IRREDUCIBLE REPRESENTATIONS OF HILBERT C^*-MODULES

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Abstract

The subject of the paper is representations of Hilbert C^*-modules. We introduce the concept of a nondegenerate, irreducible and cyclic representation and the commutant of a Hilbert C^*-module. Relations between representations of a Hilbert C^*-module, its linking algebra and underlying C^*-algebras are discussed. We also study Hilbert CCR- and GCR-modules and extend some results from CCR- and GCR-algebras. The results we obtain are similar to the results known for C^*-algebras.

1. Introduction

It is well known that every Hilbert C^*-module can be isometrically embedded into a Banach space of all bounded operators $\mathbf{B}(H_1, H_2)$ between some Hilbert spaces H_1, H_2 (as in [6; 8; 9]). This allows us to extend the notion of a representation from C^*-algebras to Hilbert C^*-modules. Following traditional ways of thinking in the C^*-algebra theory, we ask ourselves what nondegenerate, irreducible and cyclic representations of a Hilbert C^*-module are and how they are related not only to the associated representations of the underlying C^*-algebras, but also to the associated representation of its linking algebra (Lemma 3.4, Lemma 3.5, Corollary 3.7, Corollary 3.8, Note 3.14). We introduce the commutant of a Hilbert C^*-module, which extends the definition from C^*-algebras (Note 4.8), in order to see whether it characterises the irreducibility of a representation (Proposition 4.5), as is the case in the C^*-algebra theory. Since CCR- and GCR-algebras are defined by irreducible representations, it is natural to define Hilbert CCR- and GCR-modules in a similar way. We prove that a full Hilbert C^*-module is a Hilbert CCR-module (resp. GCR) precisely when its underlying C^*-algebras are CCR-algebras (resp. GCR) (Corollary 5.4, Corollary 6.4). These (expected) results are a consequence of the very strong relationship between irreducibility of the associated representations (Corollary 3.8). Then it is not difficult to extend some well-known results about CCR- and GCR-algebras.

2. Preliminaries

A Hilbert C^*-module V over a C^*-algebra A (or *a Hilbert A-module*) is by definition (see [4]) a linear space that is a right A-module, together with an A-valued inner product $\langle \cdot, \cdot \rangle$ on $V \times V$ that is A-linear in the second and conjugate linear in the first variable, such that V is a Banach space with the norm $\|v\| = \|\langle v, v\rangle\|^{1/2}$. We denote

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by $\langle V, V\rangle \subseteq A$ the ideal of A spanned by all inner products $\langle x, y\rangle$, $x, y \in V$. V is said to be *a full Hilbert A-module* if $\langle V, V\rangle = A$ (by an ideal, we always understand a closed, two-sided ideal). We denote the C^*-algebras of all adjointable and 'compact' operators on a Hilbert C^*-module V by $\mathbf{B}(V)$ and $\mathbf{K}(V)$, respectively. We also use $\mathbf{B}(V, W)$ and $\mathbf{K}(V, W)$ to denote spaces of all adjointable and 'compact' operators acting between different Hilbert C^*-modules V and W over A.

For Hilbert spaces H_1 and H_2, $\mathbf{B}(H_1, H_2)$ and $\mathbf{K}(H_1, H_2)$ denote the right Hilbert $\mathbf{B}(H_1)$-module of all bounded operators and its ideal submodule of all 'compact' operators, respectively. Furthermore, we write $\mathbf{B}(H) = \mathbf{B}(H, H)$ and $\mathbf{K}(H) = \mathbf{K}(H, H)$.

Given a Hilbert A-module V, *the linking algebra* $\mathcal{L}(V)$ is defined as the matrix algebra of the form

$$\mathcal{L}(V) = \left[\begin{array}{cc} \mathbf{K}(A) & \mathbf{K}(V, A) \\ \mathbf{K}(A, V) & \mathbf{K}(V) \end{array}\right].$$

Observe that $\mathcal{L}(V)$ is in fact the C^*-algebra of all 'compact' operators acting on $A \oplus V$. Each $v \in V$ induces the maps $r_v \in \mathbf{B}(A, V)$ and $l_v \in \mathbf{B}(V, A)$ given by $r_v(a) = va$ and $l_v(w) = \langle v, w\rangle$ such that $l_v^* = r_v$. The map $v \mapsto l_v$ is an isometric conjugate linear isomorphism of V to $\mathbf{K}(V, A)$ and $v \mapsto r_v$ is an isometric linear isomorphism of V to $\mathbf{K}(A, V)$. Furthermore, every $a \in A$ induces the map $T_a \in \mathbf{K}(A)$ given by $T_a(b) = ab$ and the map $a \mapsto T_a$ defines an isomorphism of C^*-algebras A and $\mathbf{K}(A)$. Therefore, we may write

$$\mathcal{L}(V) = \{\left[\begin{array}{cc} T_a & l_y \\ r_x & T \end{array}\right] : a \in A, x, y \in V, T \in \mathbf{K}(V)\}$$

and we can identify the C^*-algebras of 'compact' operators with the corresponding corners in the linking algebra: $\mathbf{K}(A) = \mathbf{K}(A \oplus 0) \subseteq \mathbf{K}(A \oplus V) = \mathcal{L}(V)$ and $\mathbf{K}(V) = \mathbf{K}(0 \oplus V) \subseteq \mathbf{K}(A \oplus V) = \mathcal{L}(V)$ (for details see [7, lemma 2.32, corollary 3.21]). If $V \subseteq \mathbf{B}(H_1, H_2)$, then we will write

$$\mathcal{L}(V) = \left[\begin{array}{cc} A & V^* \\ V & \mathbf{K}(V) \end{array}\right] = \{\left[\begin{array}{cc} a & y^* \\ x & T \end{array}\right] : a \in A, x, y \in V, T \in \mathbf{K}(V)\},$$

(by using the mentioned isomorphisms $a \mapsto T_a, v \mapsto l_v, v \mapsto r_v$).

Let V and W be Hilbert C^*-modules over C^*-algebras A and B, respectively, and $\varphi : A \to B$ a morphism of C^*-algebras. A map $\Phi : V \to W$ is said to be *a φ-morphism* of Hilbert C^*-modules if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ is satisfied for all $x, y \in V$. A φ-morphism $\Phi : V \to \mathbf{B}(H_1, H_2)$, where $\varphi : A \to \mathbf{B}(H_1)$ is a representation of A is called *a representation* of V. We will say that a representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ is *a faithful representation* of V if Φ is injective. Throughout the paper, when we say that Φ is a representation of V, we will assume that an associated representation of A is denoted by the same small case letter φ, so we will not explicitly mention φ.

For every Hilbert C^*-module there is a representation to $\mathbf{B}(H_1, H_2)$ for some Hilbert spaces H_1, H_2 (see [6; 8; 9]). It is easy to check that each φ-morphism is necessarily a linear operator and a module map in the sense $\Phi(va) = \Phi(v)\varphi(a)$,

$a \in A$, $v \in V$. There is also a morphism of C^*-algebras $\Phi^+ : \mathbf{K}(V) \to \mathbf{K}(W)$ such that $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$. Furthermore, every φ-morphism $\Phi : V \to W$ induces a morphism of the linking algebras $\rho_{\varphi,\Phi} : \mathcal{L}(V) \to \mathcal{L}(W)$ given by

$$\rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(a)} & l_{\Phi(y)} \\ r_{\Phi(x)} & \Phi^+(T) \end{bmatrix}.$$

Conversely, if $\rho : \mathcal{L}(V) \to \mathcal{L}(W)$ is a morphism of linking algebras such that $\rho(\mathbf{K}(A)) \subseteq \mathbf{K}(B)$ and $\rho(\mathbf{K}(V)) \subseteq \mathbf{K}(W)$, then there is a morphism of C^*-algebras $\varphi : A \to B$ and a φ-morphism $\Phi : V \to W$ such that $\rho = \rho_{\varphi,\Phi}$ [2, theorem 2.15]. In particular, if $\Phi : V \to \mathbf{B}(H_1, H_2)$ is a representation of V, then

$$\rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right) = \begin{bmatrix} \varphi(a) & \Phi(y)^* \\ \Phi(x) & \Phi^+(T) \end{bmatrix}.$$

3. Irreducible representations of a Hilbert C^*-module

In this section, we construct a representation of a Hilbert C^*-module from a representation of its linking algebra and prove that this correspondence is bijective. Then we define nondegenerate and irreducible representations of a Hilbert C^*-module and relate them to nondegenerate and irreducible representations of the linking algebra. This enables the construction of the faithful representation of a Hilbert C^*-module that is the direct sum of irreducible representations. In Note 3.13 we state, without proving, that the irreducibility of a representation is preserved when we restrict to or extend from an ideal submodule; also, the induced representation of a quotient module stays irreducible. We conclude the first section with Note 3.14, where we define a cyclic representation of a Hilbert C^*-module and give some statements about it, without proofs.

Given a representation of a Hilbert C^*-module, it is easy to construct a representation of its linking algebra. In order to fix some notation we give the following proof:

Let V be a Hilbert A-module, $\mathcal{L}(V)$ its linking algebra and $\Phi : V \to \mathbf{B}(H_1, H_2)$ a representation of V. Denote by $p_1, p_2 \in \mathbf{B}(H_1 \oplus H_2)$ the orthogonal projections on H_1 and H_2, respectively, and

$$\sigma : \mathbf{B}(H_1 \oplus H_2) \to \begin{bmatrix} \mathbf{B}(H_1) & \mathbf{B}(H_2, H_1) \\ \mathbf{B}(H_1, H_2) & \mathbf{B}(H_2) \end{bmatrix}, \sigma(x) = \begin{bmatrix} p_1 x p_1 & p_1 x p_2 \\ p_2 x p_1 & p_2 x p_2 \end{bmatrix}$$

the $*$-isomorphism. Then we get a representation of $\mathcal{L}(V)$ defined by $\pi_{\varphi,\Phi} = \sigma^{-1} \circ \rho_{\varphi,\Phi} : \mathcal{L}(V) \to \mathbf{B}(H_1 \oplus H_2)$, that is,

$$\pi_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right)(\xi_1 \oplus \xi_2) = (\varphi(a)\xi_1 + \Phi(y)^*\xi_2) \oplus (\Phi(x)\xi_1 + \Phi^+(T)\xi_2),$$

for $a \in A, x, y \in V, T \in \mathbf{K}(V), \xi_1 \in H_1, \xi_2 \in H_2$.

The converse is also true: every representation of the linking algebra has this form. We prove that in the following proposition.

Proposition 3.1. *The map* $\Phi \mapsto \pi_{\varphi,\Phi}$ *is a bijection from the set of all representations of a Hilbert* A*-module* V *onto the set of all representations of its linking algebra* $\mathcal{L}(V)$*. Further, if* V *is full, then* Φ *is faithful if and only if* $\pi_{\varphi,\Phi}$ *is faithful.*

Proof. Let $\pi : \mathcal{L}(V) \to \mathbf{B}(H)$ be a representation of $\mathcal{L}(V)$ in a Hilbert space H. We will prove that there is a decomposition $H = H_1 \oplus H_2$ and a representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ such that $\pi = \pi_{\varphi,\Phi}$.

We define $H_1 = \overline{\pi(\mathbf{K}(A))H}$ and $H_2 = H_1^{\perp}$ (if π is nondegenerate then $H_2 = \overline{\pi(\mathbf{K}(V))H}$). One easily shows that $\rho = \sigma \circ \pi : \mathcal{L}(V) \to \mathcal{L}(\mathbf{B}(H_1, H_2))$ is a $*$-morphism between linking algebras $\mathcal{L}(V)$ and $\mathcal{L}(\mathbf{B}(H_1, H_2))$, such that $\rho(\mathbf{K}(A)) \subseteq \mathbf{K}(\mathbf{B}(H_1))$ and $\rho(\mathbf{K}(V)) \subseteq \mathbf{K}(\mathbf{B}(H_2))$. From [2, theorem 2.15] we conclude that there is a representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ such that $\rho = \rho_{\varphi,\Phi}$, hence $\pi = \pi_{\varphi,\Phi}$. This proves that the map $\Phi \mapsto \pi_{\varphi,\Phi}$ is surjective, and it is obviously injective. ■

Now we define nondegenerate and irreducible representations of Hilbert C^*-modules. The first definition is [8, definition A.1].

Definition 3.2. Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V. Φ is said to be *nondegenerate* if $\overline{\Phi(V)H_1} = H_2$ and $\overline{\Phi(V)^*H_2} = H_1$, (or equivalently, if $\xi_1 \in H_1, \xi_2 \in H_2$ are such that $\Phi(V)\xi_1 = 0$ and $\Phi(V)^*\xi_2 = 0$, then $\xi_1 = 0$ and $\xi_2 = 0$).

Definition 3.3. Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V and $K_1 < H_1$ and $K_2 < H_2$ closed subspaces. A pair of subspaces (K_1, K_2) is said to be Φ-*invariant* if $\Phi(V)K_1 \subseteq K_2$ and $\Phi(V)^*K_2 \subseteq K_1$. Φ is said to be *irreducible* if $(0, 0)$ and (H_1, H_2) are the only Φ-invariant pairs.

In the following two lemmas, we describe relations between nondegeneracy and irreducibility of the associated representations Φ, φ, Φ^+ and $\pi_{\varphi,\Phi}$.

Lemma 3.4. *Let* V *be a Hilbert* A*-module, let* $\Phi : V \to \mathbf{B}(H_1, H_2)$ *be a representation of* V *and let* $\pi = \pi_{\varphi,\Phi}$*. Then the following statements hold:*

(a) π *is nondegenerate if and only if* φ *and* Φ^+ *are nondegenerate.*

(b) *If* Φ *is nondegenerate, then* φ *and* Φ^+ *are nondegenerate. If* V *is full and* φ *and* Φ^+ *are nondegenerate, then* Φ *is nondegenerate.*

(c) *If* Φ *is nondegenerate, then* π *is nondegenerate. If* V *is full and* π *is nondegenerate, then* Φ *is nondegenerate.*

(d) *If* $\Phi \neq 0$ *is irreducible, then* Φ *is nondegenerate.*

Proof. (a) Suppose π is nondegenerate. If $\xi_1 \in H_1$ is such that $\varphi(A)\xi_1 = 0$, then by the Hewitt-Cohen factorisation theorem, $\Phi(V)\xi_1 = \Phi(VA)\xi_1 = \Phi(V)\varphi(A)\xi_1 = 0$, therefore $\pi(\mathcal{L}(V))(\xi_1 \oplus 0) = 0$. Since π is nondegenerate, $\xi_1 = 0$. Nondegeneracy of Φ^+ is proved in a similar way.

Suppose that φ and Φ^+ are nondegenerate, i.e. $H_1 = \varphi(A)H_1$ and $H_2 = \Phi^+(\mathbf{K}(V))H_2$. For an arbitrary vector $\eta_1 \oplus \eta_2 \in H_1 \oplus H_2$ there are $a \in A$, $T \in$

$\mathbf{K}(V)$, $\xi_1 \in H_1$ and $\xi_2 \in H_2$, such that $\eta_1 = \varphi(a)\xi_1$ and $\eta_2 = \Phi^+(T)\xi_2$. Then we have

$$\eta_1 \oplus \eta_2 = \pi\left(\begin{bmatrix} a & 0 \\ 0 & T \end{bmatrix}\right)(\xi_1 \oplus \xi_2) \in \pi(\mathcal{L}(V))(H_1 \oplus H_2),$$

hence, $\pi(\mathcal{L}(V))(H_1 \oplus H_2) = H_1 \oplus H_2$, so π is nondegenerate.

(b) If $\varphi(A)\xi_1 = 0$ then, as in (a), $\Phi(V)\xi_1 = 0$, and if $A = \langle V, V\rangle$ then the converse is also true: $\Phi(V)\xi_1 = 0 \Rightarrow \varphi(\langle x, y\rangle\xi_1 = \Phi(x)^*\Phi(y)\xi_1 = 0$, for all $x, y \in V$, hence $\varphi(A)\xi_1 = 0$. Since V, considered as a left Hilbert $\mathbf{K}(V)$-module, is full we have: $\Phi^+(\mathbf{K}(V))\xi_2 = 0$ if and only if $\Phi(V)^*\xi_2 = 0$. This proves (b).

(c) This follows from (a) and (b).

(d) Suppose $\Phi \neq 0$ is an irreducible representation and $\xi_1 \in H_1, \xi_2 \in H_2$ are such that $\Phi(V)\xi_1 = 0$ and $\Phi(V)^*\xi_2 = 0$. Then $(< \xi_1 >, 0)$ and $(0, < \xi_2 >)$ are two pairs of Φ-invariant subspaces. Since $\Phi \neq 0$, we have $H_i \neq 0, i = 1, 2$, therefore $(< \xi_1 >, 0) = (0, < \xi_2 >) = (0, 0)$ and $\xi_1 = 0, \xi_2 = 0$. ■

Lemma 3.5. *Let V be a Hilbert A-module, let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of V and let $\pi = \pi_{\varphi,\Phi}$. Then the following statements hold:*

(a) *If $\Phi \neq 0$ is irreducible, then $\varphi \neq 0, \Phi^+ \neq 0, \pi \neq 0$ are irreducible.*

(b) *If $\pi \neq 0$ is irreducible, then $\Phi = 0$ or Φ is irreducible. If V is full, then $\pi \neq 0$ is irreducible if and only if $\Phi \neq 0$ is irreducible.*

(c) *Φ is irreducible if and only if φ and Φ^+ are irreducible.*

PROOF. (a) If $x \in V$ such that $\Phi(x) \neq 0$, then $\varphi(\langle x, x\rangle) \neq 0$, $\Phi^+(\theta_{x,x}) \neq 0$ and $\pi\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) \neq 0$, hence $\Phi \neq 0 \Rightarrow \varphi \neq 0, \Phi^+ \neq 0, \pi \neq 0$.

For a φ-invariant (resp. Φ^+-invariant) subspace $K_1 < H_1$ (resp. $K_2 < H_2$) we define $K_2 = \overline{\Phi(V)K_1}$ (resp. $K_1 = \overline{\Phi(V)^*K_2}$). It is easy to verify that (K_1, K_2) is a Φ-invariant pair of subspaces. Since Φ is irreducible, we conclude that $(K_1, K_2) = (0, 0)$ or $(K_1, K_2) = (H_1, H_2)$, hence $K_1 = 0$ or $K_1 = H_1$ (resp. $K_2 = 0$ or $K_2 = H_2$). This proves that φ (resp. Φ^+) is irreducible.

Let $K_1 \oplus K_2 < H_1 \oplus H_2$ be π-invariant. This means that $\varphi(a)\xi_1 + \Phi(x)^*\xi_2 \subseteq K_1, a \in A, x \in V$ and $\Phi(y)\xi_1 + \Phi^+(T)\xi_2 \subseteq K_2, y \in V, T \in \mathbf{K}(V)$, for all $\xi_1 \in K_1$ and $\xi_2 \in K_2$. In particular, $\Phi(V)^*K_2 \subseteq K_1$ and $\Phi(V)K_1 \subseteq K_2$, i.e. (K_1, K_2) is a Φ-invariant pair of subspaces. Since Φ is irreducible, we have $(K_1, K_2) = (0, 0)$ or $(K_1, K_2) = (H_1, H_2)$, and $K_1 \oplus K_2 = 0$ or $K_1 \oplus K_2 = H_1 \oplus H_2$.

(b) Suppose $\Phi \neq 0$. We will first prove that Φ is nondegenerate. π is an irreducible representation of the C^*-algebra $\mathcal{L}(V)$, hence it is nondegenerate, and then by Lemma 3.4(a), φ and Φ^+ are nondegenerate. From $\Phi(V)^*\xi_2 = 0$ for some $\xi_2 \in H_2$, we get $\Phi^+(\mathbf{K}(V))\xi_2 = 0$; and since Φ^+ is nondegenerate, it follows $\xi_2 = 0$. Furthermore, if $\Phi(V)\xi_1 = 0$ for some $\xi_1 \in H_1$, then for all $a, b \in A$, $x, y \in V$, $T \in \mathbf{K}(V)$ we get

$$\pi\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right)(\varphi(b)\xi_1 \oplus 0) = \varphi(ab)\xi_1 \oplus \Phi(xb)\xi_1 \in \varphi(A)\xi_1 \oplus 0,$$

hence $\overline{\varphi(A)\xi_1} \oplus 0 < H_1 \oplus H_2$ is π-invariant. Since π is irreducible, $\overline{\varphi(A)\xi_1} \oplus 0 = 0 \oplus 0$ or $\overline{\varphi(A)\xi_1} \oplus 0 = H_1 \oplus H_2$. From $\Phi \neq 0$ we have $\Phi(V)H_1 \subseteq H_2 \Rightarrow H_2 \neq 0$, so it has to be $\varphi(A)\xi_1 = 0$. φ is irreducible, particularly nondegenerate, so we conclude $\xi_1 = 0$. This proves that Φ is nondegenerate.

Finally, for an arbitrary Φ-invariant pair of subspaces (K_1, K_2) we get the π-invariant subspace $\overline{\Phi(V)^* K_2} \oplus K_2$. π is irreducible, therefore $\overline{\Phi(V)^* K_2} \oplus K_2 = 0 \oplus 0$ or $\overline{\Phi(V)^* K_2} \oplus K_2 = H_1 \oplus H_2$. If $\overline{\Phi(V)^* K_2} \oplus K_2 = 0 \oplus 0$ then $K_2 = 0$ and $\Phi(V)K_1 \subseteq K_2 = 0$. We proved that Φ is nondegenerate, hence $K_1 = 0$ and $(K_1, K_2) = (0, 0)$. If $\overline{\Phi(V)^* K_2} \oplus K_2 = H_1 \oplus H_2$ then $K_2 = H_2$ and since Φ is nondegenerate we have $K_1 \subseteq H_1 = \overline{\Phi(V)^* H_2} = \overline{\Phi(V)^* K_2} \subseteq K_1$, so $(K_1, K_2) = (H_1, H_2)$.

If V is full, then from $\Phi = 0$ we get $\varphi(\langle x, y\rangle) = \Phi(x)^*\Phi(y) = 0$ and $\Phi^+(\theta_{x,y}) = \Phi(x)\Phi(y)^* = 0$ for all $x, y \in V$, hence $\varphi = 0$ and $\Phi^+ = 0$, and then $\pi = 0$. Our statement now follows from (a) and the first statement in (b).

(c) Suppose φ and Φ^+ are irreducible. Let $(K_1, K_2) \neq (0, 0)$ be a Φ-invariant pair of subspaces. Then K_2 is a Φ^+-invariant, therefore $K_2 = H_2$. Further, $\varphi|_{\langle V,V\rangle} \neq 0$ is irreducible (since $\Phi \neq 0$ and $\langle V, V\rangle$ is the ideal in A) and $\varphi(\langle V, V\rangle)K_1 \subseteq K_1$, hence $K_1 = H_1$. The converse is proved in (a). ■

If $\Phi : V \to \mathbf{B}(H_1, H_2)$ is a representation of V such that $H_2 = \overline{\Phi(V)H_1}$, then we can improve the statement (c) in Lema 3.5 and we get the following proposition.

Proposition 3.6. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a nonzero representation of a Hilbert A-module V, such that $H_2 = \overline{\Phi(V)H_1}$. Then Φ is irreducible if and only if φ is irreducible.*

Proof. Suppose $\varphi : A \to \mathbf{B}(H_1)$ is an irreducible representation and (K_1, K_2) is a Φ-invariant pair of subspaces. Then $\varphi(\langle V, V\rangle)K_1 \subseteq K_1$ and $\varphi|_{\langle V,V\rangle} \neq 0$ is irreducible, hence $K_1 = 0$ or $K_1 = H_1$. If $K_1 = 0$ then $\Phi(V)^* K_2 \subseteq K_1 = 0$, and for every $\xi_2 \in K_2$ we have $0 = (\Phi(v)^*\xi_2, \xi_1) = (\xi_2, \Phi(v)\xi_1)$, $v \in V, \xi_1 \in H_1$; that is, $K_2 \perp \overline{\Phi(V)H_1} = H_2$ and $K_2 \subseteq H_2$, hence $K_2 = 0$. If $K_1 = H_1$ then $H_2 = \overline{\Phi(V)H_1} = \overline{\Phi(V)K_1} \subseteq K_2$, so $K_2 = H_2$. Therefore, $(K_1, K_2) = (0, 0)$ or $(K_1, K_2) = (H_1, H_2)$, so Φ is irreducible.

The converse is proved in Lemma 3.5(a). ■

Corollary 3.7. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a nonzero representation of a Hilbert A-module V, such that $H_1 = \overline{\Phi(V)^* H_2}$. Then Φ is irreducible if and only if Φ^+ is irreducible.*

Corollary 3.8. *Let V be a full Hilbert A-module and let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a nondegenerate representation of V. If any of the representations Φ, φ, Φ^+, $\pi_{\varphi,\Phi}$ is irreducible, then all of them are irreducible.*

Note 3.9. A representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ with $H_2 = \overline{\Phi(V)H_1}$ can be constructed from a representation $\varphi : A \to \mathbf{B}(H_1)$ (as in [6], [8] or [9]). Although our construction of Φ (described in Proposition 3.1) starts from a representation of its linking algebra π, we can also start from a representation of C^*-algebra A: since

A is the C^*-subalgebra of $\mathcal{L}(V)$, a representation $\varphi : A \to \mathbf{B}(H_1)$ can be extended to a representation $\pi : \mathcal{L}(V) \to \mathbf{B}(H)$ (e.g., see [3, Proposition 2.10.2]). Then, by Proposition 3.1, there is the decomposition $H = H_1' \oplus H_2'$ and ψ-morphism Ψ such that $\pi = \pi_{\psi,\Psi}$, where $\psi : A \to \mathbf{B}(H_1')$ and $\Psi : V \to \mathbf{B}(H_1', H_2')$. Then one can easily check that $\psi(a)|_{H_1'} = \varphi(a), a \in A$. We define $H_2 = \overline{\Psi(V)H_1}$ and then $\Phi : V \to \mathbf{B}(H_1, H_2)$ by $\Phi(v) = \Psi(v)|_{H_1}$.

Now we will construct a faithful representation of a Hilbert C^*-module that is a direct sum of irreducible representations. Such a result is well known in the C^*-algebra theory.

Definition 3.10. If $(\Phi_i)_{i\in I}$ is a family of representations $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$ of a Hilbert A-module V, then their *direct sum* is the representation $\Phi : V \to \mathbf{B}(H_1, H_2)$, where $H_1 = \bigoplus_i H_1^i$ and $H_2 = \bigoplus_i H_2^i$ are Hilbert sums, and $\Phi(v)((\xi_1^i)_i) = (\Phi_i(v)\xi_1^i)_i$ for all $v \in V$ and $(\xi_1^i)_i \in H_1$.

It is easy to verify that Φ is indeed a representation of V and that an associated φ is a direct sum of representation $(\varphi_i)_{i\in I}$.

Theorem 3.11. *Let V be a Hilbert A-module. There is a faithful representation of V that is the direct sum of irreducible representations of V.*

PROOF. Let $\{\pi_i : i \in I\}$ be a family of all nonzero irreducible representations of $\mathcal{L}(V)$, $\pi_i : \mathcal{L}(V) \to \mathbf{B}(H^i)$. Their direct sum $\pi := \bigoplus_{i\in I} \pi_i$ is a faithful representation of $\mathcal{L}(V)$ in the Hilbert space $H := \bigoplus_{i\in I} H^i$. There are $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$ representations of V such that $\pi_i = \pi_{\varphi_i,\Phi_i}$, $i \in I$. By Lemma 3.5(b), $\Phi_i = 0$ or Φ_i is irreducible. Let $I_1 \subseteq I$ be such that $\Phi_i \neq 0$ for $i \in I_1$ and let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be the direct sum of representations $(\Phi_i)_{i\in I_1}$. Since π is faithful, $I_1 \neq \emptyset$.

If $\Phi(v) = 0$ for some $v \in V$, then $\Phi_i(v) = 0, i \in I_1$. Since $\Phi_i = 0, i \in I\backslash I_1$, we have $\Phi_i(v) = 0, i \in I$, hence $\pi\left(\begin{bmatrix} 0 & 0 \\ r_v & 0 \end{bmatrix}\right) = 0$. Since π is injective, we get $v = 0$ and that proves the injectivity of Φ. ■

An immediate consequence of this theorem is the following corollary:

Corollary 3.12. *Let V be a Hilbert A-module. For every $v \in V$ we have*

$$\begin{aligned}\|v\| &= \sup\{\|\Phi(v)\| : \Phi \text{ is a representation of } V\}\\ &= \sup\{\|\Phi(v)\| : \Phi \text{ is an irreducible representation of } V\}\\ &= \sup\{\sqrt{\|\varphi(\langle v, v\rangle)\|} : \varphi \text{ is a representation of } A\}\\ &= \sup\{\sqrt{\|\varphi(\langle v, v\rangle)\|} : \varphi \text{ is an irreducible representation of } A\}.\end{aligned}$$

Let us conclude this section with a few additional comments. The proofs are omitted.

Note 3.13. Let $V_I = VI$ be the ideal submodule of a Hilbert A-module V associated to an ideal I of A (for the definition of an ideal submodule see [2]). If $\Phi : V \to \mathbf{B}(H_1, H_2)$ is an irreducible representation such that $\Phi(V_I) \neq 0$, then $\Phi|_{V_I}$ is an irreducible representation of V_I. Also, an irreducible representation of V_I extends uniquely to the irreducible representation of V. Further, if $\Phi : V \to \mathbf{B}(H_1, H_2)$ is an irreducible representation of V such that $V_I \subseteq \operatorname{Ker}\Phi$, then

$$\bar{\Phi} : V/V_I \to \mathbf{B}(H_1, H_2), \quad \bar{\Phi}(v + V_I) = \Phi(v)$$

is an irreducible representation of the quotient module V/V_I.

Note 3.14. We can define *a cyclic representation* of a Hilbert A-module V to be a representation $\Phi : V \to \mathbf{B}(H_1, H_2)$, such that there is a pair of vectors $(\xi_1, \xi_2) \neq (0, 0)$, $\xi_1 \in H_1, \xi_2 \in H_2$, satisfying $\overline{\Phi(V)\xi_1} = H_2$ and $\overline{\Phi(V)^*\xi_2} = H_1$. Such pair of vectors is called *a cyclic pair* for Φ.

If Φ is cyclic, then φ, Φ^+ and $\pi_{\varphi,\Phi}$ are cyclic. Every irreducible representation of a Hilbert C^*-module is cyclic and every nonzero pair of vectors is cyclic. Furthermore, every nondegenerate representation of a Hilbert C^*-module is the direct sum of cyclic representations.

4. The commutant of a Hilbert C^*-module

In this section, we extend the definition of the commutant of a C^*-algebra to a Hilbert C^*-module and prove that a representation of a Hilbert C^*-module is irreducible if and only if the only elements in its commutant are the scalar operators.

Definition 4.1. Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V. We define *the commutant of* $\Phi(V)$ as a set

$$\begin{aligned}\Phi(V)' = \{T_1 \oplus T_2 \in \mathbf{B}(H_1 \oplus H_2) \,:\, & T_1 \in \mathbf{B}(H_1), T_2 \in \mathbf{B}(H_2),\\ & T_2\Phi(v) = \Phi(v)T_1, T_1\Phi(v)^* = \Phi(v)^*T_2, v \in V\},\end{aligned}$$

where $(T_1 \oplus T_2)(\xi_1 \oplus \xi_2) = T_1\xi_1 \oplus T_2\xi_2$ for $\xi_1 \oplus \xi_2 \in H_1 \oplus H_2$.

This definition is motivated by the following lemma:

Lemma 4.2. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V. Let p_1 and p_2 be the orthogonal projections on closed subspaces $K_1 < H_1$ and $K_2 < H_2$, respectively. Then (K_1, K_2) is a Φ-invariant pair of subspaces if and only if $p_1 \oplus p_2 \in \Phi(V)'$.*

Proof. If (K_1, K_2) is Φ-invariant then $\Phi(V)K_1 \subseteq K_2$, hence for $\xi_1 \in K_1$ we get $\Phi(v)p_1\xi_1 = \Phi(v)\xi_1 = p_2\Phi(v)\xi_1$. Since $\Phi(V)^*K_2 \subseteq K_1$ is equivalent to $\Phi(V)K_1^\perp \subseteq K_2^\perp$, we get $\Phi(v)p_1\xi_1 = 0 = p_2\Phi(v)\xi_1$, for $\xi_1 \in K_1^\perp$. Finally, since $p_1^* = p_1$ and $p_2^* = p_2$, the second relation from the definition of $\Phi(V)'$ is proved by taking the adjoints in the first one.

The converse is obvious. ■

Lemma 4.3. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V. Then $\Phi(V)'$ is a C^*-algebra.*

PROOF. An elementary calculation shows that for a complex number α and $T_1 \oplus T_2, S_1 \oplus S_2 \in \Phi(V)'$ we have $\alpha T_1 \oplus \alpha T_2, (T_1 + S_1) \oplus (T_2 + S_2), T_1^* \oplus T_2^*, T_1 S_1 \oplus T_2 S_2 \in \Phi(V)'$. Further, $\Phi(V)'$ is obviously closed in $\mathbf{B}(H_1 \oplus H_2)$, therefore a C^*-algebra. ■

In the next lemma we will see how $\Phi(V)'$ is related to commutants of underlying C^*-algebras and the linking algebra.

Lemma 4.4. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a representation of a Hilbert A-module V and $\pi = \pi_{\varphi,\Phi}$.*

(a) *If $T_1 \oplus T_2 \in \Phi(V)'$, then $T_1 \in \varphi(\langle V, V\rangle)', T_2 \in \Phi^+(\mathbf{K}(V))'$.*

(b) *$\pi(\mathcal{L}(V))' \subseteq \Phi(V)'$. If V is full and Φ is nondegenerate, then $\pi(\mathcal{L}(V))' = \Phi(V)'$.*

PROOF. (a) Straightforward.
(b) A direct calculation shows that for a nondegenerate representation π we have

$$\pi(\mathcal{L}(V))' = \{T_1 \oplus T_2 \in \Phi(V)' \,:\, T_1 \in \varphi(A)', T_2 \in \Phi^+(\mathbf{K}(V))'\} \subseteq \Phi(V)'.$$

If V is full, then (a) implies $\pi(\mathcal{L}(V))' = \Phi(V)'$. ■

In the following proposition we extend the result that an irreducible representation of a C^*-algebra can be characterised through its commutant (cf., [5, theorem 5.1.5]).

Proposition 4.5. *Let $\Phi : V \to \mathbf{B}(H_1, H_2)$ be a nonzero representation and I_1, I_2 identities on H_1, H_2, respectively. Then Φ is irreducible if and only if $\Phi(V)' = \mathbf{C} \cdot (I_1 \oplus I_2)$.*

PROOF. The statement follows from Lemma 3.5, Lemma 4.2 and [5, theorem 5.1.5]. ■

Note 4.6. For a nondegenerate representation $\Phi : V \to \mathbf{B}(H_1, H_2)$, an operator $T_1 \oplus T_2 \in \Phi(V)'$ is completely determined by knowing only one of the operators T_1, T_2. Indeed, if T_1 (resp. T_2) is given, then a relation $T_2(\Phi(v)\xi_1) = \Phi(v)T_1\xi_1, \xi_1 \in H_1$ (resp. $T_1(\Phi(v)^*\xi_1) = \Phi(v)^*T_2\xi_2, \xi_2 \in H_2$) determines T_2 (resp. T_1) on a dense subset $\Phi(V)H_1$ of H_2 (resp. $\Phi(V)^*H_2$ of H_1). In particular, $T_1 = \lambda I_1$ or $T_2 = \lambda I_2$ implies $T_1 \oplus T_2 = \lambda(I_1 \oplus I_2)$, and therefore

$$\varphi(\langle V, V\rangle)' = \mathbf{C} \cdot I_1 \Leftrightarrow \Phi(V)' = \mathbf{C} \cdot (I_1 \oplus I_2) \Leftrightarrow \Phi^+(\mathbf{K}(V))' = \mathbf{C} \cdot I_2.$$

Since $\varphi|_{\langle V,V\rangle} \neq 0$ is irreducible if and only if φ is irreducible, the above statement is equivalent to

$$\varphi \text{ is irreducible } \Leftrightarrow \Phi \text{ is irreducible } \Leftrightarrow \Phi^+ \text{ is irreducible.}$$

This was already proved in Proposition 3.6 and Corollary 3.7.

Note 4.7. Let Φ be a nondegenerate representation of a Hilbert A-module V and let K_1, K_2, p_1, p_2 be as in Lemma 4.2. The preceding remark says that p_1 and p_2, that is, K_1 and K_2, are determined by each other. Of course, this can be seen directly from the definitions of a nondegenerate representation and a Φ-invariant pair of subspaces: $\Phi(V)K_1 \subseteq K_2, \Phi(V)K_1^\perp \subseteq K_2^\perp$ and $\overline{\Phi(V)H_1} = H_2$ imply $\Phi(V)K_1 = K_2$ and $\Phi(V)^*K_2 = K_1$.

Now it is easy to prove that the definition of the commutant of a Hilbert C^*-module extends the definition of the commutant of a C^*-algebra.

Note 4.8. If we consider a C^*-algebra A as a Hilbert A-module V, and if Φ is a nondegenerate representation of A, then $\Phi = \varphi = \Phi^+$ and $\Phi(V)' = \{T_1 \oplus T_2 \in B(H_1 \oplus H_1) : \varphi(a)T_1 = T_2\varphi(a), T_1\varphi(a) = \varphi(a)T_2\}$. For $T_1 \oplus T_2 \in \Phi(V)'$ we have

$$T_2\varphi(a) = \varphi(a)T_1 = (\text{since } T_1 \in \varphi(A)') = T_1\varphi(a), a \in A,$$

hence $T_2 = T_1$. Then $\Phi(V)' = \{T_1 \oplus T_1 \in B(H_1 \oplus H_1) : \varphi(a)T_1 = T_1\varphi(a)\} = \varphi(A)'$.

5. Hilbert *CCR*-modules

Recall that a C^*-algebra A is a CCR-algebra (or liminal C^*-algebra) if $\varphi(A) \subseteq \mathbf{K}(H)$ (equivalently, $\varphi(A) = \mathbf{K}(H)$), for every nonzero irreducible representation $\varphi : A \to \mathbf{B}(H)$. We define Hilbert CCR-modules in a similar way.

In order to simplify our discussion, we restrict ourselves in this section to full Hilbert C^*-modules. However, the reader can easily convince himself that similar results can be obtained for general Hilbert C^*-modules.

Definition 5.1. A Hilbert A-module V is said to be *a Hilbert CCR-module* if for every nonzero irreducible representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ we have $\Phi(V) \subseteq \mathbf{K}(H_1, H_2)$.

It is easy to see that in the case of an irreducible representation Φ of V satisfying $\Phi(V) \cap \mathbf{K}(H_1, H_2) \neq \{0\}$, we have $\mathbf{K}(H_1, H_2) \subseteq \Phi(V)$. Therefore, in Definition 5.1, we can put $\Phi(V) = \mathbf{K}(H_1, H_2)$ instead of $\Phi(V) \subseteq \mathbf{K}(H_1, H_2)$.

Proposition 5.2. *Let V be a Hilbert A-module. Then V is a Hilbert CCR-module if and only if A is a CCR-algebra.*

Proof. If A is a CCR-algebra and $\Phi : V \to \mathbf{B}(H_1, H_2)$ is an arbitrary irreducible nonzero representation of V, then by Lemma 3.5(a), φ is an irreducible representation of A, therefore $\varphi(A) \subseteq \mathbf{K}(H_1)$. Applying the Hewit-Cohen factorisation theorem we have $V = VA$; hence $\Phi(V) = \Phi(V)\varphi(A) \subseteq \mathbf{K}(H_1, H_2)$.

Suppose V is a Hilbert CCR-module and $\varphi : A \to \mathbf{B}(H_1)$ is an arbitrary irreducible representation. By Note 3.9, there is a representation $\Phi : V \to \mathbf{B}(H_1, H_2)$ such that $H_2 = \overline{\Phi(V)H_1}$. By Proposition 3.6, Φ is irreducible. Then $\Phi(V) \subseteq \mathbf{K}(H_1, H_2)$, therefore $\varphi(A) = \Phi(V)^*\Phi(V) \subseteq \mathbf{K}(H_1)$. ∎

From [3, proposition 4.2.4] and Proposition 5.2 we get the following statement:

Corollary 5.3. *Let V be a Hilbert CCR-module. Every submodule, ideal submodule and quotient module of V is also a Hilbert CCR-module.*

Corollary 5.4. *Let V be a Hilbert A-module.*
V is a Hilbert CCR-module $\Leftrightarrow$ A is a CCR-algebra $\Leftrightarrow$ $\mathbf{K}(V)$ is a CCR-algebra $\Leftrightarrow$ $\mathcal{L}(V)$ is a CCR-algebra.
In particular, if A and B are Morita equivalent C^*-algebras with A a CCR-algebra, then B is also a CCR-algebra.

PROOF. The first equivalence is proved in Proposition 5.2. The second one can be seen in the same way. Equivalence $\mathcal{L}(V)$ is a CCR-algebra if and only if A is a CCR-algebra is proved using Lemma 3.5. ∎

The last statement of the preceding corollary is already known. It was proved in [10, corollary 3.3] by using induced representations of a Hilbert C^*-module.

In order to extend some results from CCR-algebras to Hilbert CCR-modules, we first need to extend the definition of equivalent representations of a C^*-algebra.

Definition 5.5. Let V be a Hilbert A-module. Two representations $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$ of V, $i = 1, 2$ are said to be *(unitarily) equivalent,* and we write $\Phi_1 \sim \Phi_2$, if there are unitary operators $U_i : H_i^1 \to H_i^2, i = 1, 2$ such that $\Phi_1(x) = U_2^*\Phi_2(x)U_1$, for all $x \in V$.

The proof of the next lemma is elementary, hence it is omitted.

Lemma 5.6. *Let V be a Hilbert A-module, let $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$ be a nondegenerate representation and let $\pi_i = \pi_{\varphi_i,\Phi_i}$, $i = 1, 2$. If $\pi_1 \sim \pi_2$ then $\Phi_1 \sim \Phi_2$. If $\Phi_1 \sim \Phi_2$, then $\varphi_1 \sim \varphi_2, \Phi_1^+ \sim \Phi_2^+$ and $\pi_1 \sim \pi_2$.*

Observe that every nonzero irreducible representation of a Hilbert $\mathbf{K}(H_1)$-module $\mathbf{K}(H_1, H_2)$ is unitarily equivalent to the identical representation.

Proposition 5.7. *Let V be a Hilbert CCR-module over a C^*-algebra A and let $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i), i = 1, 2,$ be irreducible nonzero representations of V. If* $\operatorname{Ker}\Phi_1 \subseteq \operatorname{Ker}\Phi_2$ *then $\Phi_1 \sim \Phi_2$.*

PROOF. Since Φ_1 and Φ_2 are irreducible, $\pi_1 = \pi_{\varphi_1,\Phi_1}$ and $\pi_2 = \pi_{\varphi_2,\Phi_2}$ are irreducible by Lemma 3.5(a). Furthermore, by Corollary 5.4, $\mathcal{L}(V)$ is a CCR-algebra. From $\operatorname{Ker}\Phi_1 \subseteq \operatorname{Ker}\Phi_2$, it follows that $\operatorname{Ker}\pi_1 \subseteq \operatorname{Ker}\pi_2$. Now it only remains to apply [1, proposition 1.5.2] and then Lemma 5.6. ■

Proposition 5.8. *Let V be a Hilbert CCR-module over a C^*-algebra A, let $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$ be mutually inequivalent irreducible representations of V and $T_i \in \mathbf{K}(H_1^i, H_2^i)$ for $i = 1, \ldots, n$. Then there is $x \in V$ such that $\Phi_i(x) = T_i, i = 1 \ldots, n$.*

Proof. By applying Lemma 3.5(a), Corollary 5.4 and Lemma 5.6 we prove that $\pi_i = \pi_{\varphi_i,\Phi_i}$, $i = 1, \ldots, n$ are mutually inequivalent irreducible representations of a CCR-algebra $\mathcal{L}(V)$. Then the statement follows easily from [3, proposition 4.2.5]. ■

6. Hilbert GCR-modules

We begin by recalling that a C^*-algebra A is a GCR-algebra (or postliminal C^*-algebra) if $\mathbf{K}(H) \subseteq \varphi(A)$ (equivalently, $\varphi(A) \cap \mathbf{K}(H) \neq \{0\}$), for every nonzero irreducible representation $\varphi : A \to \mathbf{B}(H)$. We define Hilbert GCR-modules in a similar way.

As in the preceding section, our discussion will be restricted to full Hilbert C^*-modules.

Definition 6.1. A Hilbert A-module V is said to be a *Hilbert GCR-module* if for every nonzero irreducible representation $\Phi : V \to \mathbf{B}(H_1, H_2)$, we have $\mathbf{K}(H_1, H_2) \subseteq \Phi(V)$, (or, equivalently, $\Phi(V) \cap \mathbf{K}(H_1, H_2) \neq \{0\}$).

It is obvious that every Hilbert CCR-module is also a Hilbert GCR-module. The converse is not true.

Proposition 6.2. *Let V be a Hilbert A-module. Then V is a Hilbert GCR-module if and only if A is a GCR-algebra.*

Proof. Similar to the proof of Proposition 5.2, hence it is omitted. ■

From [5, theorem 5.6.2] and Proposition 6.2 we get the following statement:

Corollary 6.3. *Let V be a Hilbert A-module and I an ideal in A. Then V is a Hilbert GCR-module if and only if V_I and V/V_I are GCR-modules.*

Proof. Since V is a full A-module, V_I is a full I-module and V/V_I is a full A/I-module. Now [5, theorem 5.6.2] together with Proposition 6.2 completes the proof. ■

As a consequence of Proposition 6.2 we get the following result. The statement about Morita equivalence was also proved by Zettl in [10] using different techniques.

Corollary 6.4. *Let V be a Hilbert A-module.*
V is a Hilbert GCR-module $\Leftrightarrow$ A is a GCR-algebra $\Leftrightarrow$ $\mathbf{K}(V)$ is a GCR-algebra $\Leftrightarrow$ $\mathcal{L}(V)$ is a GCR-algebra.
In particular, if A and B are Morita equivalent C^*-algebras, with A a GCR-algebra, then B is also a GCR-algebra.

This relation between Hilbert GCR-modules and GCR-algebras enables us to define notions that are known in the context of GCR-algebras for Hilbert GCR-modules, for example a composition sequence. Also, it will not be difficult to extend some well-known results about GCR-algebras.

Definition 6.5. Let V be a Hilbert A-module. We define a set $CCR(V)$ as a set of all $v \in V$ such that $\Phi(v)$ is compact for every irreducible representation Φ of V.

Proposition 6.6. *Let V be a Hilbert A-module. $CCR(V)$ is the largest closed ideal Hilbert CCR-submodule of V, and it is a full Hilbert C^*-module over $CCR(A)$, where $CCR(A)$ is the set of all $a \in A$ such that $\varphi(a)$ is compact for every irreducible representation φ of A.*

PROOF. We first prove that $CCR(V)$ is the ideal submodule of V associated to the ideal $CCR(A)$ in A. Then, by applying [3, proposition 4.2.6], we prove the other statements. The proof is elementary, hence it is omitted. ■

Now we can state another characterisation of Hilbert GCR-modules.

Proposition 6.7. *Let V be a Hilbert A-module. V is a Hilbert GCR-module if and only if $CCR(V/V_I) \neq 0$, for every ideal submodule $V_I \neq V$ of V.*

PROOF. Since V is a full A-module, V/V_I is a full A/I-module. Then by Proposition 6.6, $CCR(V/V_I) = (V/V_I)\, CCR(A/I)$ is a full $CCR(A/I)$-module. Hence $CCR(V/V_I) \neq 0 \Leftrightarrow CCR(A/I) \neq 0$. Now we have: V is a Hilbert GCR-module $\Leftrightarrow$ A is a GCR-algebra $\Leftrightarrow$ $CCR(A/I) \neq 0$, for all ideals $I \neq A$ $\Leftrightarrow$ $CCR(V/V_I) \neq 0$, for all ideal submodules $V_I \neq V$. ■

Definition 6.8. *A composition sequence* for a Hilbert A-module V is a family of ideal submodules $\{V_\alpha : 0 \leq \alpha \leq \alpha_0\}$ of V, indexed by the ordinals lying between 0 and some fixed ordinal α_0, and possessing the following properties:

1) $\alpha < \alpha_0 \Rightarrow V_\alpha \subset V_{\alpha+1}$,
2) $V_0 = 0$, $V_{\alpha_0} = V$,
3) $V_\beta = \overline{\bigcup_{\alpha<\beta} V_\alpha}$, (where $^-$ denotes the norm closure).

Now we can characterise a Hilbert GCR-module using composition sequences.

Proposition 6.9. *For every Hilbert GCR-module V over a C^*-algebra A there* *exists a unique composition sequence* $\{V_\alpha : 0 \leq \alpha \leq \alpha_0\}$ *such that*

$$V_{\alpha+1}/V_\alpha = CCR(V/V_\alpha), \quad 0 \leq \alpha < \alpha_0.$$

Conversely, if a Hilbert A*-module* V *possesses a composition sequence* $\{V_\alpha : 0 \leq \alpha \leq \alpha_0\}$ *such that all* $V_{\alpha+1}/V_\alpha$ *are* CCR*-modules, then* V *is a Hilbert* GCR*-module.*

Proof. Let $V_\alpha = VI_\alpha$, where I_α is an ideal in A, for all $0 \leq \alpha \leq \alpha_0$. An elementary calculation shows that $\{V_\alpha : 0 \leq \alpha \leq \alpha_0\}$ is a composition sequence for V if and only if $\{I_\alpha : 0 \leq \alpha \leq \alpha_0\}$ is a composition sequence for A. Then we use Proposition 6.2 and [1, theorem 1.5.5] to complete the proof. ■

We end with a proposition that proves that an irreducible representation of a Hilbert GCR-module is determined (up to unitary equivalence) by its kernel. That result is an extension of [5, theorem 5.6.3]. We omit the proof.

Proposition 6.10. *Let* V *be a Hilbert* GCR*-module over a* C^**-algebra* A *and let* $\Phi_i : V \to \mathbf{B}(H_1^i, H_2^i)$, i *= 1,2 be nonzero irreducible representations of* V*. Then* $\Phi_1 \sim \Phi_2$ *if and only if* $\operatorname{Ker}\Phi_1 = \operatorname{Ker}\Phi_2$.

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