2^r),$$

and then

$$\kappa_r^{r-1}(\alpha) = (1, 1 + 2^r - 2^{r-1}, 2, 2 + 2^r - 2^{r-1}, \dots, 2^{r-1}, 2^r)$$

and

$$\kappa_r^r(\alpha) = (1, 2, \dots, 2^r). \quad \blacksquare$$

2.4. The kernel injection

We recall the concept of the kernel injection. For a vector bundle $q: E \to M$ we have two vector bundle structures on TE, namely $p: TE \to E$ and $Tq: TE \to TM$. The *heart* HE of the vector bundle $E \to M$ is defined as the vector bundle $Vp \cap VTq \to M$. We can identify E with HE and there is the canonical *kernel injection* $\iota: E \approx HE \to TE$.

Now we take ${}_a\pi_b^1: \underbrace{T \dots T}_{s\text{-times}} M \to \underbrace{T \dots T}_{s-1\text{-times}} M$ in the role of the mentioned vector bundle, and we denote the kernel injection by ${}_{a+1}\iota_b: \underbrace{T \dots T}_{s\text{-times}} M \to \underbrace{T \dots T}_{s+1\text{-times}} M$.

2.5. Quasijets and quasijet bundles

Let $x \in M$, $y \in N$. A map

$$\phi: (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r\text{-times}} M)_{\bar{x}}$$

is said to be a *quasijet* of order r, with the source x and the target $\bar{x}$, if it is a vector bundle morphism with respect to all vector bundle structures ${}_a\pi_b^1: (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{(r-1)\text{-times}} M)_x$ and ${}_a\pi_b^1: (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}} \to (\underbrace{T \dots T}_{(r-1)\text{-times}} N)_{\bar{x}}$, $a + b = r - 1$. The set of all such quasijets is denoted by $QJ_x^r(M, N)_{\bar{x}}$ and $QJ^r(M, N)$ means the set of all quasijets from M to N. There is a bundle structure $QJ_x^r(M, N) \to M \times N$.

If x^i are local coordinates on M, $i = 1, \dots, m = \dim M$, and if $\bar{x}^p$ are local coordinates on N, $p = 1, \dots, n = \dim N$, then x^i, $\bar{x}^p$ and the induced coordinates $c_{i_1 \dots i_q}^{p\gamma_1 \dots \gamma_q}$, $q = 1, \dots, r$, $i_1, \dots, i_q = 1, \dots, m$, where $\gamma_1, \dots, \gamma_q$ are multiindexes of the length r with elements from $\{0, 1\}$ representing an ordered decomposition of every multiindex of this form. The ordering of them respects the rule of the increasing number of left zeros in $\gamma_1, \dots, \gamma_q$.

2.6. Projections in quasijet and jet bundles

The following three propositions recall properties proved in [1] and [7].

For every s, $0 < s < r$ and for every $a, b \geq 0$ such that $a + b + s = r$, we denote by ${}_a\phi_b^s$ the underlying map with respect to the projection ${}_a\pi_b^s: \underbrace{T \dots T}_{r\text{-times}} \to \underbrace{T \dots T}_{(r-s)\text{-times}}$, i.e.

$$ {}_a\phi_b^s \circ {}_a\pi_b^s = {}_a\pi_b^s \circ \phi. $$

Proposition 2. ${}_a\phi_b^s \in QJ_x^{r-s}(M,N)_{\bar{x}}$.

For every s, $0 < s < r$ and for every $a, b \geq 0$ such that $a+b+s=r$, we define the map ${}^{r-s-1}_{\;\;a}\psi_b$ by

$$ \underbrace{T \dots T}_{(r-s-1)\text{-times}} {}_a\iota_b \circ {}^{r-s-1}_{\;\;a}\psi_b = \phi \circ \underbrace{T \dots T}_{(r-s-1)\text{-times}} {}_a\iota_b. $$

Proposition 3. ${}^{r-s-1}_{\;\;a}\psi_b \in QJ_x^{r-1}(M,N)_{\bar{x}}$.

We have the induced projections of quasijet spaces now. We define

$$ {}_a\Pi_b^s : QJ^r(M,N) \to QJ^{r-s}(M,N) $$

by

$$ {}_a\Pi_b^s(\phi) = {}_a\phi_b^s, $$

and we define

$$ {}^{r-s-1}_{\;\;a}\Pi_b : QJ^r(M,N) \to QJ^{r-1}(M,N) $$

by

$$ {}^{r-s-1}_{\;\;a}\Pi_b(\phi) = {}^{r-s-1}_{\;\;a}\psi_b. $$

Now, we are able to describe a wide class of subbundles of $QJ^r(M,N)$ by identifications of some of these projections. We intend to determine the most significant subbundle now. We know that nonholonomic jets can be viewed as special quasijets using μ.

Proposition 4. *$\phi = \mu Z$ for a nonholonomic jet $Z \in \tilde{J}^r(M,N)$ if and only if*

$$ {}_a\Pi^1_{r-a-1}(\phi) = {}^a_h\Pi_{r-a-h-1}(\phi) $$

for all $a = 0, \dots, r-2$ and for all $h = 1, \dots, r-a-1$.

If the condition from the proposition comes true, we obtain the identification of coordinates, in which it turns out that only the first units in each of multi-indexes $\gamma_1, \dots, \gamma_q$ are essential. So, we remove all units except the first ones in each of the multiindexes, and we obtain only one unit in each multiindex in this way. We then replace the remaining units by related $i_1, \dots, i_q$ and take the sum of these multiindexes. This is how the transfer from quasijet local coordinates to the nonholonomic coordinates occurs. This requires the following new proposition:

Proposition 5. *If an r-th order quasijet ϕ is an image of a nonholonomic jet, then all quasijets ${}_a\Pi_b^s(\phi)$, $0 < s < r$, $a+b+s=r$, are also images of nonholonomic jets.*

Proof. An insight to coordinate expressions makes the proof easy. As Proposition 2 holds, ${}_a\Pi_b^s(\phi) = {}_a\phi_b^s$ is a quasijet. We suppose that ϕ is an image of a nonholonomic jet and thus we can transfer from quasijet local coordinates to the nonholonomic coordinates for ϕ. Then applying the projection ${}_a\Pi_b^s$ also gives a result in nonholonomic coordinates. Clearly, ${}_a\phi_b^s$ is an image of a nonholonomic jet. ∎

As to the proposition, we have induced projections in the nonholonomic jet bundles. We denote these by the same symbols as the projections in the iterated tangent bundles, i.e. by ${}_a\pi_b^s$.

2.7. Semiholonomic jets and ω-holonomic jets

We recall that a nonholonomic jet $Z \in \tilde{J}^r(M, N)$ satisfying the condition

$$\pi^1_{r-1}(Z) = {}_{q-1}\pi^1_{r-q}(Z)$$

for all q, $q = 1, \dots, r$ is called a *semiholonomic jet*, and the bundle of all semiholonomic jets is denoted by $\bar{J}^r(M, N)$.

Let $\omega \in \{1, \dots, r\}$. By a generalisation of the previous definition, a nonholonomic jet $Z \in \tilde{J}^r(M, N)$ satisfying the condition

$${}_{\omega-1}\pi^1_{r-\omega}(Z) = {}_{q-1}\pi^1_{r-q}(Z)$$

for all q, $q = \omega, \dots, r$ is called an *ω-holonomic jet*. Such jets were studied in [8].

2.8. Involutions in quasijet bundles

The r-involutions in $\underbrace{T \dots T}_{r\text{-times}} M$ induce r-involutions in $QJ^r(M, N)$ and we shall denote them by the same symbols. More precisely, the r-involution $\kappa_r\phi$ of a quasijet $\phi : (\underbrace{T \dots T}_{r\text{-times}} M)_x \to (\underbrace{T \dots T}_{r\text{-times}} N)_{\bar{x}}$ is a morphism

$$\kappa_r \circ \phi \circ \kappa_r.$$

Of course, it is also a quasijet. Nevertheless, if a quasijet ϕ is an image of a nonholonomic jet, i.e. $\phi = \mu Z$, then $\kappa_r\phi$ is not an image of another nonholonomic jet, in general. This fact is known for the second-order case: we read it that κ_2 does not induce an involution in the second-order nonholonomic jet bundles (however, it induces the involution in the second-order semiholonomic jet bundles). This will also be seen in the evaluation 2.10.

2.9. Natural transformations of nonholonomic and semiholonomic jet functors

We present the concordance of our approach with another one. Adopting the categorical point of view, we observe that several differential geometric operations can be interpreted as natural transformations of corresponding functors. Presented jet functors are bundle functors on the product category $\mathcal{M}f_m \times \mathcal{M}f$, where $\mathcal{M}f_m$ is the category of m-dimensional manifolds and local diffeomorphism and $\mathcal{M}f$ is the category of manifolds and smooth maps.

All natural transformations of the nonholonomic and semiholonomic second-order jet functor into itself, i.e. natural transformations $\tilde{J}^2 \to \tilde{J}^2$ and $\bar{J}^2 \to \bar{J}^2$, are described in [6]. The yield of the results is that the canonical involution of the iterated tangent bundle provides the involution only for the semiholonomic case $\bar{J}^2 \to \bar{J}^2$, but not for the nonholonomic case $\tilde{J}^2 \to \tilde{J}^2$.

Vosmanská has continued research in this direction for the third-order jet functors. In [15], all natural transformations $\bar{J}^3 \to \bar{J}^3$ and $\bar{J}^{3,2} \to \bar{J}^{3,2}$ are described ($\bar{J}^{3,2}$ denotes the bundle functor of the third-order semiholonomic jets, underlying second-order jets, which are holonomic). The results are interesting: there is no transformation of an involution type for semiholonomic jets, whereas a richer natural transformation $\bar{J}^{3,2} \to \bar{J}^{3,2}$ is demonstrated. Namely, if the choice $d = f = g = h = 0$ and $e = 1$ is taken in the second 5-parameter family of the classification (see [15, formula 18]), then the involution κ_3 is obtained. Nevertheless, is there any jet space wider than $\bar{J}^{3,2}(M, N)$ in which the involution κ_3 operates?

2.10. *The evaluation*

We take $r = 3$ and give the answer for the above formulated question. If we have local coordinates on $TTTM$ as $x^i, y^i_{100} = y^i, y^i_{010} = X^i, y^i_{110} = Y^i, y^i_{001} = \xi^i, y^i_{101} = \eta^i, y^i_{011} = \Xi^i, y^i_{111} = H^i$, and on $TTTN$ as $\bar{x}^p, \bar{y}^p_{100} = \bar{y}^p, \bar{y}^p_{010} = \bar{X}^p, \bar{y}^p_{110} = \bar{Y}^p, \bar{y}^p_{001} = \bar{\xi}^p, \bar{y}^p_{101} = \bar{\eta}^p, \bar{y}^p_{011} = \bar{\Xi}^p, \bar{y}^p_{111} = \bar{H}^p$, a quasijet $\phi : (TTTM)_x \to (TTTN)_{\bar{x}}$ has the coordinate expression $\bar{x}^p = f^p(x^i)$ and, in fibre coordinates,

$$\begin{aligned}
\bar{y}^p &= c_i^{p100} y^i \\
\bar{X}^p &= c_i^{p010} X^i \\
\bar{Y}^p &= c_{ij}^{p100\,010} y^i X^j + c_i^{p110} Y^i \\
\bar{\xi}^p &= c_i^{p001} \xi^i \\
\bar{\eta}^p &= c_{ij}^{p100\,001} y^i \xi^j + c_i^{p101} \eta^i \\
\bar{\Xi}^p &= c_{ij}^{p010\,001} X^i \xi^j + c_i^{p011} \Xi^i \\
\bar{H}^p &= c_{ijk}^{p100\,010\,001} y^i X^j \xi^k + c_{ij}^{p101\,010} \eta^i X^j + c_{ij}^{p100\,011} y^i \Xi^j + c_{ij}^{p110\,001} Y^i \xi^j + c_i^{p111} H^i.
\end{aligned} \tag{2.1}$$

By a direct evaluation, we find an expression of $\kappa_3 \phi$ as

$$\begin{aligned}
\bar{y}^p &= c_i^{p001} y^i \\
\bar{X}^p &= c_i^{p100} X^i \\
\bar{Y}^p &= c_{ji}^{p100\,001} y^i X^j + c_i^{p101} Y^i \\
\bar{\xi}^p &= c_i^{p010} \xi^i \\
\bar{\eta}^p &= c_{ji}^{p010\,001} y^i \xi^j + c_i^{p011} \eta^i \\
\bar{\Xi}^p &= c_{ij}^{p100\,010} X^i \xi^j + c_i^{p110} \Xi^i \\
\bar{H}^p &= c_{jki}^{p100\,010\,001} y^i X^j \xi^k + c_{ji}^{p100\,011} \eta^i X^j + c_{ji}^{p110\,001} y^i \Xi^j + c_{ij}^{p101\,010} Y^i \xi^j + c_i^{p111} H^i.
\end{aligned} \tag{2.2}$$

In view of Proposition 4, ϕ represents an image of a nonholonomic jet if and only if

$$\Pi_2^1 = {}_1\Pi_1 = {}_2\Pi$$
$${}_1\Pi_1^1 = {}_1^1\Pi.$$

Therefore the quasijet coordinates are identified as

$$\begin{aligned}
&c_i^{p100} = c_i^{p110} = c_i^{p101} = c_i^{p111}\\
&c_i^{p010} = c_i^{p011}\\
&c_{ij}^{p100\,010} = c_{ij}^{p101\,010} = c_{ij}^{p100\,011}\\
&c_{ij}^{p100\,001} = c_{ij}^{p110\,001}.
\end{aligned}$$

In the nonholonomic case, we see that we make do only with coordinates c_i^{p100}, c_i^{p010}, $c_{ij}^{p100\,010}$, c_i^{p001}, $c_{ij}^{p100\,001}$, $c_{ij}^{p010\,001}$, $c_{ijk}^{p100\,010\,001}$. They represent nonholonomic local coordinates b_{i00}^p, b_{0i0}^p, b_{ij0}^p, b_{00i}^p, b_{i0j}^p, b_{0ij}^p, b_{ijk}^p, respectively. Then ϕ has a coordinate form

$$\begin{aligned}
\bar{y}^p &= b_{i00}^p y^i\\
\bar{X}^p &= b_{0i0}^p X^i\\
\bar{Y}^p &= b_{ij0}^p y^i X^j + b_{i00}^p Y^i\\
\bar{\xi}^p &= b_{00i}^p \xi^i\\
\bar{\eta}^p &= b_{i0j}^p y^i \xi^j + b_{i00}^p \eta^i\\
\bar{\Xi}^p &= b_{0ij}^p X^i \xi^j + b_{0i0}^p \Xi^i\\
\bar{H}^p &= b_{ijk}^p y^i X^j \xi^k + b_{ij0}^p \eta^i X^j + b_{ij0}^p y^i \Xi^j + b_{i0j}^p Y^i \xi^j + b_{i00}^p H^i,
\end{aligned} \tag{2.3}$$

and $\kappa_3\phi$ has a form

$$\begin{aligned}
\bar{y}^p &= b_{00i}^p y^i\\
\bar{X}^p &= b_{i00}^p X^i\\
\bar{Y}^p &= b_{j0i}^p y^i X^j + b_{i00}^p Y^i\\
\bar{\xi}^p &= b_{0i0}^p \xi^i\\
\bar{\eta}^p &= b_{0ji}^p y^i \xi^j + b_{0i0}^p \eta^i\\
\bar{\Xi}^p &= b_{ji0}^p X^i \xi^j + b_{i00}^p \Xi^i\\
\bar{H}^p &= b_{jki}^p y^i X^j \xi^k + b_{ji0}^p \eta^i X^j + b_{j0i}^p y^i \Xi^j + b_{ij0}^p Y^i \xi^j + b_{i00}^p H^i.
\end{aligned} \tag{2.4}$$

Then $\kappa_3\phi$ is not an image of a nonholonomic jet. If we want $\kappa_3\phi$ to be an image of a nonholonomic jet, we must add further conditions:

$$b_{i00}^p = b_{0i0}^p = b_{00i}^p$$
$$b_{ij0}^p = b_{i0j}^p = b_{0ji}^p.$$

Let us notice, that the nonholonomic jets become semiholonomic ones if and only if

$$b_{i00}^p = b_{0i0}^p = b_{00i}^p$$
$$b_{ij0}^p = b_{i0j}^p = b_{0ij}^p.$$

Furthermore, if the coordinates are invariant with respect to any transposition of subscripts, i.e. fully symmetric, we obtain the holonomic coordinates $a^p_i, a^p_{ij}, a^p_{ijk}$ by deleting zeros in the subscripts.

With these conditions and with notation $a^p_i := b^p_{i00} = b^p_{0i0} = b^p_{00i}$, $\check{b}^p_{ij} := b^p_{ij0} = b^p_{i0j} = b^p_{0ji}$, we have ϕ as

$$\begin{aligned}
\bar{y}^p &= a^p_i y^i \\
\bar{X}^p &= a^p_i X^i \\
\bar{Y}^p &= \check{b}^p_{ij} y^i X^j + a^p_i Y^i \\
\bar{\xi}^p &= a^p_i \xi^i \\
\bar{\eta}^p &= \check{b}^p_{ij} y^i \xi^j + a^p_i \eta^i \\
\bar{\Xi}^p &= \check{b}^p_{ji} X^i \xi^j + a^p_i \Xi^i \\
\bar{H}^p &= b^p_{ijk} y^i X^j \xi^k + \check{b}^p_{ij} \eta^i X^j + \check{b}^p_{ij} y^i \Xi^j + \check{b}^p_{ij} Y^i \xi^j + a^p_i H^i,
\end{aligned} \tag{2.5}$$

and we have $\kappa_3\phi$ as

$$\begin{aligned}
\bar{y}^p &= a^p_i y^i \\
\bar{X}^p &= a^p_i X^i \\
\bar{Y}^p &= \check{b}^p_{ji} y^i X^j + a^p_i Y^i \\
\bar{\xi}^p &= a^p_i \xi^i \\
\bar{\eta}^p &= \check{b}^p_{ij} y^i \xi^j + a^p_i \eta^i \\
\bar{\Xi}^p &= \check{b}^p_{ji} X^i \xi^j + a^p_i \Xi^i \\
\bar{H}^p &= b^p_{jki} y^i X^j \xi^k + \check{b}^p_{ji} \eta^i X^j + \check{b}^p_{ji} y^i \Xi^j + \check{b}^p_{ij} Y^i \xi^j + a^p_i H^i.
\end{aligned} \tag{2.6}$$

Our 'anti-symmetry' in the condition

$$b^p_{ij0} = b^p_{i0j} = b^p_{0ji}$$

is due to the involution κ_2 operating in one of the underlying second-order bundles. Surely, in

$$b^p_{ij0} = b^p_{i0j}$$

the choice $i = 0$ yields

$$b^p_{0j0} = b^p_{00j},$$

which means that the second-order jets after application of ${}_2\pi^1$ are semiholonomic. It also means that κ_2 is operable for them.

Now, κ_3 is operating in special quasijets according to (2.5) and (2.6). But κ_3 is not operating in jet spaces with coordinates $x^i, y^p, a^p_i, \check{b}^p_{ij}, b^p_{ijk}$ as yet, because we have no well-defined image of $\check{b}^p_{ij}$. The provision of such an image requires that

$$\check{b}^p_{ij} = \check{b}^p_{ji},$$

and we have

$$b^p_{ij0} = b^p_{ji0} = b^p_{i0j} = b^p_{j0i} = b^p_{0ij} = b^p_{0ji}.$$

Hence, the widest jet space in which the involution κ_3 operates is exactly $\bar{J}^{3,2}(M,N)$. (See Proposition 6.)

2.11. Three new spaces

The following definitions are evoked by 2.10. Mentioned jet spaces with coordinates $x^i, y^p, a_i^p, \breve{b}_{ij}^p, b_{ijk}^p$ can come into being in three ways. We define these ways as follows

$$\breve{J}^{3,{}_2\pi^1}(M,N) := \{Z \in \tilde{J}^3(M,N);\ \pi_2^1(Z) = {}_1\pi_1^1(Z) = \kappa_2 \circ {}_2\pi^1(Z)\}$$
$$\breve{J}^{3,{}_1\pi_1^1}(M,N) := \{Z \in \tilde{J}^3(M,N);\ \pi_2^1(Z) = \kappa_2 \circ {}_1\pi_1^1(Z) = {}_2\pi^1(Z)\}$$
$$\breve{J}^{3,\pi_2^1}(M,N) := \{Z \in \tilde{J}^3(M,N);\ \kappa_2 \circ \pi_2^1(Z) = {}_1\pi_1^1(Z) = {}_2\pi^1(Z)\}.$$

As we have suggested in 2.9, $\bar{J}^{3,2}$ can be defined as

$$\bar{J}^{3,2}(M,N) := \{Z \in \bar{J}^3(M,N);\ \pi_2^1(Z) \in J^2(M,N)\}.$$

This definition is used e.g. in [4]. We now present:

Proposition 6.

$$\{Z \in \bar{J}^3(M,N);\ \pi_2^1(Z) \in J^2(M,N)\}$$
$$= \{Z \in \bar{J}^3(M,N);\ {}_1\pi_1^1(Z) \in J^2(M,N)\}$$
$$= \{Z \in \bar{J}^3(M,N);\ {}_2\pi^1(Z) \in J^2(M,N)\}.$$

Proof. For $Z \in \bar{J}^3(M,N)$,

$$b_{ij0}^p = b_{i0j}^p = b_{0ij}^p$$

is satisfied. The condition $\pi_2^1(Z) \in J^2(M,N)$ gives

$$b_{ij0}^p = b_{ji0}^p.$$

In common,

$$b_{ij0}^p = b_{ji0}^p = b_{i0j}^p = b_{j0i}^p = b_{0ij}^p = b_{0ji}^p;$$

it is easy to see that the application of ${}_1\pi_1^1(Z) \in J^2(M,N)$ and ${}_2\pi^1(Z) \in J^2(M,N)$, respectively, gives the same. ■

Hence, we consider it advisable to use local coordinates $a_i^p, a_{ij}^p, b_{ijk}^p$ (the 'a's are symmetric in subscripts) for $\bar{J}^{3,2}(M,N)$. Further, we have:

Proposition 7.

$$\breve{J}^{3,\pi_2^1}(M,N) \cap \breve{J}^{3,{}_1\pi_1^1}(M,N)$$
$$= \breve{J}^{3,{}_1\pi_1^1}(M,N) \cap \breve{J}^{3,{}_2\pi^1}(M,N)$$
$$= \breve{J}^{3,{}_2\pi^1}(M,N) \cap \breve{J}^{3,\pi_2^1}(M,N) = \bar{J}^{3,2}(M,N).$$

Proof. We have $Z \in \breve{J}^{3,\pi_2^1}(M,N)$

$$b_{ji0}^p = b_{i0j}^p = b_{0ij}^p$$

for $Z \in \breve{J}^{3,\pi_2^1}(M,N)$, and we have

$$b^p_{ij0} = b^p_{j0i} = b^p_{0ij}$$

for $Z \in \breve{J}^{3,{}_1\pi_1^1}(M,N)$. In common, it is

$$b^p_{ij0} = b^p_{ji0} = b^p_{i0j} = b^p_{j0i} = b^p_{0ij} = b^p_{0ji}$$

and we obtain the same for $\breve{J}^{3,{}_1\pi_1^1}(M,N) \cap \breve{J}^{3,{}_2\pi^1}(M,N)$ as well as for $\breve{J}^{3,{}_2\pi^1}(M,N) \cap \breve{J}^{3,\pi_2^1}(M,N)$. ■

3. Symmetrised nonholonomic jets

3.1. Transposition rules

We lead up to the classification of jet spaces now. In the first subsection, we explain what we mean by the symmetrisation of nonholonomic jets. We dwell on sequences $i_1 \dots i_r$ of r indexes $i_1, \dots, i_r$ having values in the set $\{0, 1, \dots, m\}$.

Let $s \in \mathbb{N} \cup \{0\}$, $s < r$. By the *transposition rule* we mean the property

$$\sigma_1(i_1 \dots i_{r-s} \underbrace{0 \dots 0}_{s\text{-times}}) = \sigma_2(i_1 \dots i_{r-s} \underbrace{0 \dots 0}_{s\text{-times}})$$

satisfied for all possible values of $i_1, \dots i_{r-s} \in \{1, \dots, m\}$, where σ_1 and σ_2 are two permutations of r elements yielding different sequences.

The *symmetrised nonholonomic jets* are nonholonomic jets having subscripts of coordinates $b^p_{i_1 \dots i_r}$ $(i_1, \dots, i_r = 0, 1, \dots, m)$ under one or more transposition rules.

3.2. Second-order case

It is evident that there is no transposition rule for $r = 1$. We demonstrate the situation in the second order. Let Z be an element of $\tilde{J}^2(M,N)$. We take one transposition rule.

2_A. $s = 1$ enables only the rule $i0 = 0i$, i.e. $b^p_{i0} = b^p_{0i}$, which gives $Z \in \bar{J}^2(M,N)$.

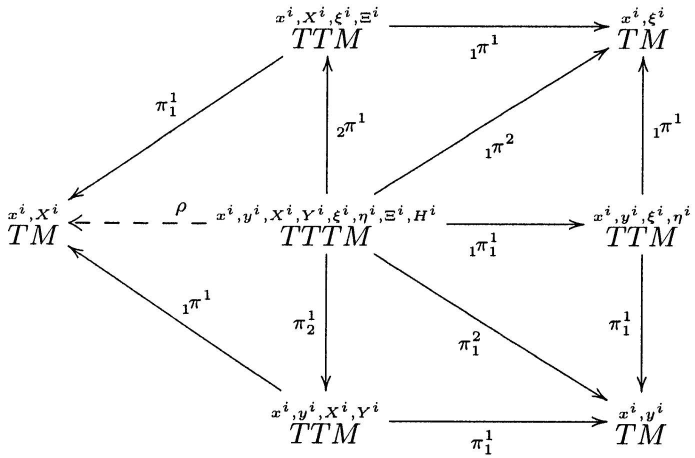
2_B. $s = 0$ enables only the rule $ij = ji$, i.e. $b^p_{ij} = b^p_{ji}$, which gives $Z \in J^2(M,N)$.

Two transposition rules give no new case. So, these are all possibilities and we have proved:

Theorem 1. *Only $\bar{J}^2(M,N)$ and $\tilde{J}^2(M,N)$ are second-order symmetrised nonholonomic jet bundles.*

3.3. Third-order case: Setup

We now demonstrate the situation for $r = 3$. Let x^i be local coordinates on M, let y^i be the induced coordinates on TM, let X^i, Y^i be the induced coordinates on TTM and let $\xi^i, \eta^i, \Xi^i, H^i$ be the induced coordinates on $TTTM$. We have the following diagram:



We denote by ρ the projection $\pi_1^1 \circ {}_2\pi^1 = {}_1\pi^1 \circ \pi_2^1$, which is not of the type ${}_a\pi_b^s$. Further,

$$\kappa_3 : (x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) \mapsto (x^i, X^i, \xi^i, \Xi^i, y^i, Y^i, \eta^i, H^i).$$

3.4. Third-order case: Classification

Now, we are prepared to present the description of all subbundles of $\tilde{J}^3(M, N)$ obtained by the statement of a condition of a transposition of subscripts.

Let Z be an element of $\tilde{J}^3(M, N)$. Now, we obtain the classification of Z satisfying various conditions for symmetry directly from local coordinates. One transposition rule yields:

$\mathbf{3_A}$. $b^p_{i00} = b^p_{0i0}$. This means $\pi_2^1(Z) \in \bar{J}^2(M, N)$.
$\mathbf{3_B}$. $b^p_{i00} = b^p_{00i}$. This means ${}_1\pi_1^1(Z) \in \bar{J}^2(M, N)$.
$\mathbf{3_C}$. $b^p_{0i0} = b^p_{00i}$. This means ${}_2\pi^1(Z) \in \bar{J}^2(M, N)$.
$\mathbf{3_D}$. $b^p_{ij0} = b^p_{ji0}$. This means $\pi_2^1(Z) = \kappa_2 \circ \pi_2^1(Z)$ or $\pi_2^1(Z) \in J^2(M, N)$.
$\mathbf{3_E}$. $b^p_{i0j} = b^p_{j0i}$. This means ${}_1\pi_1^1(Z) = \kappa_2 \circ {}_1\pi_1^1(Z)$ or ${}_1\pi_1^1(Z) \in J^2(M, N)$.
$\mathbf{3_F}$. $b^p_{0ij} = b^p_{0ji}$. This means ${}_2\pi^1(Z) = \kappa_2 \circ {}_2\pi^1(Z)$ or ${}_2\pi^1(Z) \in J^2(M, N)$.
$\mathbf{3_G}$. $b^p_{ij0} = b^p_{i0j}$. This means $\pi_2^1(Z) = {}_1\pi_1^1(Z)$.
$\mathbf{3_H}$. $b^p_{ij0} = b^p_{0ij}$. This means $\pi_2^1(Z) = {}_2\pi^1(Z)$.
$\mathbf{3_I}$. $b^p_{i0j} = b^p_{0ij}$. This means ${}_1\pi_1^1(Z) = {}_2\pi^1(Z)$.
$\mathbf{3_J}$. $b^p_{ij0} = b^p_{j0i}$. This means $\pi_2^1(Z) = \kappa_2 \circ {}_1\pi_1^1(Z)$.
$\mathbf{3_K}$. $b^p_{ij0} = b^p_{0ji}$. This means $\pi_2^1(Z) = \kappa_2 \circ {}_2\pi^1(Z)$.
$\mathbf{3_L}$. $b^p_{i0j} = b^p_{0ji}$. This means ${}_1\pi_1^1(Z) = \kappa_2 \circ {}_2\pi^1(Z)$.
$\mathbf{3_M}$. $b^p_{ijk} = b^p_{ikj}$. This means the identification of Z with the jet obtained by the induced operating of κ_2 (as to the projection ${}_2\pi^1$).

$\mathbf{3_N}$. $b^p_{ijk} = b^p_{kji}$. This means the identification of Z with the jet obtained by the induced operating of κ_2 (as to the projection ${}_1\pi^1_1$).

$\mathbf{3_O}$. $b^p_{ijk} = b^p_{jik}$. This means the identification of Z with the jet obtained by the induced operating of κ_2 (as to the projection π^1_2).

$\mathbf{3_P}$. $b^p_{ijk} = b^p_{jki}$. ($= b^p_{kij}$ after second application). This means $Z = \kappa_3(Z)$.

In addition, new cases can come into being by combinations of the previous cases.

So, we have:

Theorem 2. *All third-order symmetrised nonholonomic jet bundles are presented by bundles in the list* $\mathbf{3_A}$ — $\mathbf{3_P}$ *and by their intersections.*

PROOF. Practically, one can derive the theorem by a direct evaluation and all interpretations are already described. We now explain the three-subscripts cases more. As to the case 3_O, it is easy to show that

$$\kappa_2 : (x^i, y^i, X^i, Y^i) \mapsto (x^i, X^i, y^i, Y^i)$$

on TTM induces

$$\kappa_{2,\pi^1_2} : (x^i, y^i, X^i, Y^i, \xi^i, \eta^i, \Xi^i, H^i) \mapsto (x^i, X^i, y^i, Y^i, \xi^i, \Xi^i, \eta^i, H^i)$$

on $TTTM$. Similarly, as in 2.10, one can transfer κ_{2,π^1_2} to jet bundles and $\mathbf{3_O}$ can be read as $Z = \kappa_{2,\pi^1_2}(Z)$. It can easily be demonstrated that involutions κ_{2,π^1_2} and κ_3 (their compositions, of course) are sufficient to the interpretation of the cases $\mathbf{3_M}$ and $\mathbf{3_N}$. ■

We can now describe some important bundles obtained if more of the transposition rules are satisfied together.

(i) If for $Z \in \tilde{J}^r(M, N)$ two conditions from $\mathbf{3_G}$, $\mathbf{3_H}$, $\mathbf{3_I}$ are satisfied, we have $\bar{J}^3(M, N)$.

(ii) If for $Z \in \tilde{J}^r(M, N)$ two conditions from $\mathbf{3_J}$, $\mathbf{3_K}$, $\mathbf{3_L}$ or from $\mathbf{3_G}$, $\mathbf{3_K}$, $\mathbf{3_L}$ or from $\mathbf{3_H}$, $\mathbf{3_J}$, $\mathbf{3_L}$ or from $\mathbf{3_I}$, $\mathbf{3_J}$, $\mathbf{3_K}$ are satisfied, we have a space of a type $\check{J}^3$ described in 1.11.

(iii) As to ω-holonomic jets [8], 1-holonomic jets are semiholonomic, 2-holonomic jets are determined by $\mathbf{3_I}$ and 3-holonomic jets are nonholonomic.

(iv) If for $Z \in \tilde{J}^r(M, N)$ two conditions from $\mathbf{3_M}$, $\mathbf{3_N}$, $\mathbf{3_O}$, $\mathbf{3_P}$ are satisfied together, we have $J^3(M, N)$.

Remark 1. By the identification of nonholonomic jet bundles with iterated jet bundles, as routine, we can read $\mathbf{3_G}$ as $\bar{J}^2(M, J^1(M, N))$, $\mathbf{3_I}$ as $J^1(M, \bar{J}^2(M, N))$, $\mathbf{3_M}$ as $J^2(M, J^1(M, N))$ and $\mathbf{3_O}$ as $J^1(M, J^2(M, N))$.

Remark 2. Kolář has introduced the concept of the jet functor on category $\mathcal{M}f_m \times \mathcal{M}f$ and has it generalised for the category $\mathcal{FM}_m$ of fibred manifolds, with m-dimensional bases and fibre preserving maps with local diffeomorphisms as base maps. The classification of such second-order jet functors is presented in [4].

Acknowledgements

The author was supported by GA CR, grant No. 201/02/0225. The paper represents the extended version of the contribution at the Joint Meeting of the 56th British Mathematical Colloquium and the 17th Annual Meeting of the Irish Mathematical Society, Belfast 2004.

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