ON IDEALS OF GENERALISED INVERTIBLE ELEMENTS IN BANACH ALGEBRAS

By

R.M. Brits
Department of Mathematics and Statistics, Rand Afrikaans University, South Africa

L. Lindeboom*
Department of Mathematics, Applied Mathematics and Astronomy,
University of South Africa, South Africa

and

H. Raubenheimer
Department of Mathematics and Statistics, Rand Afrikaans University, South Africa

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Abstract

We characterise ideals consisting of generalised invertible elements in a semisimple Banach algebra.

1. Introduction

Throughout A is a complex Banach algebra with identity 1. We denote the group of invertible elements by A^{-1} and the connected component of 1 in A^{-1} by $\mathrm{Exp}(A)$ Recall that $\mathrm{Exp}(A)$ is a normal open and closed subgroup of A^{-1} generated by the elements e^a, $a \in A$ If $a \in A$ then a is said to be *central* if a commutes with every element of A. Accordingly, the set of central elements of A is called the *centre* of A. The spectrum of a in A is denoted by $\sigma(a, A) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$, and when no confusion can arise we simply write $\sigma(a)$. We denote the isolated points of $\sigma(a)$ by iso $\sigma(a)$. A subset J of A is a left (right) *multiplicative ideal* if $AJ \subseteq J$ ($JA \subseteq J$). If J is also a vector space then, as usual, J is just called a left (right) ideal. Whenever both the left and the right conditions hold, we shall only use the term ideal. If J is any subset of A, then $\bar{J}$ denotes the closure of J in A. An ideal J in A is said to be *inessential* [2, p. 106] if for every $x \in J$, $\sigma(x)$ is either a finite set or a sequence converging to zero. A is said to be it *semiprime* if $0 \neq u \in A$ implies there is $x \in A$ such that $uxu \neq 0$. All semisimple Banach algebras are semiprime. An element $a \neq 0$ in a semiprime Banach algebra A is called *rank one* if there exists a linear functional τ_a on A such that $axa = \tau_a(x)a$ for all $x \in A$. For properties of these elements we refer to [13]. The *finite elements* of A, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$

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*Corresponding author; e-mail: lindel@unisa.ac.za of the form $a = \sum_{i=1}^{n} a_i$ with each a_i a rank one element. For semiprime Banach algebras the set of finite elements coincides with the socle of A that is, by definition, the set of all finite sums of minimal left ideals. It is well-known that the socle of A is also the set of all finite sums of minimal right ideals. If we denote the socle of A by Soc(A), then it is clear that Soc(A) is an ideal of A. We call an element $a \in A$ *regular* if it has a generalised inverse in A, that is, there is $b \in A$ for which $a = aba$. We write

$$\hat{A} = \{a \in A : a \in aAa\}$$

for the set of regular elements. This set includes A^{-1} as well as the set of idempotents

$$A^{\bullet} = \{a \in A : a^2 = a\}.$$

The *decomposably regular* elements are defined to be those regular elements that admit invertible generalised inverses. The terminology is appropriate as the decomposably regulars are precisely those elements that can be written as the product of an invertible and an idempotent:

$$A^{-1}A^{\bullet} = A^{\bullet}A^{-1} = \{a \in A : a \in aA^{-1}a\} \subseteq \hat{A}.$$

It can also be shown [7, theorem 7.3.4] that

$$A^{-1}A^{\bullet} = \hat{A} \cap \overline{A^{-1}}. \tag{1.1}$$

Following [12] and [15] we call an element $a \in A$ *group invertible* if there exists $b \in A$ such that $a = aba$, $b = bab$ and $ab = ba$. The term group refers to the fact that $\{a, b\}$ generates an Abelian group (with identity ab); naturally b is called the group inverse of a, and a is called the group inverse of b. We denote the set of group invertible elements by $\mathcal{G}(A)$.

An element $a \in A$ is called *Drazin invertible* if there is $b \in A$, $k \in \mathbb{Z}^+$ such that $a^k ba = a^k$, $bab = b$ and $ab = ba$. It is easy to see that a is Drazin invertible if and only if a^k is group invertible for some $k \in \mathbb{Z}^+$. If a is Drazin invertible with Drazin inverse b, then the least non-negative integer k satisfying $a^k ba = a^k$ is called the *Drazin index* of a. We denote the set of Drazin invertible elements by $\mathcal{D}(A)$ and the subset of $\mathcal{D}(A)$ consisting of elements with index k by $\mathcal{D}^k(A)$. Note that, by definition, the sets $\mathcal{D}^k(A)$ are mutually disjoint, although it is true that if $a \in \mathcal{D}^k(A)$, then corresponding to each $n > k$ there is $b \in A$ such that $a^n ba = a^n$, $bab = b$ and $ab = ba$. Note also that a Drazin invertible element is nilpotent if and only if its Drazin inverse is 0. With the convention $x^0 = 1$ we have $\mathcal{D}^0(A) = A^{-1}$. Since every invertible element has in its inverse a group inverse, it follows that $A^{-1} \cup \mathcal{D}^1(A) = \mathcal{G}(A)$.

Recall that an element $a \in A$ is called *quasipolar* if there exists $b \in A$ such that $ab = ba = p = p^2$ with $\|a^n(1-p)\|^{\frac{1}{n}} \to 0$ [7, p. 257]. The set of quasipolar elements is denoted by $\mathcal{QP}(A)$. One may easily verify that

$$\mathcal{QP}(A) = \{a \in A : 0 \notin \operatorname{acc} \sigma(a)\},$$

where acc $\sigma(a)$ is the set of accumulation points of $\sigma(a)$. In [10] J.J. Koliha points out that the set $\mathcal{QP}(A)$ may be viewed as those elements that admit a generalised Drazin inverse, that is, $a \in \mathcal{QP}(A)$ if and only if there is $b \in A$ such that $ab = ba$, $b = bab$ and $a(1 - ab)$ is quasinilpotent. Note that in the definition of $a \in \mathcal{D}(A)$, the only difference is that we require $a(1-ab)$ to be nilpotent rather than quasinilpotent. As such we have the following:

$$A^{-1} \subseteq \mathcal{G}(A) \subseteq A^{-1}A^{\bullet} \subseteq \hat{A} \tag{1.2}$$

and

$$A^{-1} \subseteq \mathcal{G}(A) \subseteq \mathcal{D}(A) \subseteq \mathcal{QP}(A). \tag{1.3}$$

In general the sets of quasipolar and regular elements do not coincide, since by [8, example 2] there is a Banach algebra A and $a \in A^{-1}A^{\bullet}$ such that $0 \in \operatorname{acc}\sigma(a)$. A distinct advantage of the set $\mathcal{QP}(A)$ over $A^{-1}A^{\bullet}$ and $\hat{A}$ is the uniqueness of the Drazin inverse for elements of $\mathcal{QP}(A)$, [10, theorem 4.2]; even in finite dimensional cases it is easy to show that the generalised inverses of elements of $A^{-1}A^{\bullet}$ and $\hat{A}$ might not be unique. For a more detailed account on generalised inversion (and other types of) one may consult ([12; 15]). If A is a semisimple commutative Banach algebra the pictures (1.2) and (1.3) simplify significantly:

Proposition 1. *If A is a semisimple commutative Banach algebra then* (1.2) *and* (1.3) *reduce to*

$$A^{-1} \subseteq A^{-1}A^{\bullet} = \hat{A} = \mathcal{G}(A) = \mathcal{D}(A) = \mathcal{QP}(A). \tag{1.4}$$

PROOF: The equality of $\mathcal{G}(A)$, $\mathcal{D}(A)$ and $\mathcal{QP}(A)$ follows from [15, proposition 2]. If $a \in \hat{A}$ with $b \in A$ such that $aba = a$, then it is easily seen that bab is a group inverse for a. Hence $\hat{A} = \mathcal{G}(A)$. Since $A^{-1}A^{\bullet} = \hat{A} \cap \overline{A^{-1}}$ and $\mathcal{G}(A) \subseteq \overline{A^{-1}}$, it follows that $A^{-1}A^{\bullet} = \mathcal{G}(A) \cap \overline{A^{-1}} = \mathcal{G}(A)$. ■

It is well-known [2, lemma 3.2.1; 7, theorem 7.3.2; 8, theorem 5] that if $a, b \in A$, then

$$1 - ab \in X \quad \text{if and only if} \quad 1 - ba \in X, \tag{1.5}$$

with $X \in \{A^{-1},\ \hat{A},\ A^{-1}A^{\bullet},\ \mathcal{QP}(A)\}$.

In the next result we illustrate that (1.5) also holds for Drazin invertible elements with preservation of the index. We start with the following:

Lemma 2. *If A is a Banach algebra and $a \in A$ with $a \notin A^{-1}$ then $a \in \mathcal{D}^k(A)$ if and only if $0 \in \operatorname{iso}\sigma(a)$ and k is the least positive integer such that*

$$\frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k(\lambda - a)^{-1}\,d\lambda = 0,$$

where Γ_0 is a small circle surrounding $0 \in \mathbb{C}$ and separating 0 from $\sigma(a)\backslash\{0\}$.

PROOF: $\Rightarrow$ If $a \notin A^{-1}$ and $a \in \mathcal{D}^k(A)$, then it follows from (1.3) that $0 \in \operatorname{iso}\sigma(a)$. If $\sigma(a) = \{0\}$ then, using the fact $a \in \mathcal{D}^k(A)$, it follows that k is the least positive integer such that $a^k = 0$ (see the remarks following the definition of a Drazin inverse). But, by the Holomorphic Functional Calculus, $a^k = \frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k(\lambda - a)^{-1} d\lambda$ for any circle Γ_0 surrounding 0. Suppose $\sigma(a) \neq \{0\}$. Let U_0 be an open ball about 0 and U_1 be an open set containing $\sigma(a)\backslash\{0\}$ such that U_0 and U_1 are separated in $\mathbb{C}$. Let Γ_0 be a circle in U_0 surrounding 0, and let Γ_1 be a smooth contour in U_1 surrounding $\sigma(a)\backslash\{0\}$. By the Holomorphic Functional Calculus, the Drazin inverse of a is given by

$$b = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} g(\lambda)(\lambda - a)^{-1} d\lambda,$$

where

$$g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0. \end{cases}$$

Since k is the least positive integer satisfying

$$\frac{1}{2\pi i} \int_{\Gamma_1} \lambda^k(\lambda - a)^{-1} d\lambda = a^k ba = a^k = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} \lambda^k(\lambda - a)^{-1} d\lambda,$$

it follows that k is the least positive integer such that

$$\frac{1}{2\pi i} \int_{\Gamma_0} \lambda^k(\lambda - a)^{-1} d\lambda = 0.$$

$\Leftarrow$ If $\sigma(a) = \{0\}$, then it follows that k is the least positive integer such that $a^k = 0$ and hence $a \in \mathcal{D}^k(A)$ with Drazin inverse $b = 0$. If $\sigma(a) \neq \{0\}$ then, since $0 \in \operatorname{iso} \sigma(a)$, we may find U_0, U_1, Γ_0, Γ_1 as in the first part of the proof. If we define

$$b = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} g(\lambda)(\lambda - a)^{-1} d\lambda,$$

with

$$g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0, \end{cases}$$

then it follows by the assumption that k is the least positive integer such that

$$\frac{1}{2\pi i} \int_{\Gamma_0} \lambda^k(\lambda - a)^{-1} d\lambda = 0,$$

so that, $a \in \mathcal{D}^k(A)$ with Drazin inverse b. ■

We refer to [10] for other representations and characterisations of Drazin invertibility.

Theorem 3. *If A is a Banach algebra with $a, b \in A$, then $1 - ab \in \mathcal{D}^k(A)$ if and only if $1 - ba \in \mathcal{D}^k(A)$. In particular $1 - ab \in \mathcal{G}(A)$ if and only if $1 - ba \in \mathcal{G}(A)$.*

PROOF: Suppose $1-ab \in \mathcal{D}^k(A)$. Since $1-ab \in A^{-1}$ if and only if $1-ba \in A^{-1}$ the theorem holds for $k=0$. Assume $k \neq 0$. In view of Lemma 2 we have $0 \in \operatorname{iso}\sigma(1-ab)$ and hence also $0 \in \operatorname{iso}\sigma(1-ba)$. We consider two cases.

Case (i): If $\sigma(1-ab)=\{0\}$ then k is the least positive integer such that $(1-ab)^k=0$ and $1-ab$ has Drazin inverse $c=0$. Now

$$\begin{aligned}(1-ba)^k ba &= \left(\sum_{r=0}^{k}\binom{k}{r}(-ba)^r\right)ba\\ &= b\left(\sum_{r=0}^{k}\binom{k}{r}(-ab)^r\right)a\\ &= b(1-ab)^k a\\ &= 0.\end{aligned}$$

It follows easily from this that $(1-ba)^k$ is an idempotent, and one may verify that $d=(1-ba)^k$ is a Drazin inverse for $1-ba$. Thus $1-ba \in \mathcal{D}^l(A)$, $l \leq k$.

Case (ii): If $\sigma(1-ab) \neq \{0\}$ choose U_0, U_1, Γ_0 and Γ_1 such that the separation properties in Lemma 2 hold for both $\sigma(1-ba)$ and $\sigma(1-ab)$. In particular choose U_0 small enough such that $1 \notin U_0$. Now

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_0}\lambda^k(\lambda-(1-ba))^{-1}d\lambda &= \frac{1}{2\pi i}\int_{\Gamma_0}\frac{-\lambda^k}{1-\lambda}[b((1-\lambda)-ab)^{-1}a+1]d\lambda\\ &= b\left[\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\lambda^k}{1-\lambda}(\lambda-(1-ab))^{-1}d\lambda\right]a\\ &\quad+\frac{1}{2\pi i}\int_{\Gamma_0}\frac{-\lambda^k}{1-\lambda}d\lambda.\end{aligned}$$

Clearly, the second integral is 0. If we define

$$f(\lambda)=\begin{cases}\lambda^k, & \lambda \in U_0\\ 0, & \lambda \in U_1\end{cases}$$

and

$$g(\lambda)=\begin{cases}\frac{1}{1-\lambda}, & \lambda \in U_0\\ 0, & \lambda \in U_1,\end{cases}$$

then it follows by the Holomorphic Functional Calculus, that $1-ab \in \mathcal{D}^k(A)$, and, by Lemma 2, that the first integral is also 0. Thus $1-ba \in \mathcal{D}^l(A)$, $l \leq k$. Applying the proof to $1-ba$ it follows that $k \leq l$ and hence $k=l$. ■

2. Ideals of generalised inverses

In this section we investigate the structure of ideals consisting of generalised invertible elements. Firstly we focus on ideals of regular, decomposably regular and Drazin invertible elements, and finally we concentrate on the case of group invertible elements. The motivation for this study stems from the fact that regular and decomposably regular elements do not multiply well [8, example 2]. But also, on the other hand, if a semisimple A possesses a non zero socle then Theorem 4 shows that $\hat{A}$ and $A^{-1}A^{\bullet}$ contains the ideal $\mathrm{Soc}(A)$.

Theorem 4. *If A is semisimple the*
(i) $\mathrm{Soc}(A) \subseteq A^{-1}A^{\bullet} \subseteq \hat{A}$.
If $J \subseteq A$ is an inessential ideal then
(ii) $J \subseteq \hat{A} \Rightarrow J \subseteq \mathrm{Soc}(A)$.

Proof: (i) is given [11, lemma 6.2] by combining Puhl [13] (cf. also [3, corollary 2.10] with [8, theorem 7]). For (ii) [11, theorem 6.6] note that if $a = aba$ the idempotent ab belongs to the 'hull–kernel' of the socle, and therefore to the socle. ■

We show that we can drop the requirement that J be inessential and that we can relax the ideal properties of J somewhat. We start with a spectral description of the socle and a well-known lemma (which is barely mentioned in the literature). Recall that if D is a subset in the complex plane then $\#D$ denotes the number of elements in D.

Theorem 5. [3; 4, theorem 2.3] *If A is a semisimple Banach algebra and $a \in A$ then $a \in \mathrm{Soc}(A)$ if and only if $\#\sigma(xa) < \infty$ or, equivalently, $\#\sigma(ax) < \infty$ holds for all $x \in A$.*

Lemma 6. [1, lemma 6, p. 4] *Let A be a semisimple Banach algebra and $p \in A^{\bullet}$. Then*
(i) *pAp is a semisimple Banach algebra with identity p.*
(ii) *$\sigma(pxp, A)\backslash\{0\} = \sigma(pxp, pAp)\backslash\{0\}$ holds for all $x \in A$.*

We are now ready to prove our first main result:

Theorem 7. *If A is semisimple and $J \subseteq A$ is either a left or a right multiplicative ideal then the following are equivalent:*
(i) $J \subseteq \mathrm{Soc}(A)$
(ii) $J \subseteq A^{-1}A^{\bullet}$
(iii) $J \subseteq \hat{A}$

Proof: Implications (i) ⇒ (ii) ⇒ (iii) are clear from Theorem 4(i). For the converse, we go back to two old results of Kaplansky [9] and Puhl [13]. Let J be a left multiplicative ideal with $J \subseteq \hat{A}$. Let $a \in J$ be arbitrary with generalised inverse b, that is, $aba = a$. Now $ba = p \in A^{\bullet}$, and hence pAp is a semisimple Banach algebra with $pAp \subseteq \hat{A}$ since J is a multiplicative ideal. For $y \in A$ arbitrary, there is $z \in A$

such that $(pyp)z(pyp) = pyp = (pyp)(pzp)(pyp)$. Thus pyp has a generalised inverse belonging to pAp. But this says that pAp is von Neumann regular and hence finite dimensional [9, p. 58]. It follows from [13, theorem 3.4] that $p \in \mathrm{Soc}(A)$ and hence $ap = a \in \mathrm{Soc}(A)$. If J is a right multiplicative ideal, then, noticing that also $ab \in A^{\bullet}$, the proof follows similarly. ■

Theorem 7 solves a problem of R. Harte and H. Raubenheimer [8, problem 8]: If J is an ideal in a semisimple Banach algebra A, is there implication

$$J \subseteq A^{-1}A^{\bullet} \Rightarrow A^{-1}A^{\bullet} + J \subseteq A^{-1}A^{\bullet}?$$

In fact, as soon as we know that $A^{-1}A^{\bullet} + \mathrm{Soc}(A) \subseteq A^{-1}A^{\bullet}$, which is given by [8, theorem 7], the solution of [8, problem 8] is trivial.

Remark 8. It is worthwhile to note that if J is a left or a right multiplicative ideal such that $J \subseteq \hat{A}$, then, by Theorem 7, we have that $x, y \in J$ also implies that $x + y \in \hat{A}$.

The next result shows that Theorem 7 can be extended to Drazin invertible elements:

Theorem 9. *Let A be a semisimple Banach algebra, and let J be a left or right multiplicative ideal in A. Then $J \subseteq \mathcal{D}(A)$ if and only if $J \subseteq \mathrm{Soc}(A)$.*

PROOF: $\Leftarrow$ If $a \in \mathrm{Soc}(A)$ then $a \in \mathcal{QP}(A)$ and hence, following the comment preceding diagrams (1.2) and (1.3), there is $b \in A$ such that $ab = ba$, $bab = b$ and $a(1-ab)$ is quasinilpotent. Since $a(1-ab) \in \mathrm{Soc}(A)$, and hence algebraic, it follows easily that $a(1-ab)$ is actually nilpotent. Thus $a \in \mathcal{D}(A)$.

$\Rightarrow$ Let J be a left multiplicative ideal such that $J \subseteq \mathcal{D}(A)$ and let $a \in J$ be arbitrary. If b is the Drazin inverse of a then $ab = ba$, $bab = b$ and $a^k = a^k ba$ for some $k \in \mathbb{N}$. Let $ab = p \in A^{\bullet}$ so that pAp is a semisimple Banach algebra with $pAp \subseteq \mathcal{D}(A)$. For x arbitrary, pxp has Drazin inverse, say c. It is easy to show that pcp is a Drazin inverse for pxp in pAp. Moreover, if the Drazin index of pxp is equal to n then it follows that $(pxp)^n(pcp)^n(pxp)^n = (pxp)^n$. But this says that pAp is π regular and hence finite dimensional [16, p. 4]. For $z \in A$ arbitrary, we have that $\#\sigma(pzp, pAp) < \infty$ and hence, using Lemma 6, $\#\sigma(zp, A) < \infty$. In particular if $z = a^k$ then $\#\sigma(a^k, A) = \#\sigma(a^k ba, A) < \infty$. By the Spectral Mapping Theorem $\#\sigma(a, A) < \infty$. Since $a \in J$ was arbitrary it follows from the fact that J is a left multiplicative ideal, together with Theorem 5, that $a \in \mathrm{Soc}(A)$. For J right multiplicative the proof follows similarly. ■

If A is the Banach algebra $B(X)$ of bounded linear operators on a Banach space X, then by [6, corollary 5.2.2] and Theorems 7 and 9 it follows that $\mathrm{Soc}(A)$, which is exactly the ideal of finite rank operators, is the only ideal contained in $\hat{A}$ and $\mathcal{D}(A)$. With respect to topological considerations we have the following:

Theorem 10. *Let A be a Banach algebra and let J be an ideal of A such that $J \subseteq \hat{A}$. Then $\bar{J} \subseteq \hat{A}$ if and only if $J = \bar{J}$, that is, if and only if J is closed.*

Proof: Let J be an ideal of A. We claim that $\bar{J} \cap A^{\bullet} \subseteq J$: let p be an idempotent belonging to $\bar{J}$. Now pJp is a dense ideal of the Banach algebra $p\bar{J}p$, and since pJp is contained in a maximal ideal, which must be closed, it follows that $pJp = p\bar{J}p$. So we have that $p \in pJp \subseteq J$, which proves that J and $\bar{J}$ must have the same set of idempotents. If $x \in \bar{J} \subseteq \hat{A}$ then, by Theorem 7, $x \in A^{-1}A^{\bullet}$, and hence $\bar{J} \cap A^{\bullet} \subseteq J$ implies that $x \in J$ so that J is closed. The converse holds trivially. ■

Theorem 11. (Characterisation of ideals of group invertible elements) *Let A be a semisimple Banach algebra and let J be an ideal of A. Then*

$$J \subseteq G(A) \Leftrightarrow J \subseteq \mathrm{Soc}(A) \cap \mathrm{Centre}(A).$$

Proof: If $a \in J \subseteq \mathrm{Soc}(A) \cap \mathrm{Centre}(A)$ then it follows easily (see the proof of Proposition 1) that $a \in \mathcal{G}(A)$. For the converse, if b is the group inverse of $a \in J$ then $p = ab \Rightarrow pAp \subseteq J \subseteq \mathcal{G}(A)$. It follows from Theorem 9 that $p \in \mathrm{Soc}(A)$, and hence from [13, theorem 3.4] that pAp is finite dimensional. Since pAp is semisimple, the Wedderburn–Artin Theorem says that pAp is isomorphic (as an algebra) to a direct sum of matrix algebras. It is further easy to see that $pAp \subseteq \mathcal{G}(A)$ implies that $pAp = \mathcal{G}(pAp)$. Since for every $n \geq 2$ the matrix algebra $M_n(\mathbb{C})$ has elements that are not group invertible, it follows that pAp must in fact be isomorphic to $\mathbb{C}^n$ as an algebra. Hence we may write

$$a = \sum_{i=1}^{n} \alpha_i q_i,$$

where $\alpha_i \in \mathbb{C}$ and the q_i are orthogonal rank 1 idempotents in pAp. Observing that for each i, $pq_i = q_i p = q_i$ it follows that q_i is also rank 1 in A. Write $q_i = q$. For $v \in \mathrm{Exp}(A)$ we may write

$$q - vqv^{-1} = \alpha_1 r_1 + \alpha_2 r_2, \tag{2.1}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ and r_1, r_2 are orthogonal rank 1 idempotents in A [14, theorem 5, lemma 2; 3, p. 117]. Let us assume for the moment that $\alpha_1, \alpha_2 \neq 0$ and $r_1, r_2 \neq 0$, $r_1 \neq r_2$. Since $r_1 q r_2$ is group invertible in $A, r_1 q r_2 = 0$; and since q is rank 1 it follows from [5, corollary 2.4] that $r_1 q = 0$ or $q r_2 = 0$. If $r_1 q = 0$ then it forces $\alpha_1 = -1$, and if $q r_2 = 0$ then $\alpha_2 = -1$. So, without loss of generality, assume $r_1 q = 0$. Using a similar argument we obtain $r_1 vqv^{-1} = 0$ or $vqv^{-1} r_2 = 0$, but the first case is impossible since it would imply $\alpha_1 r_1 = 0$. So we have that $vqv^{-1} r_2 = 0$ forces $\alpha_2 = 1$. Thus

$$\begin{gathered} q - vqv^{-1} = r_2 - r_1 \\ \text{or} \\ q - vqv^{-1} = r_1 - r_2. \end{gathered} \tag{2.2}$$

Now if $v \in \mathrm{Exp}(A)$ satisfies $\|q - vqv^{-1}\| < 1$, then the form (2.2) cannot hold, because the distance between commuting idempotents is at least 1. So we have that for $v \in \mathrm{Exp}(A)$ satisfying $0 < \|q - vqv^{-1}\| < 1$ there is $0 \neq \alpha \in \mathbb{C}$ and $r \in A$ a rank 1 idempotent such that

$$q - vqv^{-1} = \alpha r. \tag{2.3}$$

Writing $q - \alpha r = vqv^{-1}$ and squaring both sides, we obtain

$$-\alpha qr - \alpha rq + \alpha^2 r = -\alpha r. \tag{2.4}$$

Multiplication of (2.4) by $1 - r$, first on the left and then on the right, yield $qr = rq = rqr$. Multiplying (2.3) by q, first on the left and then on the right, it follows that q and vqv^{-1} commute. But this contradicts $\|q - vqv^{-1}\| < 1$. So we have that for all $v \in \operatorname{Exp}(A)$ with $\|q - vqv^{-1}\| < 1$ that $q = vqv^{-1}$. Writing this as $w = e^{x_1} \dots e^{x_k}$ we have that

$$f : \lambda \mapsto (e^{\lambda x_1} \dots e^{\lambda x_k} q e^{-\lambda x_k} \dots e^{-\lambda x_1}) - q$$

is an entire function from $\mathbb{C}$ into A, with the property that there is $\epsilon > 0$ such that $\lambda \in B(0, \epsilon)$ implies $f(\lambda) = 0$. By the Scarcity Theorem [2, theorem 3.4.25] $\#\sigma(f(\lambda)) = 1$ holds for all $\lambda \in \mathbb{C}$. This argument shows that for $w \in \operatorname{Exp}(A)$ we have $\sigma(q - wqw^{-1}) = \{0\}$. Since each $q - wqw^{-1}$ has the form (2.1) it follows that $q = wqw^{-1}$ holds for all $w \in \operatorname{Exp}(A)$. Now, for $x \in A$ arbitrary choose $\lambda \in \mathbb{C}$ with $|\lambda| > \|x\|$. By the Holomorphic Functional Calculus $\lambda - x \in \operatorname{Exp}(A)$, and hence q commutes with x, that is, q belongs to the centre of A. Since $a = \sum_{i=1}^{n} \alpha_i q_i$ was arbitrary in J the result follows at once. ■

Corollary 12. (Generalised Gelfand Mazur Theorem) *If A is a Banach algebra, then every element of A is group invertible if and only if A is isomorphic to $\mathbb{C}^n$, $n \in \mathbb{N}$.*

PROOF: Following the proof of Theorem 11 it suffices to show that if $A = \mathcal{G}(A)$ then A is semisimple: If $a \in \operatorname{Rad}(A)$ and $a \in \mathcal{G}(A)$ with group inverse B, then $ab \in A^{\bullet} \cap \operatorname{Rad}(A)$. But this only possible if $ab = 0$. Thus $aba = a = 0$ and hence A is semisimple.

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