MINIMAL COMPLETELY DECOMPOSABLE GROUPS

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[Received 18 October 2003. Read 31 January 2005. Published 11 March 2005.]

Abstract

A torsion-free abelian group is called minimal if it is isomorphic to all of its finite index subgroups. We characterize the minimal completely decomposable groups of arbitrary rank. Moreover, we classify those (finite) sets of types T, such that every (almost completely decomposable) bounded completely decomposable group with critical typeset T is completely decomposable.

1. Introduction

The notion of minimality appears in several ways in the theory of abelian groups. It was Ó hÓgáin who investigated the different notions of minimality systematically in his doctoral thesis [5] for the first time. In this paper we concentrate on the following definition: A torsion-free abelian group G is called *minimal* if it is isomorphic to all of its finite index subgroups. Among the minimal groups there are extensive classes of well-known and interesting abelian groups, e.g. Whitehead groups (see [5; 6]). However, many questions on minimal groups are still unsolved, and in this note we consider one of them. Ó hÓgáin proved in [5] a characterisation theorem for minimal completely decomposable groups of finite rank. We shall extend this result to the infinite rank case, which was left open in [5]. In order to do so, we prove the relationship between completely decomposable minimal groups and bounded completely decomposable groups (almost completely decomposable groups, respectively). We obtain a characterisation of the typesets T, which have the property that every bounded completely decomposable group with critical typeset T is already completely decomposable. We will demonstrate that these are exactly the critical typesets of minimal completely decomposable groups.

Notations follow those of [1; 2], with the exception that maps are written on the right. For further details on the class of almost completely decomposable groups and bounded completely decomposable groups we refer to [3; 4]. Further investigations of minimal groups (of various kinds) can be found in [5].

2. Minimal completely decomposable groups

Recall that a *completely decomposable group* is a direct sum $\bigoplus_{i\in I} R_i$ of subgroups R_i of the rational numbers Q. An *almost completely decomposable group* (*acd-group*) is a

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Mathematics Subject Classification: 20K01, 20K10, 20K30 torsion-free finite extension of a completely decomposable group of finite rank. More generally, a torsion-free abelian group G is a *bounded completely decomposable group* (*bcd-group*) if G contains a completely decomposable group A such that G/A is bounded. Given an element $x \in G$, we denote by $type(x)$ the type of x in G and by $Tst(G) = \{type(x) : x \in G\}$ the typeset of G. Moreover, for a type τ, $G(\tau) = \langle x \in G : type(x) \geq \tau \rangle$ denotes the τ-socle of G, which is a pure, fully invariant subgroup of G. Furthermore, $G[\tau] = \langle x \in G : type(x) \not\geq \tau \rangle$ is the τ-radical of G. In [3] it was shown that any bcd-group G has a Butler decomposition for any type τ, i.e. $G(\tau) = G_\tau \oplus G^\sharp(\tau)$, where $G^\sharp(\tau) = \langle x \in G : type(x) > \tau \rangle_*$. If $G(\tau)/G^\sharp(\tau) \neq 0$, then τ is called a *critical type* of G. By $T_{cr}(G)$ we denote the set of all critical types of G. Two types τ and σ are incomparable if $\tau \not< \sigma$ and $\sigma \not< \tau$. Since there is no loss of generality we will identify types with rational subgroups containing the integers in the sequel.

Finally, recall that the *p-rank* $r_p(G)$ of a torsion-free group G (p a prime) is just the dimension of G/pG over the field with p elements.

We shall now consider minimal torsion-free abelian groups (see also [5, chapter V]).

Definition 2.1. *A torsion-free group G is called* ***minimal*** *if it is isomorphic to all of its finite index subgroups.*

Ó hÓgáin characterized in [5, chapter V, theorem 4.17] the completely decomposable minimal groups of finite rank as exactly those finite rank completely decomposable groups C, such that $r_p(R)r_p(S) = 0$ for all incomparable types $R, S \in T_{cr}(C)$ and all primes p. It is our aim to extend this result to the infinite rank. Therefore we first characterise certain classes of bounded completely decomposable groups. The following lemma is an easy consequence of Ó hÓgáin's result and the proof is left to the reader.

Lemma 2.2. *Let $T = \{R_1, \cdots, R_n\}$ be a finite set of types. Then every almost completely decomposable group G with $T_{cr}(G) = T$ is completely decomposable if and only if $r_p(R)r_p(S) = 0$ for all incomparable types $R, S \in T$ and all primes p.*

We now extend Lemma 2 to bounded completely decomposable groups.

Theorem 2.3. *Let T be a set of types. Then any bounded completely decomposable group G with critical typeset $T_{cr}(G) = T$ is completely decomposable if and only if the following two properties are satisfied:*

(i) *T contains no countable ascending chain of types $\{R_n : n \in \omega\}$ with $r_q(R_n) = 1$ for some prime q and all $n \in \omega$;*

(ii) *$r_p(R)r_p(S) = 0$ for all incomparable types $R, S \in T$ and all primes p.*

Proof. The necessity of (i) and (ii) follow from Lemma 2 and [4, example 6.2]. Note that $r_q(R_n) = 1$ implies that we may assume without loss of generality that $1/q \notin R_n$ for all $n \in \omega$.

Now assume that T is given and has the stated properties. Let G be a bcd-group with critical typeset T. Choose a completely decomposable subgroup $A \subseteq G$ such that $mG \subseteq A$ for some integer m. By a standard induction argument we may assume without loss of generality that $m = p^k$ for some prime p and some integer k. Let $\tau = \mathsf{Z}[1/p]$, the subring of Q generated by Z and $1/p$. Clearly, $A(\tau) \subseteq G(\tau)$, and since $A(\tau)$ is p-divisible we conclude that $A(\tau) = G(\tau)$. We claim that $G = F \oplus A(\tau) \oplus G[\tau]$ with F a free group. By the Butler decomposition, $G = G(\mathsf{Z}) = F \oplus G^\sharp(\mathsf{Z})$, where F is a free group. Hence we may assume without loss of generality that $G = G^\sharp(\mathsf{Z})$. Since $A(\tau) \cap G[\tau] = 0$, it suffices to prove that $G = A(\tau) + G[\tau]$. Let $g \in G$, then $p^k g = a + b$ with $a \in A(\tau)$ and $b \in A[\tau]$. Note that $A = A(\tau) \oplus A[\tau]$. Since $A(\tau)$ is p-divisible, it follows that $a = p^k a'$ for some $a' \in A(\tau)$ and thus $p^k(g - a') \in A[\tau]$. Therefore $g - a' \in G[\tau] = A[\tau]_*$ and hence $g \in A(\tau) + G[\tau]$.

By our assumption on T, the critical typeset $S = T_{cr}(G[\tau])$ of $G[\tau]$ must be linearly ordered. Note that any type $\sigma \in Tst(G[\tau])$ must be p-reduced (so $r_p(\sigma) = 1$) and hence all types in $Tst(G[\tau])$ are comparable. Since T does not contain a countable ascending chain of types R_n ($n \in \omega$) with $r_p(R_n) = 1$ for all $n \in \omega$, we conclude that $T_{cr}(G[\tau])$ must be inversely well-ordered. Hence [4, corollary 6.1] shows that $G[\tau]$—and therefore also G—is completely decomposable as well. ■

We now turn our attention to minimal groups and prove the following result, which was missing in [5].

Therorem 2.4. *Let C be a completely decomposable group. Then C is minimal if and only if $T_{cr}(C)$ satisfies the following two properties:*

(i) *$T_{cr}(C)$ does not contain a countable ascending chain of types $\{R_n : n \in \omega\}$ with $r_q(R_n) = 1$ for some prime q and all $n \in \omega$;*

(ii) *$r_p(R) r_p(S) = 0$ for all incomparable types $R, S \in T_{cr}(C)$ and all primes p.*

PROOF. Assume that $T_{cr}(C)$ does not contain a countable chain as in (i) and that $r_p(R) r_p(S) = 0$ for all incomparable types $R, S \in T_{cr}(C)$ and all primes p. Let C be given and let D be a finite index subgroup of C. Then $mC \subseteq D$ for some integer m. Thus D is a bounded extension of the completely decomposable group mC. By Theorem 2.3 it follows that D is completely decomposable and hence isomorphic to $mC \cong C$. Thus C is minimal.

Conversely, assume that C is minimal. By Lemma 2.2 condition (ii) follows. It remains to prove (i). Hence assume that there exists in T a countable ascending chain of types $S = \{R_n : n \in \omega\}$ with $r_q(R_n) = 1$ for some prime q and all $n \in \omega$. Without loss of generality we may assume that $1/q \notin R_n$ and $R_n \subseteq R_{n+1}$ for all $n \in \omega$. It is sufficient to construct a completely decomposable group with critical typeset S, which is not minimal. By [4, example 6.2] there exists a bcd-group E, with critical typeset S, which is not completely decomposable. The example is of the following form:

$$A = \bigoplus_{n=1}^{\infty} R_n v_n \text{ and } E = A + \sum_{n=1}^{\infty} \mathsf{Z}\frac{1}{q}(v_1 + (-1)^n v_{n+1}).$$

Clearly, $qE \subseteq A$, hence E/A is q-bounded. Put $D = qE = qA + \sum_{n=1}^{\infty} \mathsf{Z}(v_1 + (-1)^n v_{n+1})$. We claim that A/D is finite, hence D is a subgroup of A of finite index which is not completely decomposable. Thus A is not minimal. By definition, $\langle v_1 + D\rangle = \langle v_n + D\rangle$ for all $1 < n \in \omega$, since $v_1 + (-1)^n v_i \in D$. Thus $\left(\bigoplus_{n=1}^{\infty} \mathsf{Z}v_n\right) + D = \langle v_0 + D\rangle$. Now let $a \in A$, then there is an integer m such that $(m, q) = 1$ and $ma \in \bigoplus_{n=1}^{\infty} \mathsf{Z}v_n$. Since $(m, q) = 1$ and $qA \subseteq D$, we easily obtain that

$$a + D \in \langle v_1 + D\rangle.$$

Thus $A/D \cong \langle v_1 + D\rangle$, which is a finite group of order q. This finishes the proof. ■

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