HARTE THEOREM FOR WAELBROECK ALGEBRAS

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ABSTRACT

Let B be a locally convex Waelbroeck algebra. Let $a_1, \ldots, a_k \in B$ be an arbitrary k-tuple of mutually commuting elements. The joint spectrum $\sigma_B(a_1, \ldots, a_k)$ is defined as the set of those $(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ for which the elements $a_1 - \lambda_1, \ldots, a_k - \lambda_k$ generate a proper (left or right) ideal. Let $p : \mathbb{C}^k \to \mathbb{C}^m$ be a polynomial mapping. The spectral mapping formula

$$p(\sigma_B(a_1, \ldots, a_k)) = \sigma_B(p(a_1, \ldots, a_k))$$

is proved.

1. Introduction

The spectral mapping formula for the left and the right joint spectra on a Banach unital algebra B was proved by R. Harte in [1].

The left (right) joint spectrum of a k-tuple of mutually commuting elements $a_1, \ldots, a_k \in B$ is defined as the set of those $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ for which the left (resp. right) ideal generated in B by the elements $a_1 - \lambda_1, \ldots, a_k - \lambda_k$ is proper. The left (right) joint spectrum is denoted by $\sigma^l_B(a_1, \ldots, a_k)$ (resp. $\sigma^r_B(a_1, \ldots, a_k)$). The set

$$\sigma_B(a_1, \ldots, a_k) = \sigma^l_B(a_1, \ldots, a_k) \cup \sigma^r_B(a_1, \ldots, a_k)$$

is called the Harte spectrum of $a_1, \ldots, a_k$ in B.

The Harte spectral mapping theorem states that for an arbitrary polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$

$$p(\sigma_B(a_1, \ldots, a_k)) = \sigma_B(p(a_1, \ldots, a_k)) \tag{1.1}$$

and that this formula is valid separately for the left and for the right spectra. While the relation $p(\sigma_B(a_1, \ldots, a_k)) \subset \sigma_B(p(a_1, \ldots, a_k))$ is purely algebraic the inverse one involves both the algebraic and the topological structures of the algebra B.

The original proof of the Harte theorem makes use of the concept of the topological divisor of zero and the fact that for an arbitrary element $a \in B$ there exists

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$\lambda \in \mathbb{C}$ such that $a - \lambda$ is a topological zero divisor. If we consider a more general topological algebra B this argument fails even for such agreeable algebras as m-convex ones. In the present paper we prove the spectral mapping formula for an arbitrary unital, locally convex Waelbroeck algebra B and for the left and right joint spectra defined in an analogous way. Our proof applied in the case of a Banach algebra B leads to a new and very elementary proof of the Harte theorem.

A lot of basic properties of the joint spectra can be obtained exactly as in the case of Banach algebras even for Q-algebras B. In section 2 we present these facts as well as the notation and necessary definitions.

The difficult part of the theorem is the projection property of the family of all proper ideals of B, which in the case of the left ideals says the following:

If the mutually commuting elements $a_1, \ldots, a_k$ generate a proper left ideal in B and $c \in B$ commutes with all a_i, $i = 1, \ldots, k$, then there exists $\lambda \in \mathbb{C}$ such that the left ideal generated by the elements $a_1, \ldots, a_k, c - \lambda$ is also proper.

Section 3 is devoted to the proof of this property for locally convex Waelbroeck algebras. In Section 4 we deduce the spectral mapping theorem by applying the projection property and the elementary one-way spectral mapping property obtained in Section 2.

2. The one-way spectral mapping formula

Let B be a unital topological algebra over $\mathbb{C}$. The unit of B is denoted by e. For $\lambda \in \mathbb{C}$ and $x \in B$ we write $x - \lambda$ instead of $x - \lambda e$. A unital topological algebra B is a Q-algebra if the set $G(B)$ of invertible elements is open in B. A Waelbroeck algebra is a Q algebra in which the inverse $x \to x^{-1}$ is continuous on $G(B)$. Waelbroeck algebras are also called continuous inverse algebras in the literature.

All basic information about Q-algebras and Waelbroeck algebras we need can be found in [2].

If a Q-algebra is also a Fréchet algebra then its inverse is continuous, so it is a Waelbroeck algebra. This fails to be true for non-metrizable algebras. The field $C(T)$ of all rational functions of one variable provided with the topology defined by Williamson [4] is a Q-algebra but it has a discontinuous inverse and it is not a Waelbroeck algebra.

Every complete locally m-convex algebra has a continuous inverse. The algebra $C(\mathbb{C})$ of continuous functions on the real line provided with the topology of the almost uniform convergence is the simplest example of a locally m-convex algebra that is not a Q-algebra.

If B is a Q-algebra and $x \in B$, the spectrum

$$\sigma_B(x) = \{\lambda |\ x - \lambda \notin G(B)\}$$

is a compact (possibly empty) set. Moreover, if B is a Waelbroeck algebra then the function $\mathbb{C} \setminus \sigma_B(x) \ni \lambda \to (x - \lambda)^{-1}$ is holomorphic and vanishes at infinity, while the function $\lambda \to (e - \lambda x)^{-1}$ is holomorphic in some neighbourhood of zero. By applying the Liouville theorem in the usual way it follows that in a locally convex Waelbroeck algebra the spectrum $\sigma(x)$ is nonempty for every element of the algebra.

If B is a Q-algebra, then the closure of an arbitrary proper left ideal is a proper ideal. By $I^l_B(a_1, \ldots, a_k)$ we denote the closed left ideal of B generated by elements $a_1, \ldots, a_k \in B$. In Q-algebras $I^l_B(a_1, \ldots, a_k) = B$ if and only if there exist $b_1, \ldots, b_k \in B$ such that $e = \sum_{i=1}^k b_i a_i$.

Since the case of left and right ideals can be treated similarly, in what follows we consider only the left ideals, and left spectra.

Let us define for $a_1, \ldots, a_k \in B$ the left joint spectrum as

$$\sigma^l_B(a_1, \ldots, a_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k | \, I^l_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \neq B\}.$$

If B is a Q-algebra then $\sigma^l_B(a_1, \ldots, a_k)$ is bounded for an arbitrary k-tuple because

$$\sigma^l_B(a_1, \ldots, a_k) \subset \prod_{i=1}^k \sigma(a_i).$$

It is also a compact set.

Proposition 2.1. *Let B be a unital Q-algebra. Then for every $a_1, \ldots, a_k$ of mutually commuting elements of B and for every polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$*

$$p(\sigma^l_B(a_1, \ldots, a_k)) \subset \sigma^l_B(p(a_1, \ldots, a_k)).$$

PROOF. Every polynomial f of k variables $x = (x_1, \ldots, x_k)$ can be represented as

$$f(x) - f(\lambda) = \sum_{i=1}^k f_i(x)(x_i - \lambda_i),$$

where f_i are polynomials and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$. In particular, for every polynomial map $p(x) = (p_1(x), \ldots, p_m(x))$ and mutually commuting $a = (a_1, \ldots, a_k) \in B^k$ we obtain

$$I^l_B(p_1(a) - p_1(\lambda), \ldots, p_m(a) - p_m(\lambda)) \subset I^l_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k).$$

If $(\lambda_1, \ldots, \lambda_k) \in \sigma_B(a_1, \ldots, a_k)$ then $I^k_B(a_1 - \lambda_1, \ldots, a_k - \lambda_k)$ is proper and $I^l_B(p_1(a) - p_1(\lambda), \ldots, p_m(a) - p_m(\lambda))$ is also proper. This proves the result. ■

3. The projection property

Let B be a locally convex Waelbroeck algebra with unit e. Let $a_1, \ldots, a_k, c$ be a $k+1$-tuple of mutually commuting elements of B. Assume that the ideal $N := I^l_B(a_1, \ldots, a_k)$ is proper. Since c commutes with all a_i, $i = 1, \ldots, k$ then $Nc \subset N$.

Let us consider $M = \{b \in B | \, Nb \subset N\}$. It is a closed subalgebra of B containing N and the elements c, e. N is a two-sided closed ideal in M, hence the quotient space $E = M/N$, which is a closed subset of $X = B/N$, has the structure of the algebra with the natural product $[x][y] = [xy]$. Besides the natural action of the algebra B on X given by the formula $b[x] = [bx]$, $b \in B$, $[x] \in X$, there exists the right action of E on X given by the formula:

$$[x] \cdot [m] = [xm],$$

for $x \in B$, $m \in M$.

Let us choose an arbitrary maximal commutative subalgebra A of $E = M/N$ containing the elements $[c]$ and $[e]$.

Lemma 3.1. *The spectrum $\sigma_A([c])$ is contained in $\sigma_B(c)$. The resolvent of $[c]$ in A is equal to $[(c-\lambda)^{-1}]$ for $\lambda \notin \sigma_B(c)$.*

Proof. The algebra B is a Waelbroeck algebra, so outside the compact set $\sigma_B(c)$ there is the well-defined holomorphic and bounded resolvent $r_\lambda = (c-\lambda)^{-1}$. This element commutes with all a_i, $i = 1, \ldots, k$, hence it belongs to M. Moreover $[r_\lambda]$ is the inverse of $[c - \lambda e]$ in E so it commutes with all elements of A and, by the maximality of A, it follows that $[r_\lambda] \in A$. We obtain that outside the set $\sigma_B(c)$ there exists also a holomorphic, bounded resolvent of $[c]$ in A. In particular $\sigma_A([c]) \subset \sigma_B(c)$ is a bounded subset of the complex plane. ■

Lemma 3.2. *Suppose that for some $\lambda \in \mathbb{C}$ there exists a $r_\lambda \in B$ such that $r_\lambda(c - \lambda) - e \in N$. Then there exists e neighbourhood U of λ and a function $r : U \ni \mu \to r(\mu) \in B$, such that $[r(\mu)] \cdot [c - \mu] = [e]$. The functions $\mu \to r(\mu) \in B$ and $\mu \to [r(\mu)] \in X$ are holomorphic.*

Proof. By supposition there exists $n \in N$ such that $r_\lambda(c - \lambda) + n = e$. Then

$$r_\lambda(c - (\lambda + \delta)) + n = r_\lambda(c - \lambda) + n - \delta r_\lambda = e - \delta r_\lambda.$$

For sufficiently small $\rho > 0$ and for $|\delta| < \rho$ there exists in B the inverse $(e - \delta r_\lambda)^{-1}$. We obtain

$$[(e - \delta r_\lambda)^{-1} r_\lambda(c - (\lambda + \delta))] = [(e - \delta r_\lambda)^{-1} r_\lambda] \cdot [(c - (\lambda + \delta))] = [e].$$

The resolvent $\delta \to (e - \delta r_\lambda)^{-1}$ is holomorphic on the disc $D(0, \rho)$ in the sense that it has the compex derivative. So for $|\mu - \lambda| < \rho$ the functions $\mu \to r(\mu) = (e - (\mu - \lambda) r_\lambda)^{-1} r_\lambda \in B$ and $\mu \to [r(\mu)] \in X$ are also holomorphic. ■

Now we are able to prove the principal result of the section.

Theorem 3.3. *Let B be a unital, locally convex Waelbroeck algebra. Then for every $k + l$ tuple of mutually commuting $a_1, \ldots, a_k, c_1, \ldots, c_l \in B$, such that $a_1, \ldots, a_k$ generate a proper left ideal in B, there exists $\lambda \in \mathbb{C}^l$ such that the left ideal generated by $a_1, \ldots, a_k, c_1 - \lambda_1, \ldots, c_l - \lambda_l$ is also proper.*

Proof. Let us consider first the case $l = 1$ and denote $c_1 = c$. We use the notation introduced in this section. In particular $N = I_B^l(a_1, \dots, a_k)$. Assume that the theorem is false. For every $\lambda \in \mathbb{C}$ there exists $r_\lambda \in B$ such that $r_\lambda(c - \lambda e) - e \in N$. In particular, we have $[r_\lambda(c-\lambda)] = [e]$. If $\lambda \notin \sigma_A([c])$ then we have also $R(\lambda) \in A$ such that $[c - \lambda]R(\lambda) = [e]$. Hence $[r_\lambda] = [r_\lambda] \cdot [c - \lambda]R(\lambda) = [r_\lambda(c - \lambda)] \cdot R(\lambda) = R(\lambda)$. At least on the resolvent set of $[c]$ in A the class $[r_\lambda]$ is uniquely determined and belongs to A.

We know little about the structure of the algebra A so we do not know if $\sigma_A([c])$ is closed. However, the resolvent set $\mathbb{C} \setminus \sigma_A([c])$ contains the open set $\mathbb{C} \setminus \sigma_B(c)$ on which the resolvent $R(\mu) = [(c - \mu)^{-1}]$ is holomorphic, because B is a Waelbroeck algebra.

Denote by O the union of all open subsets of the complex plane on which $R(\lambda)$ is holomorphic. Let λ belong to the boundary ∂O of O. The intersection of O with an arbitrary neighbourhood U of λ is an open proper subset of U.

By Lemma 3.2 we know that in some disc $D(\lambda, \rho)$ there is defined a holomorphic function $[r(\mu)]$ valued in X that also satisfies the condition $[r(\mu)] \cdot [c-\mu] = [e]$. For $\mu \in O \cap D(\mu, \rho)$ this function coincides with $R(\mu)$, which is valued in A. For an arbitrary linear continuous functional $\varphi : X \to \mathbb{C}$ that vanishes on A, the function $\mu \to \varphi([r(\mu)])$ is holomorphic on $D(\mu, \rho)$ and vanishes on an open subset. It follows that this function vanishes on the whole disc $D(\lambda, \rho)$.

The space X is locally convex and A is closed, so $[r(\mu)] \in A$ for all $\mu \in D(\lambda, \rho)$. We have proved that $D(\lambda, \rho) \subset O$, which contradicts the assumption $\mu \in \partial O$. The boundary of the open set O is empty, so $O = \mathbb{C}$.

The resolvent $R(\mu)$ is a holomorphic function on $\mathbb{C}$, which for large μ coincides with $[(c-\mu)^{-1}]$. By applying in a familiar way the Liouville theorem we obtain the contradiction $R(\mu) \equiv 0$.

There exists $\lambda \in \mathbb{C}$ such that $I_B^l(a_1, \dots, a_k, c - \lambda e)$ is proper.

The theorem is proved in the case $l = 1$. One obtains the complete result by induction on l. ■

Theorem 3.3 implies the projection property of the joint spectrum σ_B.

Theorem 3.4. *Let B be a unital, locally convex Waelbroeck algebra. Then for every $k + l$ tuple of mutually commuting $a_1, \dots, a_k, c_1, \dots, c_l \in B$ and for every $(\mu_1, \dots, \mu_k) \in \sigma_B^l(a_1, \dots, a_k)$ there exists $(\lambda_1, \dots, \lambda_l) \in \mathbb{C}^l$ such that $(\mu_1, \dots, \mu_k, \lambda_1, \dots, \lambda_l) \in \sigma_B^l(a_1, \dots, a_k, c_1, \dots, c_l)$.*

Corollary 3.5. *Let B be a unital, locally convex Waelbroeck algebra. For an arbitrary l-tuple of mutually commuting $c_1, \dots, c_l \in B$ the joint spectrum $\sigma_B(a_1, \dots, a_k)$ is nonempty.*

4. From projection property to the spectral mapping formula

The projection property of the joint spectrum is a special case of the spectral mapping property corresponding to the polynomial mapping $p(x_1, \dots, x_{k+l}) = (x_1, \dots, x_k)$. In fact, both results are equivalent. It was proved in [3] in the case of a Banach algebra B for more general joint spectra. Our joint spectrum σ_B^l is of a very special form so the statement can be proved in an elementary way.

Theorem 4.1. *Let B be a unital, locally convex Waelbroeck algebra. For an arbitrary k-tuple of mutually commuting $a_1, \dots, a_k \in B$ and for every polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$ the spectral mapping formula*

$$p(\sigma_B^l(a_1, \dots, a_k)) = \sigma_B^l(p(a_1, \dots, a_k))$$

is valid.

Proof. Denote $a = (a_1, \dots, a_k) \in A^k$ and $p(a) = (p_1(a), \dots, p_m(a))$. By the one-way spectral mapping formula (Proposition 2.1) we already know that

$$p(\sigma_B^l(a)) \subset \sigma_B^l(p(a)).$$

Let us consider the $k+m$-tuple $(p(a), a)$ whose elements also commute. Suppose that $\mu = (\mu_1, \dots, \mu_m) \in \sigma_B^l(p(a))$. By Theorem 3.4 there exists $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$ such that $(\mu, \lambda) \in \sigma_B^l(p(a), a)$. The one-way spectral mapping property implies $\lambda \in \sigma_B^l(a)$. It suffices to prove that $\mu = p(\lambda)$. We know that $I_B^l(a - \lambda, p(a) - \mu)$ is proper. This means that for arbitrary $b_1, \dots, b_k \in B$ and $c_1, \dots, c_m \in B$

$$S = \sum_{i=1}^{m} c_i(p_i(a) - \mu_i) + \sum_{j=1}^{k} b_j(a_j - \lambda_j)$$

is not invertible in B. By the remainder theorem (see the proof of Proposition 2.1)

$$p_i(a) = \sum_{j=1}^{k} f_{ji}(a)(a_j - \lambda_j) + p_i(\lambda),$$

where f_{ij} are polynomials of k variables.

If $1 \le l \le m$ and we put in S the values $c_l = e$, $c_i = 0$ for $i \ne l$ end $b_j = -f_{lj}(a)$, we find that $(p_l(\lambda) - \mu_l)e$ is not invertible in B. So $p_l(\lambda) = \mu_l$ for $1 \le l \le m$ and the proof follows. ■

Let us define

$$\sigma_B^r(a_1, \dots, a_k) = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k | \; I_B^r(a_1 - \lambda_1, \dots, a_k - \lambda_k) \ne B\}.$$

The right-hand side version of Theorem 4.1 is obviously valid. Put

$$\sigma_B(a_1, \dots, a_k) = \sigma_B^l(a_1, \dots, a_k) \cup \sigma_B^r(a_1, \dots, a_k).$$

The spectrum σ_B also obeys the spectral mapping formula.

Theorem 4.2. *Let B be a unital Waelbroeck algebra. For an arbitrary k-tuple of* *mutually commuting* $a_1, \dots, a_k \in B$ *and for every polynomial mapping* $p : \mathbb{C}^k \to \mathbb{C}^m$ *one has*

$$p(\sigma_B(a_1, \dots, a_k)) = \sigma_B(p(a_1, \dots, a_k)).$$

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