EVERY WEAKLY SEQUENTIALLY HYPERCYCLIC SHIFT IS NORM HYPERCYCLIC

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Abstract

We show that bilateral shifts on $\ell^p(\mathbb{Z})$, with $1 \leq p < \infty$, are weakly sequentially hypercyclic if and only if they are norm hypercyclic, and that the same holds true for supercyclicity.

1. Introduction

A bounded linear operator $T : X \to X$ on a Banach space X is said to be a *weakly sequentially hypercyclic operator* if there is a vector x in X whose orbit $\text{orb}(T, x) = \{x, Tx, T^2x, \cdots\}$ is weakly sequentially dense in X. That is, for any vector y in X, there is a sequence $(T^{n_k}x)_k$ in $\text{orb}(T, x)$ converging to y weakly, or equivalently,

$$\lim_{k\to\infty} < T^{n_k}x - y,\, x^* > = \, 0, \quad \text{whenever } x^* \text{ is in the dual space } X^*.$$

Such a vector x is called a *weakly sequentially hypercyclic vector* for T. If there is an orbit $\text{orb}(T, x)$ that is dense in X in the weak (respectively, norm) topology of X, we say that T is *weakly hypercyclic* (respectively, *norm hypercyclic*), and that x is a *weakly* (respectively, *norm*) *hypercyclic vector* for T. Clearly if T is norm hypercyclic, then T is weakly sequentially hypercyclic, which in turn implies T is weakly hypercyclic.

Similarly, an operator T on a Banach space X is (a) *norm supercyclic* (b) *weakly supercyclic* (c) *weakly sequentially supercyclic*, provided there exists a vector x so that $\text{Orb}(\text{span}(x), T) = \{\lambda T^k x : \ \lambda \in \ \mathbb{C}, \ k \geq 0\}$ is (a) norm dense, (b) weakly dense, (c) weakly sequentially dense in X, respectively.

It is not known whether an operator can be weakly sequentially hypercyclic

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(respectively, weakly sequentially supercyclic) without being norm hypercyclic (respectively, norm supercyclic). We show here in Theorem 1 that the answer is negative (in both the hypercyclic and supercyclic cases) for the bilateral weighted shifts, as it is known in the case of unilateral weighted shifts.

We recall that a bilateral weighted shift on the space $\ell^p(\mathbb{Z})$ of $p-$th power summable bilateral sequences ($1 \leq p < \infty$) is an operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ of the form $Te_j = w_j e_{j-1}$ $(j \in \mathbb{Z})$, where the weight bilateral sequence $(w_j)_{j \in \mathbb{Z}}$ of scalars is bounded, and where each $e_j = (\ldots, 0, 0, 1, 0, 0, \ldots)$ in $\ell^p(\mathbb{Z})$ has coordinate 1 in the j^{th} position and 0 otherwise.

A necessary and sufficient condition for a bilateral weighted shift to be norm hypercyclic was obtained by Salas [7, theorem 2.1]. Also, it was shown in [4, corollary 3.3] that there exists a bilateral weighted shift on $\ell^p(\mathbb{Z})$, with $2 \leq p < \infty$, that is weakly hypercyclic and has the property that every orbit is norm increasing. Hence no orbit can contain a weakly convergent sequence that converges to a vector outside the orbit, and so such a bilateral shift cannot be weakly sequentially hypercyclic, nor norm hypercyclic.

In the case that T is a unilateral weighted backward shift on $\ell^p(\mathbb{Z}^+)$, Salas [7, corollary 2.9] gave an equivalent condition for T to be norm hypercyclic. It was shown in [4, theorem 4.1] that T being norm hypercyclic is equivalent to T being weakly hypercyclic, which in turn is equivalent to T being weakly sequentially hypercyclic.

The situation for supercyclicity is different. Hilden and Wallen [6, theorem 3] showed that every unilateral weighted backward shift on $\ell^p(\mathbb{Z}^+)$ is norm supercyclic, and so weakly sequentially supercyclic. For a bilateral weighted shift T, Salas [8, theorem 3.1] obtained a necessary and sufficient condition for T to be norm supercyclic. Then it was shown in [9, p. 151] that there exists a weakly supercyclic operator on $\ell^p(\mathbb{Z})$, with $2 \leq p < \infty$, that fails to be norm supercyclic, and furthermore in [10] that an isometric operator can be weakly supercyclic. This is in contrast with a result of Ansari and Bourdon [1] that no isometry can be norm supercyclic. More recently, Bayart and Matheron [3, Example 1'] showed that $M_z : L^2(\mu) \to L^2(\mu)$, the multiplication-by-$z$ operator on $L^2(\mu)$, is indeed an isometric weakly sequentially supercyclic operator, where μ is a continuous probability measure on the unit circle and its support is Helson with constant 1. On the other hand, such a multiplication operator on $L^2(\mu)$ is normal and hence hyponormal, which cannot be weakly hypercyclic, by [9, theorem 4.5]. All these results motivate us to provide the following statement.

Theorem 1. *Let T be a bilateral weighted shift on $\ell^p(\mathbb{Z})$, where $1 \leq p < \infty$. Then*
(i) *T is weakly sequentially hypercyclic if and only if it is norm hypercyclic.*
(ii) *T is weakly sequentially supercyclic if and only if it is norm supercyclic.*

To prove Theorem 1, we need Lemma 2 below. For each y in $\ell^p(\mathbb{Z})$, let $\widehat{y}(j) =< y, e_j >$ = the j-th term of the sequence y. Also, let $D = \{d_1, d_2, \ldots\}$ denote the countable set of all y in $\ell^p(\mathbb{Z})$ for each of which there exists a positive integer λ_y such that $\widehat{y}(j) = 0$ if $|j| > \lambda_y$, and $\widehat{y}(j)$ is rational if $|j| \leq \lambda_y$. If $y \in D$, then we call the least such integer λ_y the *length* of y.

Lemma 2. *Let $1 \leq p < \infty$. Suppose T is a bilateral weighted shift on $\ell^p(\mathbb{Z})$ that is weakly sequentially supercyclic. Let $\epsilon > 0$ and let N be a positive integer. For any vectors y, z in D, there exists an integer $n > N$, a nonzero scalar λ, and a vector v in D such that $\|v\| < \epsilon$, $\|\lambda T^n y\| < \epsilon$, and $\|\lambda T^n v - z\| < \epsilon$. If the shift T is weakly sequentially hypercyclic, the above conclusion holds with $\lambda = 1$.*

PROOF OF LEMMA 2. Let x be a weakly sequentially supercyclic vector for T. Choose a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers and a sequence $(\lambda_k)_{k=1}^{\infty}$ of nonzero scalars such that $\lambda_k T^{n_k} x \longrightarrow z$ weakly. Since weakly convergent sequences are bounded, there exists $M > 0$ such that $\|\lambda_k T^{n_k} x\| \leq M$, whenever $k \geq 1$. If y is the vector given in the statement of the lemma, then there exist $m \geq 1$ and a nonzero scalar β such that

$$|\langle \beta T^m x - Cy, e_j\rangle| < \frac{C|\widehat{y}(j)|}{2}, \quad \text{whenever } |j| \leq \gamma \text{ and } \widehat{y}(j) \neq 0,$$

where $C = 2M/\epsilon$ and γ is the length of y. Hence, $|\,\beta\,\widehat{T^m x}(j)| \geq \frac{C}{2}\,|\widehat{y}(j)|$ for every $|j| \leq \gamma$, and so for $i > m$ we have

$$\begin{aligned}
C^p\,\|T^{i-m}y\|^p &= \textstyle\sum_{j\in\mathbb{Z}} (w_{j+1}w_{j+2}\dots w_{j+i-m})^p\, C^p\,|\widehat{y}(j+i-m)|^p \\
&= \textstyle\sum_{k=-\gamma}^{\gamma} (w_{k+m-i+1}w_{k+m-i+2}\dots w_k)^p\, C^p\,|\widehat{y}(k)|^p \\
&\leq \textstyle\sum_{k=-\gamma}^{\gamma} (w_{k+m-i+1}w_{k+m-i+2}\dots w_k)^p\, 2^p\,|\beta\widehat{T^m x}(k)|^p \\
&= 2^p\,\beta^p\,\textstyle\sum_{k=-\gamma}^{\gamma} (w_{k+m-i+1}w_{k+m-i+2}\dots w_{k+m})^p\,|\widehat{x}(k+m)|^p \\
&\leq 2^p\,\beta^p\,\textstyle\sum_{j\in\mathbb{Z}} (w_{j-i+1}w_{j-i+2}\dots w_j)^p\,|\widehat{x}(j)|^p \\
&= 2^p\,\beta^p\,\|T^{ix}\|^p.
\end{aligned}$$

That is, if $i > m$ then $\|T^{i-m}y\| \leq \frac{2\beta}{C}\,\|T^i x\|$. In particular, for any $n_k > m$ we have

$$\|\frac{\lambda_k}{\beta}\,T^{n_k-m}y\| \leq \frac{2}{C}\,|\lambda_k|\,\|T^{n_k}x\| \leq \frac{2}{C}\,M = \epsilon.$$

For the given vector z in the statement of the lemma, we let μ be its length, and let $v_k = \beta T^m \left(\sum_{j=n_k-\mu}^{n_k+\mu} \widehat{x}(j)e_j\right)$ for each integer $k \geq 1$. Notice that

$$\|v_k\|^p \leq |\beta|^p\,\|T^m\|^p \sum_{n_k-\mu}^{n_k+\mu} |\widehat{x}(j)|^p \longrightarrow 0, \quad \text{as } k \to \infty.$$

Also, for any $n_k > m$ we have

$$\frac{\lambda_k}{\beta}\,\widehat{T^{n_k-m} v_k}(j) = \{\ \ \lambda_k\widehat{T^{n_k}x}(j) \quad \text{for } |j| \leq \mu \ \ \text{for } |j| > \mu.$$

Since $\lambda_k T^{n_k} x \longrightarrow z$ weakly, for all large enough k and all j with $|j| \leq \mu$,

$$|\langle \lambda_k T^{n_k} x - z, e_j\rangle| < \frac{\epsilon}{3\,\mu^{\frac{1}{p}}}.$$

It follows that

$$\|\frac{\lambda_k}{\beta}T^{n_k-m}v_k - z\|^p = \sum_{j=-\mu}^{\mu} |\langle \lambda_k T^{n_k}x - z, e_j\rangle|^p \ < \ (2\mu+1)\frac{\epsilon^p}{3^p\,\mu} \ < \ \epsilon^p.$$

So the conclusion of the lemma is satisfied if we set $v = v_k$, $\lambda = \frac{\lambda_k}{\beta}$, and $n = n_k - m$ for a large enough k.

In the case when T is weakly sequentially hypercyclic, we have the exact same argument as above with $\lambda_k = \beta = 1$, and conclude that $\lambda = 1$. ■

With Lemma 2, we are now ready to prove the theorem.

Proof of Theorem 1. Clearly T is weakly sequentially supercyclic if it is norm supercyclic. To show the converse, suppose T is weakly sequentially supercyclic and recall that $D = \{d_1, d_2, \ldots\}$. By Lemma 2, there exist an integer $n_1 \geq 1$, a nonzero scalar λ_1 and a vector v_1 in D such that $\|v_1\| < 2^{-1}$, and $\|\lambda_1 T^{n_1}v_1 - d_1\| < 2^{-1}$. Again by Lemma 2, there exist $n_2 > n_1$, $\lambda_2 \neq 0$ and $v_2 \in D$ so that $\max\{\|v_2\|,\ \|\lambda_2 T^{n_2}v_1\|,\ \|\lambda_2 T^{n_2}v_2 - d_2\|\ \} \ \leq\ \frac{1}{2^2}\ \min\{1, \frac{1}{|\lambda_1|\,\|T^{n_1}\|}\ \}$. By setting

$$\epsilon = s_k = \ \frac{1}{2^k}\min\left\{1, \frac{1}{|\lambda_1|\,\|T^{n_1}\|}, \ldots, \frac{1}{|\lambda_{k-1}|\,\|T^{n_{k-1}}\|}\right\}$$

in Lemma 2 as the k-th step in an induction process, we obtain an increasing sequence of positive integers (n_k), a sequence (λ_k) of non-zero scalars, and a sequence (v_k) of elements in D that satisfy

$$\{\ \|\ v_k\| \ < \ s_k\|\lambda_k T^{n_k}(v_1 + \ldots + v_{k-1})\| \ < \ s_k\|\lambda_k T^{n_k}v_k - d_k\| \ < \ s_k, \tag{1.1}$$

So if we let $v = \sum_{n=1}^{\infty} v_n$, then $v \in \ell^p(\mathbb{Z})$ and for $k \geq 2$

$$\begin{aligned}\|\lambda_k T^{n_k}v \quad -d_k\| &= \|\lambda_k T^{n_k}\left(\textstyle\sum_{i=1}^{k-1} v_i\right) + \lambda_k T^{n_k}v_k - d_k + \textstyle\sum_{i=k+1}^{\infty}\ \lambda_k T^{n_k}v_i\| \\ &\leq \|\lambda_k T^{n_k}\left(\textstyle\sum_{i=1}^{k-1} v_i\right)\| + \|\lambda_k T^{n_k}v_k - d_k\| + \textstyle\sum_{i=k+1}^{\infty}\ \|\lambda_k T^{n_k}v_i\| \\ &\leq\ \tfrac{1}{2^k} + \tfrac{1}{2^k} + \textstyle\sum_{i=k+1}^{\infty}\ \tfrac{1}{2^i}\ \longrightarrow 0,\ \text{as}\ \ k \to \infty.\end{aligned}$$

Hence v is a norm supercyclic vector for T, and (ii) follows.

To show (i), we simply take $\lambda_k = 1$ in the above argument. ■

A review of the proof of Theorem 1 shows that we may replace $\ell^p(\mathbb{Z})$ by $c_0(\mathbb{Z})$ in the statement of Theorem 1. On the other hand, the proof may seem to suggest a sharper result that a bilateral weighted shift would be norm hypercyclic if it had an orbit containing a weakly convergent sequence, but this is not the case. Take, for example, the weakly hypercyclic bilateral shift $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by $Te_j = 2e_{j-1}$ for positive integers j and $Te_j = e_{j-1}$ for nonpositive integers j. This operator is weakly hypercyclic but not norm hypercyclic; see [4, corollary 3.5]. Nevertheless the vector e_0 has its orbital vectors $T^n e_0 = e_{-n}$ converging weakly to the zero vector as a sequence.

An argument similar to the proof of Theorem 1 shows that for any bilateral weighted shifts $T_1, \ldots, T_r$ ($r \geq 1$) on $\ell^p(\mathbb{Z})$ ($1 \leq p < \infty$), their direct sum $T_1 \oplus \ldots \oplus T_r$ is weakly sequentially hypercyclic (respectively, weakly sequentially supercyclic) on $\ell^p(\mathbb{Z})^r$ if and only if each T_i is norm hypercyclic (respectively, supercyclic).

We now focus on hypercyclicity. First, we provide a condition for a weakly hypercyclic bilateral weighted shift to be norm hypercyclic.

Proposition 3. *Let $T : \ell^p(\mathbb{Z}) \longrightarrow \ell^p(\mathbb{Z})$ ($1 \leq p < \infty$) be a weakly hypercyclic bilateral weighted shift with weight sequence $\{w_j : j \in \mathbb{Z}\}$. If $\lim_{n\to\infty} \prod_{j=0}^{n} w_{-j} = 0$, then the operator T must be norm hypercyclic.*

PROOF. Let q be a positive integer, and $\epsilon > 0$. Choose a positive δ with $\delta < \epsilon \min\{1, \prod_{j=1}^{i} w_j^{-1}, \prod_{j=0}^{i-1} w_{-j} : 1 \leq i \leq q\}$. By our hypothesis, there exists an integer $N \geq 1$ such that

$$\prod_{j=0}^{n} w_{-j} < \delta, \qquad \text{for all } n \geq N.$$

From the choice of δ, it follows that $\prod_{j=0}^{n-1} w_{i-j} < \epsilon$ if $-q \leq i \leq q$ and $n \geq N+q+1$. By a result of Salas' [7, Ttheorem 2.1] it suffices to show that there exists n arbitrarily large for which $\prod_{j=1}^{n} w_{i+j} > 1/\epsilon$. Now, let x be a weakly hypercyclic vector for T with $\|x\| = \epsilon/2$. Then, there exists $n \geq N + q + 1$ such that

$$|\langle T^n x - \sum_{j=-q}^{q} e_j, e_i\rangle| < \frac{1}{2} \qquad \text{whenever} \quad |i| \leq q.$$

That is, $|\widehat{x}(n+i) \prod_{j=1}^{n} w_{i+j} - 1| < \frac{1}{2}$ for each i with $|i| \leq q$ and hence

$$\frac{1}{2} < |\widehat{x}(n+i)| \prod_{j=1}^{n} w_{i+j} \;\leq\; \frac{\epsilon}{2} \prod_{j=1}^{n} w_{i+j},$$

which gives $\prod_{j=1}^{n} w_{i+j} > 1/\epsilon$ whenever $|i| \leq q$. ■

The condition $\lim_{n\to\infty} \prod_{j=0}^{n} w_{-j} = 0$ in Proposition 3 is only sufficient, but not necessary; the shift with weight sequence

$$(\ldots, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 2, 2, \tfrac{1}{2}, \tfrac{1}{2}, 2, \tfrac{1}{2}, 1, 2, \tfrac{1}{2}, 2, 2, \tfrac{1}{2}, \tfrac{1}{2}, 2, 2, 2, 2, \ldots)$$

(here $w_0 = 1$) is norm hypercyclic, and $\lim_{n\to\infty} \prod_{j=0}^{n} w_{-j} = 0$ is not satisfied.

On the other hand we cannot conclude in Proposition 3 that a weakly hypercyclic bilateral weighted shift on $\ell^p(\mathbb{Z})$ ($1 \leq p < \infty$) must be norm hypercyclic if we merely assume that $\liminf_n \prod_{j=0}^{n} w_{-j} = 0$. To see this, let $(m_i)_{i=1}^{\infty}$ be a sequence of positive integers such that $m_i - m_{i-1} > i$ for each $i \geq 1$ with $m_0 = 0$, and let

$T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift whose weight sequence is given by

$$w_{-j} = \begin{cases} 1, & \text{if } m_{i-1} + (i-1) \le j \le m_i - 2, \\ 2^{-i}, & \text{if } j = m_i - 1, \\ 2, & \text{if } m_i \le j \le m_i + (i-1), \end{cases}$$

and

$$w_j = \begin{cases} 2, & \text{if } m_{i-1} + i \le j \le m_i - 1, \\ 2^{-m_i + m_{i-1} + i + 1}, & \text{if } j = m_i, \\ 1, & \text{if } m_i + 1 \le j \le m_i + i. \end{cases}$$

For any choice of $(m_i)_{i=1}^{\infty}$, we have that $\liminf_n \prod_{j=0}^{n} w_{-j} = 0$ and T fails to be norm hypercyclic. But one can choose the sequence $(m_i)_{i=1}^{\infty}$ far enough apart so that T is weakly hypercyclic; see [4, theorem 3.2].

We now show that weakly sequentially hypercyclic vectors carry some linear structure whenever they exist. Results of this kind were first obtained by Herrero [5], and independently by Bourdon [2]. They proved that every hypercyclic operator on a Banach space has an invariant, norm dense, linear subspace in which every nonzero vector is hypercyclic. Later Wengenroth [11] extended this to hypercyclic operators on non-locally convex spaces.

Proposition 4. *A bounded linear operator T on a Banach space is weakly sequentially hypercyclic if and only if there exists a norm dense, invariant, linear subspace in which every nonzero vector is a weakly sequentially hypercyclic vector for T.*

Proof. Let x be a weakly sequentially hypercyclic vector for T. The set $\{p(T)x : p \text{ is a polynomial}\}$ is a weakly dense, invariant, linear subspace. Since the set is convex, it is norm dense in X. We now show that every nonzero vector $p(T)x$ is weakly sequentially hypercyclic for T. For any $y \in X$, there exists a sequence $(T^{n_k}x)$ such that $T^{n_k}x \to y$ weakly. Hence, $T^{n_k}p(T)x = p(T)T^{n_k}x \to p(T)y$ weakly. Therefore,

$$p(T)X \subset \overline{\text{orb}(T, p(T)x)}^{\text{weak seq}}.$$

Since T is weakly hypercyclic and the weak and norm closures of a convex set coincide, $p(T)$ must have norm dense range, by a result by Wengenroth [11, lemma 1]. Hence, we have

$$X = \overline{p(T)X}^{\text{weak seq}} \subset \overline{\text{orb}(T, p(T)x)}^{\text{weak seq}},$$

which finishes the proof. ■

To conclude this note, we pose the following problem.

Problem 5. *Is every weakly sequentially hypercyclic operator on a Banach space norm hypercyclic?*

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