# TRACE FORMULAE ASSOCIATED WITH THE POLAR DECOMPOSITION OF OPERATORS

M. Chō*
Department of Mathematics, Kanagawa University, Yokohama, Japan

T. Huruya
Faculty of Education and Human Sciences, Niigata University, Niigata, Japan

and

C. Li
Institute of System Science, Northeastern University, Shenyang, P. R. China

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Abstract

Let $T = X + iY$ be the Cartesian decomposition of an invertible operator $T$ on a Hilbert space with trace class self-commutator $[T^*, T]$. Carey–Pincus introduced the principal function $g$ and proved a trace formula associated with the Cartesian decomposition $T = X + iY$. Applying the ordered $C^\infty$-functional calculus for $(X, Y)$ to their trace formula, we define the principal function $g^P$ and prove a trace formula associated with the polar decomposition $T = U|T|$. Using this formula, we show that $g(x, y) = g^P(e^{i\theta}, r)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.

## 1. Introduction

Let $B(\mathcal{H})$ be the set of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$, and let $\mathcal{C}_1$ be the set of trace-class operators of $B(\mathcal{H})$. In [4], Carey-Pincus defined the principal function $g$ and proved a trace formula associated with the Cartesian decomposition $T = X + iY$ with $[T^*, T] \in \mathcal{C}_1$ (see also [12]). It is known that the principal functions are useful for the operator theory; for example, relating the size of the principal function to the existence of cyclic vectors, Berger [3] proved that, for a hyponormal operator $T$, the operator $T^n$ has a non-trivial invariant subspace for sufficiently high $n$ (see other examples, [6; 9; 13; 14; 15; 16]. We also have two different trace formulae and the principal functions $g$ and $g^P$ associated with the decomposition $T = X + iY$ and the polar decomposition $T = U|T|$, respectively [4; 15; 16]. The relation between $g$ and $g^P$ is that if there exists a trace formula for the polar decomposition, then there exists $g$ by a transformation of variables, and $g$ essentially coincides with $g^P$. An operator $T$ is called $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$ [1]. If $p = 1$ and $\frac{1}{2}$, then $T$ is called hyponormal and semi-hyponormal, respectively. The principal function $g$ has been studied well.

*Corresponding author, e-mail: chiyom01@kanagawa-u.ac.jp

 

For example, if $T$ is hyponormal, then $g \geq 0$ (see, for example [13; 16]). If $T$ is semi-hyponormal, then $g^P \geq 0$. Applying this property for $g^P$, we have that $g \geq 0$.

The existences of the trace formulae and $g$ and $g^P$ [4] have been shown separately (see also [15; 16]). In this paper, by the ordered $C^\infty$-functional calculus, we give a trace formula of $|T|$ and $U$ for an invertible operator $T = U|T|$ such that $[T^*, T] \in \mathcal{C}_1$. Using this result, we show a trace formula of a non-invertible semi-hyponormal operator $T = U|T|$ with unitary $U$ such that $[|T|, U] \in \mathcal{C}_1$. Finally, we show a relation between two principal functions $g$ and $g^P$ for such an operator $T$. We remark that for an operator $T = U|T|$, it is easy to see that if $[|T|, U] \in \mathcal{C}_1$, then $[T^*, T] \in \mathcal{C}_1$.

Let $\mathcal{S}(\mathbf{R}^2)$ be the Schwartz space of rapidly decreasing functions at infinity. For $T = X + iY$, let $\mathcal{E}$ and $\mathcal{F}$ be the spectral measures of self-adjoint operators $X$ and $Y$, respectively. We define $\tau$ on $\mathcal{S}(\mathbf{R}^2)$ by

$$\tau(\phi) = \int\int \phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbf{R}^2)). \tag{$*$}$$

By a standard argument, we have

$$\int\int e^{itX}e^{isY}\hat{\phi}(t,s)dtds = \int\int \phi(x,y)d\mathcal{E}(x)d\mathcal{F}(y),$$

where

$$\hat{\phi}(t,s) = \frac{1}{2\pi}\int\int e^{-i(tx+sy)}\phi(x,y)dxdy$$

is the Fourier transform of the function $\phi$ (see, for example, [13, p. 237]).

Put $\nu(E) = \int\int_E \hat{\phi}(t,s)dtds$ for a measurable set $E \subset \mathbf{R}^2$. Since $\hat{\phi}(t,s) \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\int\int (1+|t|)(1+|s|)|\hat{\phi}(t,s)|dtds < \infty.$$

Following Carey–Pincus [4], put $G(x,y) = \int\int e^{itx+isy}d\nu(t,s)$ and define

$$G(X,Y) = \int\int G(x,y)d\mathcal{E}(x)d\mathcal{F}(y).$$

Then

$$\tau(\phi) = \int\int e^{itX}e^{isY}\nu(t,s)dtds = G(X,Y).$$

Note here that we have $\tau(\psi) = \tau(\phi)$ for any smooth function $\psi(x,y)$ that coincides with $\phi(x,y)$ on $\mathrm{supp}(\tau)$.

The map $\tau : \mathcal{S}(\mathbf{R}^2) \to B(\mathcal{H})$ has the following properties [13, chapter X, §2];

(1) $\tau$ is linear, continuous and $\mathrm{supp}\,(\tau) \subseteq \sigma(X) \times \sigma(Y)$,
(2) $\tau(1) = I, \tau(p+q) = p(X) + q(Y)$ for polynomials $p$ and $q$ of one variable of $x$ and $y$, respectively.
(3) $\tau(\phi)\tau(\psi) - \tau(\phi\psi) \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R^2})$,
(4) $\tau(\phi)^* - \tau(\bar{\phi}) \in \mathcal{C}_1$.

By (3) we have an important property $[\tau(\phi), \tau(\psi)] \in \mathcal{C}_1$ for $\phi, \psi \in \mathcal{S}(\mathbf{R^2})$.

Let $\mathcal{A}$ be the linear space of all Laurent polynomials $\mathcal{P}(r, z)$ with polynomial coefficients such that $\mathcal{P}(r, z) = \sum_{k=-N}^{N} p_k(r) z^k$, where $N$ is a non-negative integer and each $p_k(r)$ is a polynomial. For the polar decomposition $T = U|T|$ of $T$, let $\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|) U^k$. For differentiable functions $P, Q$ of two variables $(x, y)$, let $J(P, Q)(x, y) = \dfrac{\partial P}{\partial x}\dfrac{\partial Q}{\partial y} - \dfrac{\partial P}{\partial y}\dfrac{\partial Q}{\partial x}$. For a trace-class operator $T \in \mathcal{C}_1$, we denote the trace of $T$ by $\mathrm{Tr}(T)$.

In this paper, we prove the following trace formula of an invertible operator $T = X + iY = U|T|$ with $[|T|, U] \in \mathcal{C}_1$ by the above Cartesian functional calculus of $\tau$ with $X$ and $Y$. For $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,

$$\mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g^P(e^{i\theta}, r) dr d\theta.$$

The function $g^P$ in the above formula is called the principal function associated with the polar decomposition of $T$. As a corollary of this result, we show that the same formula holds for a non-invertible semi-hyponormal operator $T = U|T|$ with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. For an operator $T$, let $\sigma(T)$ be the spectrum of $T$. The following theorem [4, theorem 5.1] is a basis of this paper (see [12] also):

**Theorem 1** (Carey–Pincus). *Let $T = X + iY$ be an operator with $[T^*, T] \in \mathcal{C}_1$. Let $\mathcal{E}, \mathcal{F}$ be the spectral measures of $X$ and $Y$, respectively, and $\tau$ be given by $(*)$. Then there exists a summable function $g$ such that, for $\phi, \psi \in \mathcal{S}(\mathbf{R}^2)$,*

$$Tr([\tau(\phi), \tau(\psi)]) = \frac{1}{2\pi i} \int\int J(\phi, \psi)(x, y) g(x, y) dx dy.$$

*Moreover, if $T$ is hyponormal, then $g \geq 0$ and $g(x, y) = 0$ for $x + iy \notin \sigma(T)$.*

The function $g$ in Theorem 1 is called *the principal function associated with the Cartesian decomposition of $T$.*

## 2. Function calculus and trace

Let $||A||_1 = \mathrm{Tr}(|A|)$ for $A \in \mathcal{C}_1$, that is, $||A||_1$ is the trace norm of $A$. Let $A \in \mathcal{C}_1$ and $B$ be an operator. Then it holds that

$$|\mathrm{Tr}(A)| \leq ||A||_1, \mathrm{Tr}(AB) = \mathrm{Tr}(BA), ||AB||_1 \leq ||A||_1 ||B|| \text{ and } ||BA||_1 \leq ||B|| ||A||_1.$$

We use an elementary property that if operators $A, B$ and $C$ satisfy $[A, C], [B, C] \in \mathcal{C}_1$ and $A - B \in \mathcal{C}_1$, then $[AB, C], [BA, C] \in \mathcal{C}_1$ and

$$\mathrm{Tr}([AB, C]) = \mathrm{Tr}([BA, C]).$$

Our standard reference on trace is [11].

We begin with two lemmas that are key tools in this paper.

**Lemma 2.** *Let $A$ be a positive invertible operator and operators $D, E, F$ satisfying $[A, D], [E, D], [F, D] \in \mathcal{C}_1$. Then for any real number $\alpha$, we have*

$$[EA^\alpha F, D] \in \mathcal{C}_1.$$

Proof. We use the following expansion known as the binomial series: For $|z| < 1$, it holds

$$(1+z)^\alpha = \sum_{m=0}^{\infty} \begin{pmatrix} \alpha \\ m \end{pmatrix} z^m,$$

where $\begin{pmatrix} \alpha \\ m \end{pmatrix} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}$. Considering $||\beta A|| < 1$ with some positive number $\beta$, we may assume that $||A|| < 1$. Since $A$ is an invertible positive operator and $\|A\| < 1$, we have $||A - I|| < 1$ and

$$A^\alpha = (I+(A-I))^\alpha = \lim_{n\to\infty} \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} (A-I)^m. \tag{1}$$

Let $A_n = \left[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} (A-I)^m, D\right]$ for $n = 1, 2, 3, \cdots$. Then $\lim_{n\to\infty} A_n = [A^\alpha, D]$ with respect to the operator norm. By [12, p. 158 (3.3)], for a positive integer $m$, it holds that

$$||[(A-I)^m, D]||_1 \le m||A-I||^{m-1}||[A, D]||_1,$$

so that

$$||A_n||_1 \le \left( \sum_{m=1}^{n} \left| \begin{pmatrix} \alpha \\ m \end{pmatrix} \right| m||A-I||^{m-1} \right) ||[A, D]||_1.$$

Since $\|A - I\| < 1$, (1) converges absolutely. Hence $\{A_n\}$ is a Cauchy sequence with respect to the norm $|| \cdot ||_1$. Let $B$ denote the limit of the sequence $\{A_n\}$ in $\mathcal{C}_1$. For any unit vector $\xi \in \mathcal{H}$, we define an operator $C$ on $\mathcal{H}$ by $C\eta = (\eta, \xi)\xi$ for $\eta \in \mathcal{H}$. Let $\{e_j\}$ be a complete orthonormal basis of $\mathcal{H}$ such that $e_1 = \xi$. Since $\mathrm{Tr}(SC) = \displaystyle\sum_{j=1}^{\infty} (SCe_j, e_j) = (S\xi, \xi)$, then

$$(B\xi, \xi) = \mathrm{Tr}(BC) = \lim_{n\to\infty} \mathrm{Tr}(A_n C) = \lim_{n\to\infty} (A_n\xi, \xi) = ([A^\alpha, D]\xi, \xi).$$

Since $\xi$ is an arbitrary vector, it follows that

$$[A^\alpha, D] = B \in \mathcal{C}_1.$$

We have

$$[EA_nF, D] = [E, D]A_nF + E[A_n, D]F + EA_n[F, D].$$

Since $\lim_{n\to\infty} A_n = A^\alpha$ with respect to the operator norm,

$$\lim_{n\to\infty}[E, D]A_nF = [E, D]A^\alpha F, \quad \lim_{n\to\infty} E[A_n, D]F = E[A^\alpha, D]F,$$

$$\text{and } \lim_{n\to\infty} EA_n[F, D] = EA^\alpha[F, D],$$

so that

$$\lim_{n\to\infty}[EA_nF, D] = [EA^\alpha F, D]$$

with respect to $\mathcal{C}_1$. ■

The proof of Lemma 2 is based on an idea of [8, theorem 2].

Let $T = X + iY$ be the Cartesian decomposition of $T$. For the spectral measures $\mathcal{E}$ and $\mathcal{F}$ of self-adjoint operators $X$ and $Y$, respectively, we recall

$$\tau(\phi) = \int\int \phi(x, y)d\mathcal{E}(x)d\mathcal{F}(y) \quad (\phi \in \mathcal{S}(\mathbf{R}^2)).$$

**Lemma 3.** [4, p. 158] *Let $T = X + iY$ be an invertible operator such that $[T^*, T] \in \mathcal{C}_1$. Let $\psi \in \mathcal{S}(\mathbf{R}^2)$, $D = \tau(\psi)$ and operators $E, F$ satisfy $[E, D], [F, D] \in \mathcal{C}_1$. Then, for $\phi(x, y) = (x^2 + y^2)^\alpha$ with a real number $\alpha$,*

$$Tr([E\tau(\phi)F, D]) = Tr([E|T|^{2\alpha}F, D]).$$

PROOF. We may assume that $||T|| < d < \frac{1}{2}$. Then $||X^2+Y^2|| = |||T|^2-\frac{1}{2}[T^*, T]|| < 1$. Hence, $X^2 + Y^2 < I$. Since $T$ is invertible, we choose a positive number $c$ such that $0 < c \le X^2 + Y^2$. Hence, we may assume that $f$ of $\tau(f)$ is a function on $\{(x, y) \mid c \le x^2 + y^2 < 1\}$. Also we choose $\varphi \in C_0^\infty(\mathbf{R}^2)$ and $d_1$ such that $d < d_1 < 1$, $\varphi(x, y) = 1$ on $\{(x, y) \mid c \le x^2 + y^2 \le d\}$ and $\text{supp}(\varphi) \subset \{(x, y) \mid x^2 + y^2 < d_1\}$. Then

$$\tau(\phi\varphi) = \sum_{m=0}^{\infty} \binom{\alpha}{m} \int\int ((x^2+y^2)-1)^m d\mathcal{E}(x)d\mathcal{F}(y) = \sum_{m=0}^{\infty} \binom{\alpha}{m} \tau\big(((x^2+y^2)-1)^m\big)$$

with respect to the operator norm. Since

$$\tau((x^2 + y^2) - 1) = X^2 + Y^2 - I \text{ and } |T|^2 = X^2 + Y^2 + \tfrac{1}{2}[T^*, T],$$

we get

$$\tau((x^2 + y^2) - 1) - (|T|^2 - I) \in \mathcal{C}_1.$$

Since by property (3) of $\tau$ and the above it holds that

$$\tau\big(((x^2 + y^2) - 1)^m\big) - \tau\big((x^2 + y^2) - 1\big)^m \in \mathcal{C}_1,$$

we have for $m > 0$

$$\tau\big(\big((x^2+y^2)-1\big)^m\big) - (|T|^2 - I)^m$$
$$= \tau\big(\big((x^2+y^2)-1\big)^m\big) - \tau\big((x^2+y^2)-1\big)^m + \tau((x^2+y^2)-1)^m - (|T|^2-I)^m \in \mathcal{C}_1.$$

Hence, it holds that

$$\mathrm{Tr}([\tau\big(\big((x^2+y^2)-1\big)^m\big), D]) = \mathrm{Tr}([(|T|^2 - I)^m, D])$$

and

$$\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} \tau\big(\big((x^2+y^2)-1\big)^m\big), D\Big] \in \mathcal{C}_1.$$

Therefore, we see

$$\mathrm{Tr}\Big(\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} \tau\big(\big((x^2+y^2)-1\big)^m\big), D\Big]\Big) = \mathrm{Tr}\Big(\Big[\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} ((X^2+Y^2)-I)^m, D\Big]\Big).$$

Let

$$\varphi_\infty(r) = r^\alpha = \sum_{m=0}^{\infty} \begin{pmatrix} \alpha \\ m \end{pmatrix} (r-1)^m \qquad (0 < |r| < 1),$$
$$\varphi_n(r) = \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} (r-1)^m,$$
$$\phi_n(x,y) = \varphi_n(x^2+y^2) = \sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} ((x^2+y^2)-1)^m.$$

Put $\tilde{\phi_n} = \phi_n \varphi$ and $\tilde{\phi} = \phi\varphi$. Then for some $f_k \in C^\infty$ with $\mathrm{supp}(f_k) \subset \{(x,y) \,|\, x^2 + y^2 < 1\}$ $(k = 0, \cdots, m)$, we have

$$\frac{\partial^m}{\partial x^j \partial y^{m-j}} (\tilde{\phi_n} - \tilde{\phi})(x,y)$$
$$= (\varphi_n^{(m)}(r^2) - \varphi_\infty^{(m)}(r^2)) f_m(x,y) + (\varphi_n^{(m-1)}(r^2) - \varphi_\infty^{(m-1)}(r^2)) f_{m-1}(x,y)$$
$$+ \cdots + (\varphi_n(r^2) - \varphi_\infty(r^2)) f_0(x,y),$$

where $r^2 = x^2 + y^2$. We remark that each $f_k$ depends on $\dfrac{\partial^m}{\partial x^j \partial y^{m-j}}$ and is independent of $\tilde{\phi}_n$. Hence we obtain $\tilde{\phi}_n \to \tilde{\phi}$ in $\mathcal{S}(\mathbf{R}^2)$. By [13, chapter X, corollary 2.3], it holds that

$$[\tau(\tilde{\phi}_n), D] \to [\tau(\tilde{\phi}), D] \quad \text{in } \mathcal{C}_1.$$

Since

$$[E\tau(\tilde{\phi}_n)F, D] = [E, D]\tau(\tilde{\phi}_n)F + E[\tau(\tilde{\phi}_n), D]F + E\tau(\tilde{\phi}_n)[F, D]$$

and $\lim_{n\to\infty} \tau(\tilde{\phi}_n) = \tau(\tilde{\phi})$ with respect to the operator norm, in $\mathcal{C}_1$ it holds that

$$\lim_{n\to\infty}[E, D]\tau(\tilde{\phi}_n)F = [E, D]\tau(\tilde{\phi})F,$$
$$\lim_{n\to\infty} E[\tau(\tilde{\phi}_n), D]F = E[\tau(\tilde{\phi}), D]F,$$
$$\lim_{n\to\infty} E\tau(\tilde{\phi}_n)[F, D] = E\tau(\tilde{\phi})[F, D].$$

Hence in $\mathcal{C}_1$ we obtain

$$\lim_{n\to\infty}[E\tau(\tilde{\phi}_n)F, D] = [E\tau(\tilde{\phi})F, D].$$

Since $T \rightarrow \mathrm{Tr}(T)$ is continuous in $\mathcal{C}_1$, we have

$$\begin{aligned}\mathrm{Tr}([E\tau(\tilde{\phi})F, D]) &= \lim_{n\to\infty} \mathrm{Tr}(E[\tau(\tilde{\phi}_n)F, D]) \\ &= \lim_{n\to\infty} \mathrm{Tr}(E\sum_{m=0}^{n} \begin{pmatrix} \alpha \\ m \end{pmatrix} [(|T|^2 - I)^m F, D]) \\ &= \mathrm{Tr}([E|T|^{2\alpha}F, D]).\end{aligned}$$

■

## 3. Main theorem

First we show the following:

**Theorem 4.** *Let $T = U|T|$ be an invertible operator with $[T^*, T] \in \mathcal{C}_1$ and let $g$ be the principal function associated with the Cartesian decomposition of $T = X + iY$. Then there exists a summable function $g^P$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$Tr([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta,$$

*and $g^P(e^{i\theta}, r) = g(x, y)$ almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.*

PROOF. Since $T$ is invertible, there exists a number $c > 0$ such that $c \le X^2 + Y^2$. Then $\frac{c}{2} \le X^2$ or $\frac{c}{2} \le Y^2$, so that, if $\zeta \in \mathcal{S}(\mathbf{R}^2)$ satisfies $\zeta(x, y) = 0$ for $\frac{c}{2} > |x|^2$ or $\frac{c}{2} > |y|^2$, then $\tau(\zeta) = 0$. With $g(x, y)$ in Theorem 1, we know that $g(x, y) = 0$ for $x + iy$ with $x^2 + y^2 < \frac{c}{2}$. Let $w(x, y)$ and $h(x, y)$ be in $\mathcal{S}(\mathbf{R}^2)$ such that $w(x, y) = (x + iy)(x^2 + y^2)^{-\frac{1}{2}}$ and $h(x, y) = (x^2 + y^2)^{\frac{1}{2}}$ on the support of $g$. For $\psi, \phi_l, \phi_r \in \mathcal{S}(\mathbf{R}^2)$, let $D = \tau(\psi), E = \tau(\phi_l)$ and $F = \tau(\phi_r)$. By property (3) of $\tau$ and Lemma 3, for a positive integer $k$ we obtain

$$\begin{aligned}\mathrm{Tr}([EU^kF, D]) &= \mathrm{Tr}([E(T|T|^{-1})^kF, D]) = \mathrm{Tr}([E(T\tau(h^{-1}))^kF, D]) \\ &= \mathrm{Tr}\big([E\big(\tau(x + iy)\tau(h^{-1})\big)^kF, D]\big) \\ &= \mathrm{Tr}\big([E\tau\big(\big((x + iy)(x^2 + y^2)^{-\frac{1}{2}}\big)^k\big)F, D]\big) = \mathrm{Tr}\big([E\tau(w^k)F, D]\big)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Tr}\big([EU^{-k}F,D]\big) &= \operatorname{Tr}\big([E(|T|T^{-1})^kF,D]\big) = \operatorname{Tr}\Big(\big[E\big(\tau(h)\tau\big(1/(x+iy)\big)\big)^kF,D\big]\Big)\\ &= \operatorname{Tr}\Big(\big[E\tau\Big(\big((x-iy)(x^2+y^2)^{-\frac{1}{2}}\big)^k\Big)F,D\big]\Big) = \operatorname{Tr}\big([E\tau(w^{-k})F,D]\big).\end{aligned}$$

Then for integers $m, s$ and non-negative integers $n, t$, we have

$$\begin{aligned}\operatorname{Tr}([U^m|T|^n, U^s|T|^t]) &= \operatorname{Tr}([\tau(w^m)\tau(h^n), \tau(w^s)\tau(h^t)])\\ &= \operatorname{Tr}([\tau(w^mh^n), \tau(w^sh^t)]).\end{aligned}$$

By Theorem 1, there exists a summable function $g$ such that

$$\operatorname{Tr}([\tau(w^mh^n), \tau(w^sh^t)]) = \frac{1}{2\pi i}\int\int J(w^mh^n, w^sh^t)(x,y)g(x,y)dxdy.$$

By the transformation $x = r\cos\theta$ and $y = r\sin\theta$,

$$\begin{aligned}&\frac{1}{2\pi i}\int\int J(w^mh^n, w^sh^t)(x,y)g(x,y)dxdy\\ =&\frac{1}{2\pi i}\int\int J(w^mh^n, w^sh^t)(r\cos\theta, r\sin\theta)g(r\cos\theta, r\sin\theta)rdrd\theta.\end{aligned}$$

Hence we have, for Laurent polynomials $\mathcal{P}$ and $\mathcal{Q}$,

$$\operatorname{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)])$$

$$= \frac{1}{2\pi i}\int\int J(\mathcal{P}(h,w), \mathcal{Q}(h,w))(r\cos\theta, r\sin\theta)rg(r\cos\theta, r\sin\theta)drd\theta.$$

For $x+iy \in \sigma(T)$, let $x = r\cos\theta$ and $y = r\sin\theta$. Since $w(x,y) = (x+iy)(x^2+y^2)^{-\frac{1}{2}}$ and $h(x,y) = (x^2+y^2)^{\frac{1}{2}}$, then $w(r\cos\theta, r\sin\theta) = e^{i\theta}$, $h(r\cos\theta, r\sin\theta) = r$,

$$\frac{\partial(h,w)}{\partial(r,\theta)} = \frac{\partial(r,e^{i\theta})}{\partial(r,\theta)} = ie^{i\theta} \quad\text{and}\quad \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(r\cos\theta, r\sin\theta)}{\partial(r,\theta)} = r.$$

Also, it holds that

$$\frac{\partial\big(\mathcal{P}(r,e^{i\theta}), \mathcal{Q}(r,e^{i\theta})\big)}{\partial(r,\theta)} = \frac{\partial\big(\mathcal{P}(h,w), \mathcal{Q}(h,w)\big)}{\partial(h,w)}(r,e^{i\theta})\cdot\frac{\partial(h,w)}{\partial(r,\theta)}$$

$$= ie^{i\theta}\cdot\frac{\partial\big(\mathcal{P}(h,w), \mathcal{Q}(h,w)\big)}{\partial(h,w)}(r,e^{i\theta}) = ie^{i\theta}\cdot J\big(\mathcal{P},\mathcal{Q}\big)(r,e^{i\theta})$$

and

$$\frac{\partial\big(\mathcal{P}\big(h(r\cos\theta, r\sin\theta), w(r\cos\theta, r\sin\theta)\big), \mathcal{Q}\big(h(r\cos\theta, r\sin\theta), w(r\cos\theta, r\sin\theta)\big)\big)}{\partial(r,\theta)}$$

$$= \frac{\partial\big(\mathcal{P}(h(x,y), w(x,y)), \mathcal{Q}(h(x,y), w(x,y))\big)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= r \cdot \frac{\partial\big(\mathcal{P}(h(x,y), w(x,y)), \mathcal{Q}(h(x,y), w(x,y))\big)}{\partial(x,y)}$$

$$= r \cdot J\big(\mathcal{P}(h,w), \mathcal{Q}(h,w)\big)(r\cos\theta, r\sin\theta).$$

Hence we have

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi i}\int\int i\frac{\partial(\mathcal{P}(h,w), \mathcal{Q}(h,w))}{\partial(h,w)}(r, e^{i\theta})e^{i\theta}g(r\cos\theta, r\sin\theta)drd\theta.$$

Put $g^P(e^{i\theta}, r) = g(r\cos\theta, r\sin\theta)$. Then

$$\mathrm{Tr}([\mathcal{P}(|T|,U), \mathcal{Q}(|T|,U)]) = \frac{1}{2\pi}\int\int J(\mathcal{P},\mathcal{Q})(r, e^{i\theta})e^{i\theta}g^P(e^{i\theta}, r)drd\theta.$$

■

The function $g^P$ in Theorem 4 is called *the principal function associated with the polar decomposition* $T = U|T|$ *of* $T$. An invertible operator $T$ is said to be log-hyponormal if $\log T^*T \geq \log TT^*$ [10]. Lemma 2 and Theorem 4 give another proof of a trace formula of log-hyponormal operators in [5].

For the proof of the next result, we need the following two lemmas. For an operator $T$, let $\sigma_{ap}(T)$ and $\sigma_p(T)$ be the approximate point spectrum and the point spectrum of $T$, respectively.

**Lemma 5.** *Let* $T = U|T|$ *be an invertible semi-hyponormal operator with* $[|T|, U] \in \mathcal{C}_1$. *Then the principal function* $g^P$ *associated with the polar decomposition* $T = U|T|$ *of* $T$ *satisfies* $g^P(e^{i\theta}, r) = 0$ *for* $re^{i\theta} \notin \sigma(T)$.

PROOF. Put $S = U|T|^{\frac{1}{2}}$. Then $S$ is hyponormal and $[S^*, S] = [|T|, U]U^* \in \mathcal{C}_1$. Let $g_S^P$ be the principal function associated with the polar decomposition of $S = U|T|^{\frac{1}{2}}$. Then by Theorems 1 and 4 it holds that $g_S^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(S)$. By [16, lemma VI 3.6] and $S = U|T|^{\frac{1}{2}}$, we also have

$$\sigma(T) = \{r^2e^{i\theta} : re^{i\theta} \in \sigma(S)\} \quad \text{and} \quad g^P(e^{i\theta}, r) = g_S^P(e^{i\theta}, r^2).$$

Hence, $g^P$ has the desired property. ■

**Lemma 6.** *Let $T = U|T|$ be an operator with unitary $U$ and put $S = U(|T| + I)$. If $z \in \partial\sigma(S)$, then $|z| \geq 1$. Therefore, if $z \in \sigma(S)$, then $|z| \geq 1$.*

Proof. Since $U$ and $|T| + I$ are invertible, so is $S$. Since $z \in \partial\sigma(S)$ and $\partial\sigma(S) \subseteq \sigma_{ap}(S)$, we have $z \in \sigma_{ap}(S)$. Hence, let $\pi : B(\mathcal{H}) \rightarrow B(\mathcal{K})$ denote the Berberian representation [2]. Since $\sigma_{ap}(S) = \sigma_p(\pi(S))$, there exists $\mathbf{x} \in \mathcal{K}$ such that

$$z\boldsymbol{x} = \pi(S)\boldsymbol{x} = \pi(U)\pi(|T| + I)\boldsymbol{x}.$$

Since $\pi(U)$ is unitary, there exists $\mathbf{y} \in \mathcal{K}$ such that $\pi(U)^*\mathbf{y} = \boldsymbol{x}$. Hence

$$||\boldsymbol{y}||^2 = (\boldsymbol{y}, \boldsymbol{y}) \leq (\pi(U)\pi(|T| + I)\pi(U)^*\boldsymbol{y}, \boldsymbol{y}) = (z\boldsymbol{x}, \boldsymbol{y}) \leq |z|\, ||\boldsymbol{x}||\, ||\boldsymbol{y}|| = |z|\, ||\boldsymbol{y}||^2,$$

so that $1 \leq |z|$. Let $z_0 \in \sigma(S)$ such that $|z_0| = \inf\{|\mu| : \mu \in \sigma(S)\}$. Since $S$ is invertible, we have

$$z_0 \in \partial\sigma(S).$$

By the above argument, we obtain $1 \leq |z_0|$. ■

Now we give another proof of [7, theorem 9].

**Theorem 7.** *Let $T = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. Then there exists a summable function $g^P$ such that, for $\mathcal{P}, \mathcal{Q} \in \mathcal{A}$,*

$$Tr([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) = \frac{1}{2\pi} \int\int J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta} g^P(e^{i\theta}, r)drd\theta.$$

Proof. Since by the assumption $[|T|, U] \in \mathcal{C}_1$ it holds that $[T^*, T] \in \mathcal{C}_1$, by Theorem 4 we may only prove the theorem when $T$ is not invertible. Put $|\tilde{T}| = |T| + I$ and $\tilde{T} = U|\tilde{T}|$. Then $\tilde{T}$ is semi-hyponormal. For Laurent polynomials $\mathcal{P}$ and $\mathcal{Q}$, put $\tilde{\mathcal{P}}(r, z) = \mathcal{P}(r - 1, z)$ and $\tilde{\mathcal{Q}}(r, z) = \mathcal{Q}(r - 1, z)$. Then

$$\begin{aligned} \mathrm{Tr}([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]) &= \mathrm{Tr}([\mathcal{P}(|\tilde{T}| - I, U), \mathcal{Q}(|\tilde{T}| - I, U)]) \\ &= \mathrm{Tr}([\tilde{\mathcal{P}}(|\tilde{T}|, U), \tilde{\mathcal{Q}}(|\tilde{T}|, U)]). \end{aligned}$$

Since $\tilde{T}$ is invertible and $[|\tilde{T}|, U] = [|T|, U] \in \mathcal{C}_1$, by Theorem 4 there exists a summable function $\tilde{g}^P$ such that

$$\mathrm{Tr}([\tilde{\mathcal{P}}(|\tilde{T}|, U), \tilde{\mathcal{Q}}(|\tilde{T}|, U)]) = \frac{1}{2\pi} \int\int J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta} \tilde{g}^P(e^{i\theta}, r)drd\theta.$$

By Lemma 5, it holds that $\tilde{g}^P(e^{i\theta}, r) = 0$ for $re^{i\theta} \notin \sigma(\tilde{T})$. We have

$$\frac{1}{2\pi} \int\int J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta} \tilde{g}^P(e^{i\theta}, r)drd\theta$$

$$= \frac{1}{2\pi} \int\int_{\sigma(\tilde{T})} J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}})(r, e^{i\theta})e^{i\theta} \tilde{g}^P(e^{i\theta}, r)drd\theta$$

$$= \frac{1}{2\pi}\int\int_{\sigma(\tilde{T})} J(\mathcal{P},\mathcal{Q})(r-1,e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta},r)drd\theta$$

$$= \frac{1}{2\pi}\int\int_{A} J(\mathcal{P},\mathcal{Q})(\rho,e^{i\theta})e^{i\theta}\tilde{g}^P(e^{i\theta},\rho+1)d\rho d\theta \quad \text{(by the transformation } \rho = r-1\text{)},$$

where $A = \{(r-1)e^{i\theta} : re^{i\theta} \in \sigma(\tilde{T})\}$. We remark that, by Lemma 6, $r - 1 \geq 0$ for $re^{i\theta} \in \sigma(\tilde{T})$. We define $g^P$ by $g^P(e^{i\theta}, r) = \tilde{g}^P(e^{i\theta}, r+1)$. Then $g^P$ is the desired function. ■

Finally, we show a relation between $g$ and $g^P$.

**Theorem 8.** *Let $T = X + iY = U|T|$ be a semi-hyponormal operator with unitary $U$ and $[|T|, U] \in \mathcal{C}_1$. If $g$ and $g^P$ are the principal function associated with the Cartesian decomposition of $T$ and the summable function in Theorem 7, respectively, then*

$$g(x, y) = g^P(e^{i\theta}, r)$$

*almost everywhere $x + iy = re^{i\theta}$ on $\mathbf{C}$.*

PROOF. Since $[|T|, U] \in \mathcal{C}_1$, by Lemma 2 we have $[|T|^2, U] \in \mathcal{C}_1$. Hence

$$2i[X, Y] = T^*T - TT^* = |T|^2 - U|T|^2U^* = [|T|^2, U]U^* \in \mathcal{C}_1.$$

Let $\mathcal{Q}_0(x, y) = y$. For the polynomial $\mathcal{Q}_0(x, y) = y$ and an arbitrary polynomial $\mathcal{P}(x, y)$, by Theorem 1 and [4, theorem 5.2] we have

$$\begin{aligned}\mathrm{Tr}([\mathcal{P}(X,Y), \mathcal{Q}_0(X,Y)]) &= \frac{1}{2\pi i}\int\int_{\sigma(T)} J(\mathcal{P},\mathcal{Q}_0)g(x,y)dxdy \\ &= \frac{1}{2\pi i}\int\int_{\sigma(T)} \mathcal{P}_x(x,y)g(x,y)dxdy \\ &= \frac{1}{2\pi i}\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta)g(r\cos\theta,\, r\sin\theta)rdrd\theta, \quad (2)\end{aligned}$$

where $\mathrm{M} = \{(r, \theta) : re^{i\theta} \in \sigma(T),\ 0 \leq \theta < 2\pi\}$. Let

$$\tilde{\mathcal{P}}(r, z) = \mathcal{P}\left(\frac{zr + rz^{-1}}{2}, \frac{zr - rz^{-1}}{2i}\right) \quad \text{and} \quad \tilde{\mathcal{Q}}_0(r, z) = \frac{zr - rz^{-1}}{2i}.$$

Then

$$\begin{aligned}J(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_0) = &\left(\mathcal{P}_x \cdot \frac{z + z^{-1}}{2} + \mathcal{P}_y \cdot \frac{z - z^{-1}}{2i}\right)\left(\frac{r}{2i}(1 + \frac{1}{z^2})\right) \\ &- \frac{r}{2}\left\{\mathcal{P}_x \cdot (1 - \frac{1}{z^2}) + \frac{1}{i}\mathcal{P}_y \cdot (1 + \frac{1}{z^2})\right\}\frac{z - z^{-1}}{2i}.\end{aligned}$$

Hence

$$\begin{aligned} J\big(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_0\big)(r, e^{i\theta}) \cdot e^{i\theta} &= (\mathcal{P}_x \cdot \cos\theta + \mathcal{P}_y \cdot \sin\theta)(-ir\cos\theta) - r(i\mathcal{P}_x \cdot \sin\theta - i\mathcal{P}_y \cdot \cos\theta)\sin\theta \\ &= -ir\mathcal{P}_x. \end{aligned}$$

Theorem 7 implies

$$\begin{aligned} &\mathrm{Tr}\Bigg(\Bigg[\mathcal{P}\Big(\frac{U|T| + |T|U^{-1}}{2}, \frac{U|T| - |T|U^{-1}}{2i}\Big), \frac{U|T| - |T|U^{-1}}{2i}\Bigg]\Bigg) \\ &= \frac{1}{2\pi}\int\int_{\mathrm{M}} J\big(\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_0\big)(r, e^{i\theta})e^{i\theta} g^P(e^{i\theta}, r)\, dr d\theta \\ &= \frac{1}{2\pi}\int\int_{\mathrm{M}} -ir\mathcal{P}_x(r\cos\theta,\, r\sin\theta) g^P(e^{i\theta}, r)\, dr d\theta \\ &= \frac{1}{2\pi i}\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta) g^P(e^{i\theta}, r)\, r dr d\theta. \qquad (3) \end{aligned}$$

Since

$$\mathrm{Tr}\big([\mathcal{P}(X, Y), \mathcal{Q}_0(X, Y)]\big) = \mathrm{Tr}\Bigg(\Bigg[\mathcal{P}\Big(\frac{U|T| + |T|U^{-1}}{2}, \frac{U|T| - |T|U^{-1}}{2i}\Big), \frac{U|T| - |T|U^{-1}}{2i}\Bigg]\Bigg),$$

we have (2) = (3) and

$$\begin{aligned} &\int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta) g(r\cos\theta,\, r\sin\theta)\, r dr d\theta \\ &= \int\int_{\mathrm{M}} \mathcal{P}_x(r\cos\theta,\, r\sin\theta) g^P(e^{i\theta}, r)\, r dr d\theta. \end{aligned}$$

Since $\mathcal{P}$ is an arbitrary polynomial, we obtain the desired relation between $g$ and $g^P$. ■


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